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Research Paper

BI-INTERIOR, QUASI-INTERIOR AND BI-QUASI-INTERIOR Γ-HYPERIDEAL IN Γ-SEMIHYPERRING

Chaitanya B. Kumbharde, Kishor F. Pawar and Safoora J.Ansari

Abstract. The concept of a Γ-semihyperring is a generalization of a semiring, semihyperring, Γ-semiring. In this Paper we introduce the notion of bi-interior Γ-hyperideals, quasi-interior Γ-hyperideals and bi-quasi-interior Γ-hyperideals in a Γ-semihyperring as a generalization of Γ-hyperideal, left-Γ-hyperideal, right-Γ- hyperideals, bi Γ-hyperideal, quasi Γ-hyperideal, interior Γ-hyperideals of Γ-semihyperring. We studied the properties of these Γ-hyperideals and characterized them in simple Γ-semihyperring and regular Γ-semihyperring.

Key Words: Γ-semihyperring, bi-interior Γ- hyperideals, quasi-interior Γ- hyperideals, bi-quasi-interior Γ- hyperideals.

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1. Introduction

In normal algebraic structures the notion of *operation* is fundamental to develop the theory. It can be extended to define *hyperoperation* which in consequence provide a rout to the development of *hyperstructures*. The generalization is already established in the year 1934 for an

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∗Address correspondence to S. J. Ansari; E-mail: ansari.sjacs@snjb.org.

algebraic structure called group theory by the French Mathematician Frédéric Marty when he published the paper [7]. In this paper, Marty put forward the notion of *hypergroups* as a generalization of groups from the analysis of their properties. Study of hyperstructures become more exible as it is different from usual binary and ternary operations and produces generalized results in addition to those existing in classical algebraic theory. There are many researchers across the world who study hyperstructures and extend their contributions through research articles and books.

Davvaz et. al. defined the notion of [8] Γ-semihyperring as a generalization of semiring, semihyperring and Γ-semiring. Following which Pawar et. al. [12] introduced regular (strongly regular) Γ-semihyperrings and presented it's characterizations using ideals of Γ-semihyperrings.

The quasi ideals are a generalization of left ideals and right ideals whereas the bi-ideals are generalization of quasi ideals. The notion of quasi-ideals in semirings without zero was introduced by K. Isaki [4]. He studied some properties of semirings with reference to quasi-ideals. Shabir, Ali, and Batool [16] used the notion of quasi-ideals to characterize semirings. Jagtap and Pawar[5] thoroughly discussed the concept of quasi ideals in Γ-semirings. As a generalization of the concept of quasi-ideal in different algebraic systems, bi-ideal is introduced. The notion bi-ideals in semigroups is first given by Good and Hughes [3]. In 1970, Lajos and Szsz [6] introduced the concept of bi-ideals in associative rings. As a further generalization of ideals, M.K. Rao [15] introduced the notion of bi-interior ideal in semigroup. He has also introduced bi-quasi ideals in semiring, bi-quasi-ideals and fuzzy bi-quasi-ideals in Γ-semigroups [14] and explored some of their properties [13, 14]. S. J. Ansari and K.F. Pawar further studied various kinds of Γ-hyperideals in Γ-semihypergroups [1, 11]

The authors observed that ideals play important role in the study of various properties of algebaric structures. Deriving a huge motivation from the research work of Jagtap and Pawar, M. K. Rao and the fascinating generalization of classical algebraic structures to that of various hyperstructrues by Davvaz et. al, a need was felt to introduce and study few more ideals in Γ-semihyperrings.

In this paper we introduced bi-interior Γ-hyperideals, quasi-interior Γ-hyperideals and bi-quasi-interior Γ-hyperideals in a Γ-semihyperring as a generalization of Γ-hyperideal, one-sided Γ-hyperideals, bi Γ-hyperideal, quasi Γ-hyperideal and interior Γ-hyperideals in Γ- semihyperring. We study the properties of these ideals and later characterized them in simple Γ-semihyperrings and regular Γ-semihyperrings.

2. Preliminaries

In this section we recall some fundamental concepts and definitions which are necessary for this paper. For more details it is suggested to refer [8, 12].

Definition 2.1. [2] Let H be a non-empty set. A hyperoperation on H is map $\circ : H \times H \rightarrow \wp^*(H)$ where $\wp^*(H)$ is collection of all non-empty subset of H.
The pair $(H, \circ)$ is called a hypergroupoid.

For any two non-empty subsets A and B of H and $x \in H$,

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b,\ A \circ \{x\} = A \circ x \ and\ \{x\} \circ A = x \circ A.$$

Definition 2.2. [2] A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all $a, b, c \in H$, $(a \circ b) \circ c = a \circ (b \circ c)$, that is

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

In addition, if for every $a \in H, a \circ H = H = H \circ a$, then $(H, \circ)$ is called a hypergroup.

Definition 2.3. [8] Let R be a commutative semihypergroup and Γ be a commutative group. Then R is called a Γ-semihyperring if there is a map $R \times \Gamma \times R \rightarrow \wp^*(R)$ (images to be denoted by $a\alpha b$ for all $a, b \in R$, $\alpha \in \Gamma$) satisfying the following conditions:

(1) $a\alpha(b + c) = a\alpha b + a\alpha c$

(2) $(a + b)\alpha c = a\alpha c + b\alpha c$

(3) $a(\alpha + \beta)c = a\alpha c + a\beta c$

(4) $a\alpha(b\beta c) = (a\alpha b)\beta c;$ for all $a, b, c \in R$ and for all $\alpha, \beta \in \Gamma$.

Example 2.4. [8] Let $(R, +, \cdot)$ be a semihyperring such that $x \cdot y = x \cdot y + x \cdot y$ and Γ be a commutative group. We define $x\alpha y \rightarrow x \cdot y$ for every $x, y \in R$ and $\alpha \in \Gamma$ then R is a Γ-semiyperring.

Definition 2.5. A Γ-semihyperring R is said to be commutative if $a\alpha b = b\alpha a$ for all $a, b \in R$ and $\alpha \in \Gamma$.

Definition 2.6. [8] A Γ-semihyperring R is said to be with zero if there exists $0 \in R$ such that $a \in a + 0$ and $0 \in 0\alpha a$, $0 \in a\alpha 0$ for all $a \in R$ and $\alpha \in \Gamma$.

Let A and B be two non-empty subsets of a Γ-semihyperring R and $x \in R$ then

$$A + B = \{x \mid x \in a + b, a \in A, b \in B\}$$
$$A\Gamma B = \{x \mid x \in a\alpha b, a \in A, b \in B, \alpha \in \Gamma\}$$

Definition 2.7. [8] A non-empty subset R_1 of a Γ-semihyperring R is called a Γ-subsemihyperring if it is closed with respect to the multiplication and addition that is, $R_1 + R_1 \subseteq R_1$ and $R_1\Gamma R_1 \subseteq R_1$.

Definition 2.8. [8] A right (left) ideal I of a Γ-semihyperring R is an additive Γ-subsemihyperrring of $(R, +)$ such that $I\Gamma R \subseteq I$ $(R\Gamma I \subseteq I)$. If I is both right and left ideal of R, then we say that I is a two-sided ideal or simply an ideal of R.

Definition 2.9. [9] A non-empty set B of a Γ-semihyperring R is a bi-Γ-hyperideal of R if B is a Γ-subsemihyperring of R and $B\Gamma R\Gamma B \subseteq B$.

Definition 2.10. [10] A subsemihypergroup Q of $(R, +)$ is a quasiΓ-hyperideal of Γ-semihyperring R if $(R\Gamma Q) \cap (Q\Gamma R) \subseteq Q$.

Definition 2.11. A non-empty subset J of a Γ-semihyperring R is an interior Γ-hyperideal of R, if J is Γ-semihypergroup and $R\Gamma J\Gamma R \subseteq J$.

Definition 2.12. [12] A subset A of a Γ-semihyperring R is said to be regular, if there exist $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $B \subseteq R$ such that $A \subseteq A\Gamma_1 B\Gamma_2 A$. A singleton set $\{a\}$ of a Γ-semihyperring is regular if there exist $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $B \subseteq R$ such that

$$\{a\} = a \in a\Gamma_1 B\Gamma_2 a = \{xinR \mid xina\alpha b\beta a, \alpha in\Gamma_1, \beta \in \Gamma_2, b \in B\}.$$

That is, a singleton set $\{a\}$ of Γ-semihyperring is regular if there exists $\alpha, \beta \in \Gamma, b \in R$ such that $a \in a\alpha b\beta a$.

Definition 2.13. [12] A Γ-semihyperring R is said to be regular Γ-semihyperring, if every element of R is regular.

3. Bi-Interior Γ-hyperideal

Good and Hughes [3] defined the concept of bi-ideals of a semigroup. In this section we introduce the notion of bi-interior Γ-hyperideals in Γ-semihyperrings as a generalization of bi Γ-hyperideals and interior Γ-hyperideals. Moreover we characterize bi-Γ-hyperideal in regular as well as in simple interior-Γ-semihyperring.

Definition 3.1. A non-empty Γ-subsemihyperring B of a Γ- semihyperring R is a bi-interior Γ-hyperideal of R if $R\Gamma B\Gamma R \cap B\Gamma R\Gamma B \subseteq B$.

Definition 3.2. A Γ-semihyperring R is a bi-interior-simple Γ- semihyperring if R has no proper bi-interior Γ-hyperideal of R.

Theorem 3.3. *In a* Γ*-semihyperring* R *the following statements are true.*

(1) *Every quasi* Γ*-hyperideal of* R *is a bi-interior* Γ*-hyperideal of* R*.*
(2) *Every* Γ*-hyperideal of* R *is a bi-interior* Γ*-hyperideal of* R*.*
(3) *Every bi-*Γ*-hyperideal of* R *is a bi-interior* Γ*-hyperideal of* R*.*
(4) *Every interior* Γ*-hyperideal of* R *is a bi-interior* Γ*-hyperideal of* R*.*
(5) $B = R_1\Gamma R_2$ *is a bi-interior* Γ*-hyperideal of* R *where* R_1 *and* R_2 *are* Γ*-subsemihyperrings of* R*.*

Remark 3.4. The converse of statement (1) in the Theorem 3.3 need not be true. That is a bi-interior Γ-hyperideal need not be a quasi Γ-hyperideal of R.

Example 3.5. Let $R = \{a, b, c, d\}$. Then R is a commutative semihyperring with hyperoperations $\oplus$ and $\otimes$ on R are defined as follows:

$\oplus$	a	b	c	d
a	{a}	{a, b}	{a, c}	{a, d}
b	{a, b}	{b}	{b, c}	{b, d}
c	{a, c}	{b, c}	{c}	{c, d}
d	{a, d}	{b, d}	{c, d}	{d}

$\otimes$	a	b	c	d
a	{a}	{a}	{a}	{a}
b	{a}	{a}	{a}	{a}
c	{a}	{a}	{a}	{a, b}
d	{a}	{a, d}	{a, b}	{c}

Define a mapping $R \times \Gamma \times R \to \wp^*(R)$ by $p\gamma q = p \otimes q$ for every $p, q \in R$ and $\gamma \in \Gamma$. Then R is a Γ-semihyperring. Observe that $B_1 = \{a, b\}$ and $B_2 = \{a, c\}$ are bi-interior Γ-hyperideals but B_2 is not a quasi Γ-hyperideal. Since $(R\Gamma B_2) \cap (B_2 \Gamma R) = \{a, b\} \nsubseteq \{a, c\}$.

Remark 3.6. The converse of a statement (2) in the Theorem 3.3 need not be true. That is a bi-interior Γ-hyperideal of R need not be a Γ-hyperideal of R.

Example 3.7. Consider the following sets:

$$R = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, y, z, w \in \mathbb{R} \right\}$$

$$\Gamma = \{z \mid z \in \mathbb{Z}\}, \quad A_\alpha = \left\{ \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha d \end{pmatrix} \mid a, d \in \mathbb{R}, \alpha \in \Gamma \right\}$$

Then R is a Γ-semihyperring under matrix addition and the hyperoperation $M\alpha N \to MA_\alpha N$, for all $M, N \in R$ and $\alpha \in \Gamma$. Let $B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \mid d \in R \right\} \subseteq R$. Then B is a bi-interior Γ-hyperideal of R but B is not a Γ-hyperideal of R because it is neither left nor a right Γ-hyperideal of R.

Remark 3.8. The converse of the statements (3) and (4) of the Theorem 3.3 are also not true in general. That is a bi-interior Γ-hyperideal of R need not be a bi Γ-hyperideal of R. Similarly, a bi-interior Γ-hyperideal of R need not be an interior Γ- hyperideal of R.

Example 3.9. Consider the following sets:

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}, \qquad \Gamma = R$$

Then R is a Γ-semihyperring under matrix addition and the hyperoperation $A\alpha B \to A\Gamma B$ for all $A, B \in R$ and $\alpha \in \Gamma$.
Let $B = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mid a, c \in R \right\}$. It is easy to see that $B\Gamma R\Gamma B \cap R\Gamma B\Gamma R \subseteq B$. Hence B is a bi-interior Γ-hyperideal of R, but $B\Gamma R\Gamma B \nsubseteq B$ this implies that B is not a bi Γ-hyperideal of R. Also $R\Gamma B\Gamma R \nsubseteq B$ this implies that B is not an interior Γ-hyperideal of R, but $B\Gamma R\Gamma B \cap R\Gamma B\Gamma R \subseteq B$. Hence B is bi-interior Γ-hyperideal of R.

Theorem 3.10. *Let R be a simple Γ-semihyperring. Then every bi-interior Γ-hyperideal of R is a bi Γ-hyperideal of R.*

Proof. Let B be a bi-interior Γ-hyperideal of a simple Γ-semihyperring R. Then $R\Gamma B\Gamma R \cap B\Gamma R\Gamma B \subseteq B$. As $R\Gamma B\Gamma R$ is a Γ-hyperideal of R, by definition of a simple Γ-semihyperring, we obtain $R\Gamma B\Gamma R = R$. Therefore $R\Gamma B\Gamma R\Gamma B \subseteq B$, hence $B\Gamma R\Gamma B \subseteq B$. This implies that B is a bi Γ-hyperideal of R. □

Theorem 3.11. *Let R be a Γ-semihyperring. Then for all $a \in R$, $R\Gamma a\Gamma R \cap a\Gamma R\Gamma a = R$ if and only if R is a bi-interior simple Γ- semihypering.*

Proof. Let R be a Γ-semihyperring and for all $a \in R$, $R\Gamma a\Gamma R \cap a\Gamma R\Gamma a = R$. Let B be a bi-interior Γ-hyperideal of R and $a \in B$. We have $R = R\Gamma a\Gamma R \cap a\Gamma R\Gamma a \subseteq R\Gamma B\Gamma R \cap B\Gamma R\Gamma B$ hence $R \subseteq B$ and we obtain $R = B$. Conversely assume that R is a bi-interior simple Γ-semihyperring and $a \in R$. Then $R\Gamma a\Gamma R \cap a\Gamma R\Gamma a$ is a bi-interior Γ-hyperideal of R, but R is bi-interior simple Γ-semihyperring. Therefore $(R\Gamma a\Gamma R) \cap (a\Gamma R\Gamma a) = R$, for all $a \in R$. □

Theorem 3.12. *Let R' be a Γ-subsemihyperring of a Γ-semihyperring R and B is bi-interior Γ-hyperideal of R. Then $B \cap R'$ is a bi-interior Γ-hyperideal of R.*

Proof. Let R' be a Γ-subsemihyperring of a Γ-semihyperring R and B is a bi-interior Γ-hyperideal of R. Suppose $A = B \cap R'$ then $A\Gamma R'\Gamma A \subseteq R'\Gamma R'\Gamma R' \subseteq R'$ since R' is a Γ-subsemihypering and $A \subseteq R'$. Also $A\Gamma R'\Gamma A \subseteq B\Gamma R'\Gamma B \subseteq B\Gamma R\Gamma B$ and $R\Gamma A\Gamma R \subseteq R\Gamma B\Gamma R$. Therefore $A\Gamma R'\Gamma A \cap R\Gamma A\Gamma R \subseteq B\Gamma R\Gamma B \cap R\Gamma B\Gamma R \subseteq B$. Since B is a bi-interior Γ-hyperideal of R and $A\Gamma R'\Gamma A \cap R\Gamma A\Gamma R \subseteq A\Gamma R'\Gamma A \subseteq R'$ hence $(A\Gamma R'\Gamma A) \cap (R\Gamma A\Gamma R) \subseteq B \cap R' = A$. Therefore $B \cap R'$ is a bi-interior Γ-hyperideal of R. □

Theorem 3.13. *If R is a regular Γ-semihperring. Then the following statements are true in R.*

1. *Every bi-interior Γ-hyperideal of a Γ-semihyperring R is a Γ-hyperideal of R.*
2. *B is a bi-interior Γ-hyperideal of R if and only if $R\Gamma B\Gamma R \cap B\Gamma R\Gamma B = B$ for all bi-interior Γ-hyperideals of R.*
3. *Every bi-interior Γ-hyperideal of R is a bi Γ-hyperideal of R.*

Proof. (1) Let R be a regular Γ-semihyperring and B be a bi-interior Γ-hyperideal of R. Then $B\Gamma R\Gamma B \cap R\Gamma B\Gamma R \subseteq B$. Since R is a regular Γ-semihyperring we have $B\Gamma R \subseteq B\Gamma R\Gamma B$. Also

$B\Gamma R \subseteq R\Gamma B\Gamma R$, therefore $B\Gamma R \subseteq B\Gamma R\Gamma B \cap R\Gamma B\Gamma R \subseteq B$. Similarly, we can prove that $R\Gamma B \subseteq B$. Hence B is Γ-hyperideal of R.

(2) Suppose $R\Gamma B\Gamma R \cap B\Gamma R\Gamma B = B$ for all bi-interior Γ- hyperideals B of R. Let $B = J \cap I$, where I is a left Γ-hyperideal and J is a right Γ-hyperideal of R. Since B is bi-interior Γ-hyperideal of R we have, $R\Gamma(J \cap I)\Gamma R \cap (J \cap I)\Gamma R\Gamma(J \cap I) = (J \cap I)$. $(J \cap I) \subseteq J \cap I\Gamma R\Gamma J \cap I \subseteq J\Gamma R\Gamma I \subseteq J\Gamma I \subseteq J \cap I$. Thus $J \cap I \subseteq I$ and $J \cap I \subseteq J$ hence $J \cap I = J\Gamma I$ and R is a regular Γ-semihyperring. Conversely assume that R is a regular Γ- semiyperring and B is a bi-interior Γ-hyperideal of R. Let $b \in B$ then $R\Gamma B\Gamma R \cap B\Gamma R\Gamma B \subseteq B$ and there exist $a \in R$ and $\alpha, \beta \in \Gamma$ such that $b = a\alpha b\beta a \in B\Gamma R\Gamma B$. Therefore $b \in R\Gamma BR \cap B\Gamma R\Gamma B$ hence $R\Gamma B\Gamma R \cap B\Gamma R\Gamma B = B$.

(3) Let R be a regular Γ-semihyperring and B be a bi-interior Γ-hyperideal of R. Then $B\Gamma R\Gamma B \cap R\Gamma B\Gamma R = B$ by statement (2). Therefore $B\Gamma R\Gamma B \subseteq R\Gamma B\Gamma R\Gamma B\Gamma R \subseteq R\Gamma B\Gamma R$ and $B\Gamma R\Gamma B \cap R\Gamma B\Gamma B = B$. Hence $B\Gamma R\Gamma B = B$. Therefore B is a bi Γ-hyperideal of R.

□

Theorem 3.14. *Let R be a Γ-semihyperring. Then the intersection of all bi-interior Γ-hyperideals of R is a bi-interior Γ-hyperideal of R.*

Definition 3.15. An element i of a Γ-semihyperring R is said to be an α-idempotent if $i \in i\alpha i$. In general an element i of a Γ-semihyperring R is said to be a Γ-idempotent or simply an idempotent if $i \in i\alpha i$ for all $\alpha \in \Gamma$ that is $i \in i\Gamma i$.

Theorem 3.16. *Let R be a Γ-semihyperring and B be a bi-interior Γ-hyperideal of R. Then the following statements are true in R.*

(1) *$a\Gamma B$ is a bi-interior Γ-hyperideal where a is a β-idempotent element and $a\Gamma B \subseteq B$.*
(2) *$a\Gamma R$ and $R\Gamma a$ are bi-interior Γ-hyperideals of R where a is an α-idempotent element.*
(3) *if a is an α-idempotent and b is a β-idempotent element then $a\Gamma R\Gamma b$ is a bi-interior Γ-hyperideal of R.*

Proof. (1) Let R be a Γ-semihyperring and B be an interior Γ-hyperideal. Suppose $m \in B \cap a\Gamma R$ then $m \in B$ and $m = a\alpha n$ where $\alpha \in \Gamma, n \in R$. As $m = a\alpha n = a\beta a\alpha n = a\beta(a\alpha n) =$

$a\beta m \in a\Gamma B$ hence $B \cap a\Gamma R \subseteq a\Gamma B$. We have $a\Gamma B \subseteq B$ and $a\Gamma B \subseteq a\Gamma R$. Therefore $a\Gamma B \subseteq B \cap a\Gamma R$. Thus $a\Gamma B = a\Gamma R$. Hence $a\Gamma B$ is a bi-interior Γ-hyperideal.

(2) Straightforward.

(3) Let a be an α-idempotent and b be a β-idempotent elements of a Γ-semihyperring R. Then $a\Gamma R\Gamma b \subseteq a\Gamma R$ and $a\Gamma R\Gamma b \subseteq R\Gamma b$, this implies that $a\Gamma R\Gamma b \subseteq a\Gamma R \cap R\Gamma b$. Let $x \in a\Gamma R \cap R\Gamma b$ then $x = a\alpha r = r\beta b$, where $r \in R$. Observe that $x = a\alpha r = a\alpha a\alpha r = a\alpha r\beta b \in a\Gamma R\Gamma b$. Therefore $a\Gamma R \cap R\Gamma b \subseteq a\Gamma R\Gamma b$. Hence $a\Gamma R \cap R\Gamma b = a\Gamma R\Gamma b$.

□

4. Quasi-interior-Γ-hyperideals

Patil and Pawar [5] studied quasi Γ-hyperideal in Γ-semihyperring. In this section we define quasi-interior-Γ-hyperideals in Γ-semihyperring and study their properties in regular Γ-semihyperring. We also analyze converse statements and find suitable counter examples to support the claim.

Definition 4.1. A non-empty Γ-subsemihyperring Q of a Γ- semihyperring R is said to be a left quasi-interior-Γ-hyperideal of R if $R\Gamma Q\Gamma R\Gamma Q \subseteq Q$ and a right quasi-interior-Γ-hyperideal of R if $Q\Gamma R\Gamma Q\Gamma R \subseteq Q$.
Q is said to be a quasi-interior if it is both left and right quasi-interior-Γ-hyperideal of R.

Remark 4.2. A quasi-interior-Γ-hyperideal of a Γ-semihyperring R need not be a quasi Γ-hyperideal, interior Γ-hyperideal and bi-interior Γ-hyperideal of R.

Example 4.3. In the example 3.5 $B_2 = \{a, c\}$ is a quasi-interior Γ-hyperideal of R but not a quasi Γ-hyperideal of R.

Example 4.4. Consider the following sets:

$$R = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \mid x, y \in \mathbb{R} \right\},$$

$$\Gamma = \{z \mid z \in \mathbb{Z}\}, \;\; A_\alpha = \left\{ \begin{pmatrix} \alpha a & 0 \\ \alpha b & 0 \end{pmatrix} \mid a, b \in \mathbb{R}, \alpha \in \Gamma \right\}$$

Then R is a Γ-semihyperring under matrix addition and the hyperoperation $M\alpha N \to MA_\alpha N$, for all $M, N \in R$ and $\alpha \in \Gamma$. Let $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid \right.$

$a \in R \Big\} \subseteq R$. Then Q is a right quasi-interior Γ-hyperideal of R but Q is not a interior Γ-hyperideal of R. Also Q is not bi-interior Γ-hyperideal of R.

Theorem 4.5. *Let R be a Γ-semihyperring. Then the following statements hold in R.*

(1) *Every left (respectively right) Γ-hyperideal of R is a left (respectively right) quasi Γ-hyperideal of R.*
(2) *Every Γ-hyperideal of R is a quasi-interior Γ-hyperideal of R.*
(3) *The intersection of a right Γ-hyperideal and a left Γ-hyperideal of R is a quasi-interior Γ-hyperideal of R.*
(4) *If B is a quasi-interior Γ-hyperideal and R' is a Γ- subsemihyperring of R then $B \cap R'$ is a quasi-interior Γ-hyperideal of R.*
(5) *If B_r and B_l are right and left quasi-interior Γ-hyperideals of R respectively then $B_r \cap B_l$ is a quasi-interior Γ-hyperideal of R .*

Theorem 4.6. *Every quasi-interior Γ-hyperideal of a Γ-semihyperring R is a bi-quasi Γ-hyperideal of R.*

Theorem 4.7. *Let R be a Γ-semihyperring then the following statements hold in R.*

(1) *Every left quasi-interior Γ-hyperideal B of R is a bi-interior Γ-hyperideal of R.*
(2) *Every right quasi-interior Γ-hyperideal B of R is a bi-interior Γ-hyperideal of R.*
(3) *Every interior Γ-hyperideal of R is a left quasi-interior-Γ- hyperideal of R.*
(4) *Let B be a left quasi-interior Γ-hyperideal and J be a right Γ-hyperideal of R then $B \cap J$ is always a left quasi-interior Γ-hyperideal of R.*
(5) *Let B be a right quasi-interior Γ-hyperideal and J be a right Γ-hyperideal of R then $B \cap J$ is always a right quasi-interior Γ-hyperideal of R.*
(6) *Intersection of a quasi-interior Γ-hyperideal and a Γ- hyperideal of R is always a quasi-interior Γ-hyperideal of R.*

Proof. Let R be a Γ-semihyperring.

(1) Let B be a left quasi-interior-Γ-hyperideal of R. Then $R\Gamma B\Gamma R\Gamma B \subseteq B$. We have $R\Gamma B\Gamma R \cap B\Gamma R\Gamma B \subseteq B\Gamma R\Gamma B \subseteq R\Gamma B\Gamma R\Gamma B \subseteq B$. Hence B is an interior-Γ-hyperideal of R.
(2) On the same line of proof (1). We can prove for right quasi-interior Γ-hyperideal.
(3) Let I be an interior-Γ-hyperideal of R. Then $R\Gamma I\Gamma R\Gamma I \subseteq R\Gamma I\Gamma R \subseteq I$. Hence I is a left-quasi-Γ-hyperideal of R.
(4) Suppose $A = B \cap J$. Then $R\Gamma A\Gamma R\Gamma A \subseteq R\Gamma B\Gamma R\Gamma B \subseteq B$. Also $R\Gamma A\Gamma R\Gamma A \subseteq R\Gamma J\Gamma R\Gamma J \subseteq J$. Since J is left Γ-hyperideal of R. Therefore $R\Gamma A\Gamma R\Gamma A \subseteq B \cap J = A$. Hence A is left quasi-interior-Γ-hyperideal of R.
(5) On the same line we can prove for right quasi-interior-Γ-hyperideal of R.
(6) On similar lines as above.

□

Theorem 4.8. *Let R be a Γ-semihyperring. Then $a\Gamma B$ is a left quasi-interior Γ-hyperideal of R where B is a left quasi-interior Γ-hyperideal and $a \in B$ is a β-idempotent element of R.*

Proof. Let B be a left quasi-interior Γ-hyperideal of R. Suppose $x \in B \cap a\Gamma R$ then $x \in B$ and $x = a\alpha y$, where $\alpha \in \Gamma$ and $y \in R$. COnsider $x = a\alpha y = a\beta a\alpha y = a\beta(a\alpha y) = a\beta x \in a\Gamma B$. Therefore $B \cap a\Gamma R \subseteq a\Gamma B$ hence $a\Gamma B \subseteq B$ and $a\Gamma B \subseteq a\Gamma R$. This implies that $a\Gamma B \subseteq B \cap a\Gamma R$ and hence $a\Gamma B = B \cap a\Gamma R$. Thus $a\Gamma B$ is a left-quasi-interior Γ-hyperideal of R. □

Theorem 4.9. *Let R be a simple Γ-semihyperring. Then every left (respectively right) quasi-interior Γ-hyperideal of R is a left (respectively right) Γ-hyperideal of R.*

Proof. Let R be a simple Γ-semihyperring and B be a left quasi-interior Γ-hyperideal of R. Then $R\Gamma B\Gamma R\Gamma B \subseteq B$ and $R\Gamma B\Gamma R$ is Γ-hyperideal of R. Since R is a simple Γ-semihyperring. we have $R\Gamma B\Gamma R = R$. Therefore $R\Gamma B\Gamma R\Gamma B \subseteq B$ and hence $R\Gamma B \subseteq B$. This implies that B is a left Γ-hyperideal of R. □

Theorem 4.10. *Let R be a regular Γ-semihyperring. Then every left (respectively right) quasi-interior-Γ-hyperideal of R is a left (respectively right) Γ-hyperideal of R.*

Proof. Let B be a quasi-interior Γ-hyperideal of R where R is a regular Γ-semihyperring. Then $R\Gamma B\Gamma R\Gamma B \subseteq B$. Consider $R\Gamma B \subseteq R\Gamma B\Gamma R\Gamma R\Gamma B \subseteq$

$R\Gamma B\Gamma R\Gamma B \subseteq B$ since R is a regular Γ- semihyperring, which implies that R is a left Γ-hyperideal of R.

□

Theorem 4.11. *A Γ-semihyperring R is regular if and only if $B\Gamma R\Gamma B\Gamma R = B$ and $R\Gamma B\Gamma R\Gamma B = B$ for all quasi-interior Γ-hyperideals B of R.*

Proof. Let R be a regular Γ-semihyperring and B be a quasi-interior Γ-hyperideal of R. Let $b \in B$ then $R\Gamma B\Gamma R\Gamma B \subseteq B$. Also for any element $b \in B$ there exist elements $a \in R$, $\alpha, \beta \in \Gamma$ such that $b = b\alpha a\beta b\alpha a\beta b \in R\Gamma B\Gamma R\Gamma B$. Hence $R\Gamma B\Gamma R\Gamma B = B$. Similarly one can prove that $B\Gamma R\Gamma B\Gamma R = B$. Conversely, assume that $B\Gamma R\Gamma B\Gamma R = B$ and $R\Gamma B\Gamma R\Gamma B = B$ for all quasi-interior Γ-hyperideals B of R. Let $B = J \cap I$ and $C = J\Gamma I$ where J is a right Γ-hyperideal and I is a left Γ-hyperideal of R. Then B and C are quasi-interior-Γ-hyperideals of R. Therefore $(J \cap I)\Gamma R\Gamma(J \cap I)\Gamma R = (J \cap I)$. Moreover, $J \cap I = (J\cap I)\Gamma R\Gamma(J\cap I)\Gamma R \subseteq J\Gamma R\Gamma I\Gamma R \subseteq J\Gamma I\Gamma R$ and $J\cap I = (J\cap I)\Gamma R\Gamma(J\cap I)\Gamma R \subseteq J\Gamma I\Gamma R\Gamma J\Gamma I\Gamma R \subseteq J\Gamma I \subseteq J \cap I$. The fact that $J \cap I \subseteq J$ and $J \cap I \subseteq I$ implies that $J \cap I = J\Gamma I$. Hence R is a regular Γ-semihyperring. □

5. Bi-quasi-interior Γ-hyperideal

M. M. Rao [14] has done an extensive research in the field of ideal theory in semirings. His research work on bi-quasi-interior in semiring motivated us to study the parallel notion in Γ-semihyperrings. In this section we introduced the notion of a bi-quasi-interior Γ-hyperideals in Γ-semihyperring as a generalization of bi Γ-hyperideal, quasi Γ-hyperideal and interior Γ-hyperideal of a Γ-semihyperring. We study the properties of bi-quasi-interior Γ-hyperideals and presented few results.

Definition 5.1. A non-empty Γ-subsemihyperring Q of a Γ- semihyperring R is said to be a bi-quasi-interior Γ-hyperideal of R if $Q\Gamma R\Gamma Q\Gamma R\Gamma Q \subseteq Q$.

Remark 5.2. Every bi-quasi-interior Γ-hyperideal of Γ- semihyperring R need not be a bi Γ-hyperideal, quasi Γ-hyperideal, interior Γ-hyperideal, bi-interior Γ-hyperideal and bi-quasi Γ-hyperideal of a Γ-semihyperring R.

Example 5.3. Consider the following sets:

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{N} \right\}, \quad \Gamma = \{ z \mid z \in \mathbb{Z} \} \quad \text{and}$$

$$M_\alpha = \left\{ 4 \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} \mid a, b, c, d \in \mathbb{N}, \alpha \in \Gamma \right\}.$$

Then R is a Γ-semihyperring under matrix addition and the hyperoperation $X\alpha Y \to X + M_\alpha + Y$, for all $X, Y \in R$ and $\alpha \in \Gamma$.
Let $B = \left\{ \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} \mid a, b, c, d \in \mathbb{N} \right\} \subseteq R$. Then B is a bi-quasi-interior Γ-hyperideal of R but B is not a bi Γ-hyperideal, not a quasi Γ-hyperideal, not an interior Γ-hyperideal, not a bi-interior Γ-hyperideal and not a bi-quasi Γ-hyperideal of R.

Theorem 5.4. *Every bi Γ-hyperideal of a Γ-semihyperring R is a bi-quasi-interior Γ-hyperideal of R.*

Proof. Let R be a Γ-semihyperring and B be a bi Γ-hyperideal of R. Then $B\Gamma R\Gamma B \subseteq B$. Therefore $B\Gamma R\Gamma B\Gamma R\Gamma B \subseteq B\Gamma R\Gamma B \subseteq B$. Hence B is a bi-quasi-interior Γ-hyperideal of R. □

Theorem 5.5. *Every Interior Γ-hyperideal of a Γ-semihyperring R is a bi-quasi-interior Γ-hyperideal of R.*

Proof. Let R be a Γ-semihyperring and I be an interior Γ-hyperideal of R. Then $I\Gamma R\Gamma I\Gamma R\Gamma I \subseteq R\Gamma I\Gamma R \subseteq I$. Hence I is a bi-quasi-interior Γ-hyperideal of R. □

Proposition 5.6. *Let R be a bi-quasi-interior simple Γ-semihyperring if and only if $a\Gamma R\Gamma a\Gamma R\Gamma a = R$, for all $a \in R$.*

Proof. Let R be a bi-quasi-interior simple Γ-semihyperring and $a \in R$. Therefore $a\Gamma R\Gamma a\Gamma R\Gamma a$ is a bi-quasi-interior Γ-hyperideal of R. Hence $a\Gamma R\Gamma a\Gamma R\Gamma a = R$, For all $a \in R$. Conversely, let $a\Gamma R\Gamma a\Gamma R\Gamma a = R$ for all $a \in R$. Let Q be a bi-quasi-interior Γ-hyperideal of R and $a \in Q$. Now $R = a\Gamma R\Gamma a\Gamma R\Gamma a \subseteq Q\Gamma R\Gamma Q\Gamma R\Gamma Q \subseteq Q$. This implies that $R \subseteq Q$ and $Q \subseteq R$ holds always. Hence $R = Q$, then R is simple bi-quasi-interior Γ-semihyperring. □

Theorem 5.7. *Let R be a Γ-semihyperring. The notions of bi Γ-hyperideals and bi-quasi Γ-hyperideals in R coincides if and only if R is a simple Γ-semihyperring.*

Proof. Let R be a simple Γ-semihyperring and Q be a bi-quasi-interior Γ-hyperideal of R. Then $Q\Gamma R\Gamma Q\Gamma R\Gamma Q \subseteq Q$ and $R\Gamma Q\Gamma R$ is a Γ-hyperideal of R. As R is a simple Γ-semihyperring, therefore $R\Gamma Q\Gamma R = R$. Hence $Q\Gamma R\Gamma Q \subseteq Q$, this implies that Q is a bi-Γ-hyperideal of

R. Converse part of this theorem we can prove by using proposition (5.6). □

Theorem 5.8. *Let R be a Γ-semihyperring. Then R is a bi-quasi-interior simple Γ-semihyperring if and only if $a\Gamma R\Gamma a\Gamma R\Gamma a = R$ for all $a \in R$.*

Proof. Let R be a bi-quasi-interior simple Γ-semihyperring and $a \in R$. Therefore $a\Gamma R\Gamma a\Gamma R\Gamma a$ is a bi-quasi-interior Γ-hyperideal of R. Hence $a\Gamma R\Gamma a\Gamma R\Gamma a = R$ for all $a \in R$. Conversely, let $a\Gamma R\Gamma a\Gamma R\Gamma a = R$ for all $a \in R$ and Q be a bi-quasi-interior Γ-hyperideal of R with $a \in Q$. We obtain $R = a\Gamma R\Gamma a\Gamma R\Gamma a \subseteq Q\Gamma R\Gamma a\Gamma R\Gamma a \subseteq Q$ and hence $R = Q$. Therefore R is a simple bi-quasi interior Γ-semihyperring. □

Theorem 5.9. *In a regular Commutative Γ-semihyperring R. Following are holds*

(1) *Every bi-quasi-interior Γ-hyperideal is a Γ-hyperideal.*
(2) *Let I be an interior Γ-hyperideal 0f R. Then $R\Gamma I\Gamma R = I$.*
(3) *Let Q is a bi-quasi-interior Γ-hyperideal of R if and only if $Q\Gamma R\Gamma Q\Gamma R\Gamma Q = Q$.*

Proof. (1) Let R be a Γ-semihyperring with B is bi-quasi-interior Γ-hyperideal of R. $B\Gamma R\Gamma B\Gamma R\Gamma B \subseteq B$. This implies that $B\Gamma R \subseteq B\Gamma R\Gamma B$, since R is regular. Hence $B\Gamma R \subseteq B\Gamma R\Gamma B\Gamma R\Gamma B \subseteq B$.

(2) Trivial

(3) Let B is a bi-quasi-interior Γ-hyperideal of R. Then $B\Gamma R\Gamma B\Gamma R\Gamma B \subseteq B$. For $b \in B$, there exist $r \in R$ and $\alpha, \beta \in \Gamma$. We get $x = x\alpha r\beta x\alpha r\beta x \subseteq B\Gamma R\Gamma B\Gamma R\Gamma B$. Therefore $x \in B\Gamma R\Gamma B\Gamma R\Gamma B$. Hence $B\Gamma R\Gamma B\Gamma R\Gamma B = B$. Conversely, suppose $B\Gamma R\Gamma B\Gamma R\Gamma B = B$. Let $B = R' \cap L'$ where R' is a right Γ-hyperideal and L' is a left Γ-hyperideal of R. Then B is bi-interior Γ-hyperideal. Then $(R' \cap L')\Gamma(R' \cap L')\Gamma R\Gamma(R' \cap L') = (R' \cap L')$. $R' \cap L' = (R' \cap L')\Gamma R\Gamma(R' \cap L')\Gamma R\Gamma(R' \cap L') \subseteq R'\Gamma R\Gamma L'\Gamma R\Gamma L' \subseteq R'\Gamma L' \subseteq R'\Gamma \cap L'$. As $R'\Gamma L' \subseteq R'$ and $R'\Gamma L' \subseteq L'$ this implies that $R' \cap L' = R'\Gamma L'$. Hence R is regular Γ-semihyperring.

□

Theorem 5.10. *In a Γ-semihyperring R. Following are holds.*

(1) *Intersection of a bi-Γ-hyperideal and an interior Γ-hyperideal is a bi-quasi- interior Γ-hyperideal of R.*
(2) *Intersection of bi-quasi-interior Γ-hyperideals is a bi-quasi- interior Γ-hyperideal of R.*

(3) *Every bi-interior* Γ*-hyperideal of* R *is a bi-quasi-interior* Γ*- hyperideal of* R*.*
(4) *Intersection of a bi-quasi-interior* Γ*-hyperideal and a right* Γ*-hyperideal is always bi-quasi-interior* Γ*-hyperideal of* R

Proof. (1) Let B is a bi-Γ-hyperideal and I is an interior-Γ-hyperideal of R. Suppose $Q = B \cap I$, then Q is a Γ-subsemihyperring of R. Then $Q\Gamma R\Gamma Q\Gamma R\Gamma Q \subseteq BQ\Gamma R\Gamma Q\Gamma R\Gamma Q\Gamma R \subseteq I$. Hence we get $Q\Gamma R\Gamma Q\Gamma RQ\Gamma R \subseteq Q$. Therefor $Q = B \cap I$ is a bi-quasi-interior Γ-hyperideal of R.

(2) Trivial.

(3) Suppose B is bi-interior Γ-hyperideal of a Γ-semihyperring R. Then $B\Gamma R\Gamma B\Gamma R\Gamma B \subseteq R\Gamma B\Gamma R$ and $B\Gamma R\Gamma B\Gamma R\Gamma B \subseteq R\Gamma B\Gamma R$. Hence we get $B\Gamma R\Gamma B\Gamma R\Gamma B \subseteq R\Gamma B\Gamma R \cap R\Gamma B\Gamma R \subseteq B$. Therefore B is a bi- quasi-interior Γ-hyperideal of R.

(4) Let B is bi-quasi-interior Γ-hyperideal and C is a right Γ- hyperideal of a Γ-semihyperring R. Now $(B \cap C)\Gamma R\Gamma(B \cap C)\Gamma R(B \cap C) \subseteq B\Gamma R\Gamma B\Gamma R\Gamma B \subseteq B$ and $(B \cap C)\Gamma R\Gamma(B \cap C)\Gamma R(B \cap C) \subseteq C\Gamma R\Gamma C\Gamma R\Gamma C \subseteq C$. Therefore $(B \cap C)\Gamma R\Gamma(B \cap C)\Gamma R(B \cap C) \subseteq B \cap C$. Hence intersection of a bi-quasi-interior Γ-hyperideal and right Γ-hyperideal is bi-quasi-interior Γ-hyperideal of R.

□

Theorem 5.11. *Let* e *be a idempotent element of a* Γ*-semihyperring* R*. Such that* $e\Gamma B \subseteq B$*, where* B *is bi-quasi-interior* Γ*-hyperideal of* R*. Then* $e\Gamma B$ *is a bi-quasi-interior* Γ*-hyperideal of* R*.*

Proof. Let e is δ-idempotent element of R. Then $e\delta e = e$ for some $\delta \in \Gamma$. Let $x \in B \cap e\Gamma R$. Then $x \in B$ and $x = e\alpha b$ where $\alpha \in \Gamma, b \in R$. Now $x = e\alpha b = e\delta e\alpha b = e\delta(e\alpha b) = e\delta x \in e\Gamma B$. Therefore $B \cap e\Gamma R \subseteq e\Gamma B$. Now $e\Gamma B \subseteq B$ and $e\Gamma B \subseteq e\Gamma R$. Hence we get $e\Gamma B \subseteq B \cap e\Gamma R$, then $e\Gamma B = B \cap e\Gamma R$. By Theorem 4 we get $e\Gamma B$ is a bi-quasi-interior Γ-hyperideal of R. □

6. Conclusions

As a further generalization of Γ-hyperideals, we introduced the notion of bi-interior Γ-hyperideals, quasi-interior Γ-hyperideals and bi-quasi-interior Γ-hyperideals in a Γ-semihyperring. Which are generalization of Γ-hyperideal, left-Γ-hyperideal, right-Γ-hyperideals, bi Γ-hyperideal, quasi Γ-hyperideal, interior Γ-hyperideals in Γ-semihyperring. Also characterized simple Γ-semihyperring and regular Γ-semihyperring using bi-interior, quasi-interior and bi-quasi-interior Γ-hyperideals.

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Chaitanya B. Kumbharde
Department of Mathematics, SNJB's KKHA Arts, SMGL Commerce & SPHJ Science College, Chandwad, Dist. Nashik, 423 101, M.S. India
Email: kumbharde.cbacs@snjb.org

Kishor F. Pawar
Department of Mathematics, School of Mathematical Sciences, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon, 425 001 M.S., India
Email: kfpawar@nmu.ac.in

Safoora J. Ansari
Department of Mathematics, SNJB's KKHA Arts, SMGL Commerce & SPHJ Science College, Chandwad, Dist. Nashik, 423 101, M.S. India
Email: ansari.sjacs@snjb.org