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Research Paper

SEMI-DERIVATION ON PRIME HYPERRINGS

NIKHIL D. SONONE AND KISHOR F. PAWAR

Abstract. In this paper, we study the notion of semi-derivation in Krasner hyperring and present some examples of them. We introduce the concept of generalized semi-derivation in Krasner hyperring and present some examples. Then, we derive some properties of semi-derivation on Krasner hyperring which proves the commutativity of a Krasner hyperring. Later we prove if f is a non-zero semi-derivation on Krasner hyperring R, then $f^2 \neq 0$ on R. Finally, for a generalized semi-derivation F on R, if $F(u \circ v) = 0$, for all $u, v \in I$, then R is commutative.

Key Words: Hyperring, hyperideal, derivation, semi-derivation, generalized semi-derivation.
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1. Introduction

Algebraic hyperstructures are generalization of an algebraic structures. In an algebraic structures, we use usual binary operations where composition of two elements is again an element. In algebraic hyperstructures it is necessary to have atleast one hyperoperation. Hyperoperation on a set S is a little modified version of an usual binary operation where composition of two elements gives us a non-empty subset of a set S. If S is a non-empty set and $\mathcal{P}^*(S)$ is the collection of all non-empty subsets of S, then the map f, $f : S \times S \to \mathcal{P}^*(S)$ is called as hyperoperation. Using certain axioms and hyperoperations, researchers defined many algebraic

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∗Address correspondence to Nikhil D. Sonone; E-mail: nikhilsonone77@gmail.com.

hyperstructures such as semihypergroups, hypergroups, semihyperrings, hyperrings, Γ-Semihyperrings, etc.

A French mathematician Marty [11] introduced the notion of hypergroups in 1934 at the 8th Congress of Scandinavian Mathematicians. Further, mathematicians like Krasner [10] and Mittas [12] studied the concept of canonical hypergroups, hyperrings, hyperlattices and obtained some important results. Krasner [10] introduced a special type of hyperring $(R, +, \cdot)$, which is a ring like structure with modified axioms where only addition is a hyperoperation and multiplication is a usual operation. Another type of hyperring $(R, +, \cdot)$ was introduced by Rota [16], where addition is a usual operation and multiplication is a hyperoperation. Nowadays, these hyperstructures are cultivated in many countries across the globe and in many research institutes. Some of the application of hyperstructure theory can be found in [9].

In 1957, Posner [13] introduced the notion of derivation on rings and gave some important results on prime rings. The concept of generalized derivation was introduced by Bresar [7]. The notion of semi-derivation in rings was introduced by Bergen [5] while Chang [8] studied semi-derivation on prime rings and obtained some results on semi-derivation with the help of derivation on rings. Bell and Martindale III [4] proved commutativity of prime rings using semi-derivations. Many researchers have worked on commutativity of prime rings admitting derivations, semi-derivations and generalized derivations. In 2022, Yilmaz and Yazarli presented the definition of semi-derivations in Krasner hyperrings and gave some examples [19]. Ardekani and Davvaz [2], also studied derivations on hyperrings. Over last thirty years many researchers have worked on this field of derivations, generalized derivations and semi-derivations on prime rings, semirings and hyperrings.

The purpose of the present paper is to give some examples and prove some results on semi-derivation on Krasner hyperrings. We give some properties of semi-derivation and generalized semi-derivation on Krasner hyperrings showing the commutativity of Krasner hyperring. We also give some properties under which additive maps such as semi-derivation and generalized semi-derivation get vanished on Krasner hyperring R. Throughout this paper, by a hyperring we mean Krasner hyperring.

2. Preliminaries

Definition 2.1. [3] A hyperoperation $*$ on a non-empty set H is a mapping of $H \times H$ into the family of non-empty set of H i.e. $\mathcal{P}^*(H)$ [11]. A hypergroup $(H, *)$ is a non-empty set H equipped with a hyperoperation $*$ which satisfies the following axioms:

(1) $x * (y * z) = (x * y) * z$, for all $x, y, z \in H$;
(2) $x * H = H * x = H$, for all $x \in H$.

Definition 2.2. [3] A non-empty set R with a hyperaddition '+' and a multiplication '·' is called an additive hyperring or Krasner hyperring if it satisfies the following:

(1) $(R, +)$ is a canonical hypergroup, i.e.,
  (a) $x + (y + z) = (x + y) + z$, for all $x, y, z \in R$;
  (b) $x + y = y + x$, for all $x, y \in R$;
  (c) there exists $0 \in R$ such that $0 + x = x$, for all $x \in R$;
  (d) for all $x \in R$ there exists an unique element denoted by $-x \in R$ such that $0 \in x + (-x)$;
  (e) for all $x, y, z \in R$, $z \in x+y$ implies $y \in -x+z$ and $x \in z-y$.
(2) $(R, \cdot)$ is a semigroup having 0 as a bilaterally absorbing element, i.e.,
  (a) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in R$;
  (b) $x \cdot 0 = 0 \cdot x = 0$, for all $x \in R$.
(3) The multiplication $\cdot$ is distributive with respect to the hyperoperation $+$, i.e. $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(x+y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$.

A non-empty subset I of a canonical hypergroup R is called a canonical subhypergroup of R if I itself is a canonical hypergroup under the same hyperoperation as that of R. Equivalently, a non-empty subset I of a canonical hypergroup R is a canonical subhypergroup of R if for every $x, y \in I$ we have $x - y \subseteq I$. Here after we denote xy instead of $x \cdot y$. Moreover, for $A, B \subseteq R$ and $x \in R$, by $A + B$ we mean the set $\cup_{a \in A, b \in B}(a + b)$, $A + x = A + \{x\}$, $x + B = \{x\} + B$ and also $-A = \{-a : a \in A\}$.

In a hyperring R, if there exists an element $1 \in R$ such that $1a = a1 = a$ for every $a \in R$, then the element 1 is called the identity element of the hyperring R. In fact, the element 1 is unique. Further, if $ab = ba$ for every $a, b \in R$, then the hyperring R is called a commutative hyperring.

Note 2.3. If $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ and $(x + y) \cdot z \subseteq x \cdot z + y \cdot z$, for all $x, y, z \in R$ in (3) then R is called Weak Distributive Hyperring.

Example 2.4. [19] Let $R = \{0, x, y\}$ be a set with hyperoperation and multiplication as follows,

$+$	0	x	y
0	0	x	y
x	x	x	R
y	y	R	y

$\cdot$	0	x	y
0	0	0	0
x	0	x	y
y	0	y	x

$(R, +, \cdot)$ is a prime hyperring.

Definition 2.5. [2] Let $(R, +, \cdot)$ be a hyperring. The center of R is $Z(R) = \{x \in R | x \cdot y = y \cdot x, \text{ for all } y \in R\}$.

Definition 2.6. [3] Let R be a hyperring and I be a non-empty subset of R. I is called a left (resp. right) hyperideal of R if

(1) $(I, +)$ is a canonical subhypergroup of R, i.e., $x - y \subseteq I$, for all $x, y \in I$ and
(2) for all $a \in I$, $r \in R$, $ra \subseteq I$ (resp. $ar \subseteq I$). A hyperideal of R is one which is a left as well as a right hyperideal of R.

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Definition 2.7. [3] A hyperring R is said to be a prime hyperring if $aRb = 0$, for $a, b \in R$ implies either $a = 0$ or $b = 0$.

Definition 2.8. [3] A hyperring R is said to be 2-torsion free if $0 \in x+x$, for $x \in R$ implies $x = 0$.

Definition 2.9. [3] Let R and S be hyperrings, where both addition and multiplication are hyperoperations. A mapping $\phi : R \to S$ is called a homomorphism from R to S if for all $x, y \in R$,

(1) $\phi(x + y) \subseteq \phi(x) + \phi(y)$;
(2) $\phi(xy) \subseteq \phi(x)\phi(y)$ and $\phi(0) = 0$ hold.

If R is a Krasner hyperring, then the condition (2) becomes $\phi(xy) \in \phi(x)\phi(y)$. If both R and S are Krasner hyperrings, then the condition (2) is $\phi(xy) = \phi(x)\phi(y)$.

Lemma 2.10. [2] *Let R be a hyperring and $[x, y]$ denotes the set $xy - yx$, for all $x, y \in R$. Then for all $x, y, z \in R$, we have,*

(1) *$[x + y, z] = [x, z] + [y, z]$;*
(2) *$[xy, z] \subseteq x[y, z] + [x, z]y$;*
(3) *If $x \in Z(R)$, then $[xy, z] = x[y, z]$;*
(4) *If d is a derivation of R, then $d[x, y] \subseteq [d(x), y] + [x, d(y)]$.*

3. Derivation on Hyperrings

In this section we recall definition of a derivation and some examples are given. Also, we proved some of its properties.

Definition 3.1. [3] Let R be a hyperring. A map $d : R \to R$ is said to be a derivation of R if d satisfies

(1) $d(x+y) \subseteq d(x)+d(y)$;
(2) $d(xy) \in d(x)y + xd(y)$.

If the map d is such that $d(x+y) = d(x)+d(y)$, for all $x, y \in R$ and satisfies the second condition, then d is called a strong derivation of R.

Example 3.2. Consider the set $R = \{0, 1, 2\}$ with the hyperaddition and multiplication defined as follows:

$+$	0	1	2
0	0	1	2
1	1	1	R
2	2	R	2

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

Then $(R, +, \cdot)$ is a Hyperring.

Define a map $d : R \to R$ by $d(0) = 0$, $d(1) = 2$, $d(2) = 1$. Here, d is a strong derivation of R.

Theorem 3.3. [2] *Let d be a non-zero derivation on a prime hyperring R and I be a non-zero hyperideal on R. If $I \subseteq Z(R)$, then R is commutative.*

Theorem 3.4. *Let R be a prime hyperring and $d : R \to R$ be a derivation. Let I be a hyperideal of R such that $d(I) = 0$. Then $I \subseteq Z(R)$. Also R is a Commutative hyperring.*

Proof. Let $u \in I$ and $x \in R$. Then $0 = d([u, x]) = d(ux - xu) \subseteq d(ux) - d(xu) \implies 0 \in ud(x) - d(x)u$. That is $ud(x) = d(x)u$, for all $x \in R$, $u \in I \implies I \subseteq Z(R)$. Using Theorem 3.3, R is a commutative hyperring. □

4. Semi-Derivation on Hyperrings

In this section we recall definition of a semi-derivation on hyperrings and some examples are given. Then we proved some of its properties on the line of [6, 15, 17, 18].

Definition 4.1. [19] Let R be a hyperring. A map $f : R \to R$ is said to be a semi-derivation associated with a function $g : R \to R$ if for all $r, s \in R$,

(1) $f(r+s) \subseteq f(r)+f(s)$;
(2) $f(rs) \in f(r)g(s)+rf(s) = f(r)s+g(r)f(s)$;
(3) $f(g(r)) = g(f(r))$.

If the map f satisfies the conditions (2), (3) and $f(r+s) = f(r)+f(s)$, for all $r, s \in R$, then f is called a strong semi-derivation of R. Obviously, every derivation is a semi-derivation.

Example 4.2. Consider the set $R = \{0, a, b, c\}$ with the hyperaddition and multiplication defined as follows.

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	R	$\{a,b\}$	$\{a,c\}$
b	$\{b\}$	$\{a,b\}$	R	$\{b,c\}$
c	$\{c\}$	$\{a,c\}$	$\{b,c\}$	R

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

$(R, +, \cdot)$ is a (Weak Distributive)Hyperring.

Define a map $f : R \to R$ by $f(0) = 0$, $f(a) = b$, $f(b) = c$, $f(c) = a$. Also, define $g : R \to R$ by $g(0) = 0$, $g(a) = c$, $g(b) = a$, $g(c) = b$. Here, f is a strong semi-derivation of R associated with function g.

Define a map $f^* : R \to R$ by $f^*(0) = 0$, $f^*(a) = c$, $f^*(b) = c$, $f^*(c) = a$ and define $g^* : R \to R$ as identity map. Since $f^*(a+b) = \{c\} \subset f^*(a)+f^*(b) = \{0, a, b, c\}$, f^* is a semi-derivation(not strong) on R associated with function g.

Note 4.3. Let f be a semi-derivation on a hyperring R associated with the function g on R. If $g(0) = 0$, then $f(0) = 0$.

Lemma 4.4. [2] *Let d be a derivation on a prime hyperring R and I be a non-zero hyperideal on R. Then for all $x \in R$,*

(1) *If $Ix = 0$ or $xI = 0$, then $x = 0$;*
(2) *If $xIy = 0$, then $x = 0$ or $y = 0$.*

Proposition 4.5. *Let f be a nonzero semi-derivation on a prime hyperring R associated with the function g on R. If I is a non-zero hyperideal on R, then $f(I) \neq 0$.*

Theorem 4.6. *Let R be a hyprrring and f be a semi-derivation on R associated with a derivation d. Let I be a non-zero hyperideal of R contained in $ker g$, then $f(I)$ is a non-zero hyperideal of R.*

Lemma 4.7. *Let R be a prime hyperring and f be a semi-derivation associated with a surjective function g. Let I be a non-zero hyperideal on R. Then, for all $r \in R$,*

(1) *If* $f(I) = 0$, *then* $f = 0$*;*
(2) *If* $f(I)r = 0$ *or* $rf(I) = 0$, *then* $r = 0$ *or* $f = 0$.

Theorem 4.8. *Let* I *be a non-zero hyperideal of 2-torsion free prime hyperring* R *and* f *be a semi-derivation on* R *associated with an onto map* g. *If* $f^2(I) = 0$, *then* $f = 0$.

Proof. For all $u, v \in I$, we have

$$\begin{aligned} 0 = f^2(uv) &\in f(f(u)g(v) + uf(v)) \\ &\subseteq f^2(u)g^2(v)) + f(u)f(g(v)) + f(u)g(f(v)) + uf^2(v) \\ &= f(u)f(g(v)) + f(u)g(f(v)) \\ &= f(u)f(g(v)) + f(u)f(g(v)). \end{aligned}$$

Since R is a 2-torsion free hyperring, $f(u)f(g(v)) = 0$, for all $u, v \in I$. As g is an onto map, we have $f(u)f(t) = 0$, for $u \in I$, $t \in R$. This implies $f = 0$, by Lemma 4.7. □

Theorem 4.9. *Let* I *be a non-zero hyperideal of 2-torsion free hyperring* R. *Let* f_1 *and* f_2 *be semi-derivations on* R *associated with the onto map* g_1 *and* g_2, *respectively. If* $f_1 f_2(I) = 0$, *then* $f_1 = 0$ *or* $f_2 = 0$.

Proof. For all $u, v \in I$, we have

$$\begin{aligned} 0 = f_1 f_2(uv) &\in f_1(f_2(u)g_2(v) + uf_2(v)) \\ &\subseteq f_1 f_2(u)g_1 g_2(v) + f_2(u)f_1 g_2(v) + f_1(u)g_1 f_2(v) + uf_1 f_2(v) \\ &= f_2(u)f_1 g_2(v) + f_1(u)g_1 f_2(v) \end{aligned}$$

Replacing u by $f_2(u)$, $0 = f_2^2(u)f_1 g_2(v)$. Since g is onto, $0 = f_2^2(u)f_1(t)$, where $t \in R$. Replacing t by tw, $0 = f_2^2(u)f_1(tw) \in f_2^2(u)(f_1(t)g_1(w) + tf_1(w)) \implies 0 = f_2^2(u)tf_1(w)$, $u, w \in I$. Using primeness of R, we get, $f_2^2 = 0$ or $f_1 = 0$ on $I \implies f_2 = 0$ or $f_1 = 0$ on R, by Lemma 4.7 and Theorem 4.8. □

Theorem 4.10. *Let* R *be a 2-torsion free prime hyperring and* f *is a non-zero semi-derivation on* R *associated with a surjective function* g. *Then* R *is a commutative hyperring in each of the following cases:*

(1) *If center of* R *i.e.* $Z(R)$ *is a ring such that* $f(R) \subseteq Z(R)$,
(2) *If* $[f(R), f(R)] = 0$, *and* $Z(R)$ *is a ring,*
(3) *If* $0 = f([x, y])$, *for all* $x, y \in R$*(the map* g *is not necessarily surjective).*

Proof. (1) Let $f(R) \subseteq Z(R)$. Then $[f(x), y] = 0$, for all $x, y \in R$. Replacing x by xz, where $z \in R$, $0 = [f(xz), y] \subseteq [f(x)g(z), y] + [xf(z), y] \implies 0 \in f(x)[g(z), y] + [x, y]f(z)$. Replacing z by $f(z)$, $0 \in f(x)[gf(z), y] + [z, y]f^2(z) \implies 0 \in f(x)[f(g(z)), y] + [z, y]f^2(z) = [z, y]f^2(z)$, for all $y, z \in R$. So $f^2(z) = 0$ or $0 \in [z, y]$. Since $f \neq 0$, R is commutative.

(2) For $u, v, w \in R$, $0 = [f(u), f(vf(w))] \subseteq [f(u), f(v)f(w)+g(v)f^2(w)] \implies 0 \in [f(u), g(v)]f^2(w)$. Since g is surjective, we have $0 \in [f(u), v]f^2(w)$, for all $u, v, w \in R$. Since R is a prime hyperring, $f^2(R) = 0$ or $f(R) \subseteq Z$. As f is non-zero, by the above property R is commutative hyperring.

(3) Let $0 = f([x, y])$, for all $x, y \in R$. Replacing y by yx, $0 = f([x, yx]) = f([x, y]x) \subseteq [x, y]f(x) \implies xyf(x) = yxf(x)$. Now by replacing y by zy, $0 \in xzyf(x) - zyxf(x) = xzyf(x) - zxyf(x) \implies 0 \in [x, z]Rf(x)$. Since R is a prime hyperring, $0 \in [x, z]$, for all $x, z \in R$ or $f(x) = 0$. Since f is non-zero, R is commutative hyperring.

□

Theorem 4.11. *Let R be a prime hyperring and $a \in R$. Let f be a non-zero semi-derivation on R associated with g such that $[a, f(R)] = 0$. Then $a \in Z(R)$.*

Theorem 4.12. *Let f be a semi-derivation on a prime hyperring R associated with function g on R s.t. $f(R) \subseteq Z(R)$. Also, let there be a constant element $c \in R$ associated to f such that $g(c) \notin Z(R)$. Then $f = 0$.*

Proof. There is $x_0 \in R$, such that $g(c)x_0 \neq x_0g(c)$. Since $g(c) \notin Z(R)$. We have, $f(xc) \in f(x)g(c) + xf(c) = f(x)g(c)$, for all $x \in R \implies f(x)g(c) = f(xc) \in Z(R)$. Therefore, $f(x)g(c)x_0 = x_0f(x)g(c) = f(x)x_0g(c)$. That is $0 \in f(x)g(c)x_0 - f(x)x_0g(c) = f(x)[g(c), x_0] \implies f(x) = 0$ or $0 \in [g(c), x_0]$. So, $f(x) = 0$, for all $x \in R$. □

Theorem 4.13. *Let R be a 2-torsion free prime hyperring. If f is a non-zero semi-derivation of R associated with a onto map g, then $f^2 \neq 0$.*

Proof. Suppose $f^2(R) = 0$. For all $x, y \in R$, we have

$$\begin{aligned} 0 = f^2(xy) &\in f(f(x)y + g(x)f(y)) \\ &\subseteq f^2(x)y + g(f(x))f(y) + f(g(x))f(y) + g(x)f^2(y) \\ &\subseteq f(g(x))f(y) + f(g(x))f(y) \end{aligned}$$

As g is onto, $0 \in f(x)f(y) + f(x)f(y)$, for all $x, y \in R$. Since R is 2-torsion free, $f(x)f(y) = 0$, for all $x, y \in R$. Replacing y by ry, $0 = f(x)f(ry) \in f(x)(f(r)y + g(r)f(y))$, for all $x, y, r \in R \implies 0 = f(x)rf(y)$. Since R is prime, $f = 0$, a contradiction. □

Theorem 4.14. *Let R be a prime hyperring. Let f be a non-zero semi-derivation associated with a non-zero onto map g. If $f(xy) = 0$, for all $x, y \in R$, then $f = 0$.*

Theorem 4.15. *Let R be a prime hyperring such that the center of it i.e. $Z(R)$ is a ring. Let f be a semi-derivation associated with onto map g on R. Then $f(Z) \subseteq Z$.*

5. Generalized Semi-derivation on Hyperrings

In this section we introduce the notion of generalized semi-derivation on hyperrings and give some examples and some properties. Here, some results are proved on the line of [1].

Definition 5.1. Let R be a hyperring. A map $F : R \to R$ is said to be a generalized semi-derivation if there exists a semi-derivation $d : R \to R$ associated with a function $g : R \to R$ if for all $r, s \in R$,

(1) $F(r + s) \subseteq F(r) + F(s)$;
(2) $F(rs) \in F(r)s + g(r)d(s) = d(r)g(s) + rF(s)$;
(3) $F(g(r)) = g(F(r))$.

Example 5.2. Let $(R, +, \cdot)$ be a hyperring and a set $H(R) = \left\{ \begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix} | r, s \in R \right\}$. A hyperaddition $\oplus$ is defined on $H(R)$ by

$$\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} | x \in r_1 + r_2, y \in s_1 + s_2 \right\}$$

for all $\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix} \in H(R)$. Define a multiplication $\otimes$ on $H(R)$ by $\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} s_1 r_2 & s_1 s_2 \\ 0 & 0 \end{pmatrix}$ for all $\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix} \in H(R)$. Clearly, $H(R)$ is a Krasner hyperring.

Let $q \in R$ and $d : H(R) \to H(R)$ defined by $d\left(\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} -qr & 0 \\ 0 & 0 \end{pmatrix}$ and define a function g on $H(R)$ by $g\left(\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$. Here d is a strong semi-derivation. Also define $F : H(R) \to H(R)$ as $F\left(\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & qs \\ 0 & 0 \end{pmatrix}$. F is a generalized semi-derivation on hyperring R.

Example 5.3. Consider the hyperring and semi-derivation defined as in Example 4.2. Define a map $F : R \to R$ as $F(x) = x$, for all $x \in R$, then F is a generalized semi-derivation on R.

Lemma 5.4. *Let R be a prime hyperring and I be a non-zero hyperideal of R. Also, let F be a non-zero generalized semi-derivation of R with associated semi-derivation d and a map g associated with d such that $g(I) = I$. Then $a \in R$ and $aF(I) = 0$ (or F(I)a=0), implies $a = 0$.*

Proof. Let $aF(I) = 0$. Then for $u, v \in I$ and $a \in R$, $0 = aF(uv) \in aF(u)v + ag(u)d(v) = aud(v) \implies a = 0$ or $d(I) = 0$. Let $d(I) = 0$ then $0 = aF(uv) \in ad(u)g(v) + auF(v) = auF(v)$, for all $u, v \in I$. Since $F(I) \neq 0$, $a = 0$.

Similarly, we can prove it in case of $F(I)a = 0$ and also in case of $aF(R) = 0$. □

Lemma 5.5. *Let R be a prime hyperring and I be a non-zero hyperideal of R. If F is a non-zero generalized semi-derivation of R associated with d such that $g(I) = I$, then $F(u) \neq 0$ for some $u \in I$.*

Proof. Let $F(I) = 0$, for all $u \in I$. Replacing u by ux, we get $F(ux) = 0$, for all $u \in I$ and $x \in R$. Thus, $0 = F(ux) \in F(u)x + g(u)d(x) = g(u)d(x)$, for all $x \in R$, $u \in I \implies 0 = Id(x)$, for all $x \in R \implies d = 0$. Therefore we have, $F(xu) = F(x)u = 0$, for all $u \in I$, $x \in R \implies F = 0$, a contradiction. □

Theorem 5.6. *Let R be a 2-torsion free prime hyperring and I be a non-zero hyperideal of R. If R admits a non-zero generalized semi-derivation F associated with a non-zero semi-derivation d and a map g associated with d such that $g(I) = I$ and $F(I) \subseteq I$, then $F^2 \neq 0$.*

Proof. Let $F^2(I) = 0$. Then for all $u, v \in I$,

$$\begin{aligned} 0 = F^2(uv) &\in F(F(u)v + g(u)d(v)) \\ &\subseteq F^2(u)v + gF(u)d(u) + Fg(u)d(u) + g^2(u)d^2(v) \\ &= F(u)d(v) + F(u)d(v) + ud^2(v) \end{aligned}$$

Replacing u by $F(u)$, $0 = F(u)d^2(v)$. Using Lemma 5.4, $d^2(v) = 0$ or $F(u) = 0$. Since $d \neq 0$, we have $F(I) = 0$, a contradiction by Lemma 5.5. □

Theorem 5.7. *Let R be a prime hyperring and I be a non-zero hyperideal of R. Suppose that R admits a non-zero generalized semi-derivation F associated with a non-zero semi-derivation d and an additive map g associated with d such that $g(uv) = g(u)g(v)$, for all $u, v \in I$ and $g(I) = I$. If $F(u \circ v) = 0$ where $(u \circ v)$ denotes the set $uv + vu$, for all $u, v \in I$, then R is a commutative hyperring.*

Proof. Let $0 = F(u \circ v)$, for all $u, v \in I$. Replacing v by vu, $0 = F(u \circ vu) = F((u \circ v)u) \in F(u \circ v)u + g(u \circ v)d(u) = g(u \circ v)d(I) \implies 0 \in g(uv+vu) \subseteq g(uv)+g(vu) = g(u)g(v)+g(v)g(u)$. Replacing u by vu, $g(u)g(v)g(w) = -g(v)g(w)g(u)$, for all $u, v, w \in I \implies g(v)g(u)g(w) = -g(v)g(w)g(u)$, for all $u, v, w \in I \implies 0 \in [g(u), g(w)]$. Since $g(I) = I \implies 0 \in [u, w]$, for all $u, w \in I$. Replacing u by ur where $r \in R$, $0 \in [ur, w] = I[r, w] \implies 0 \in [r, w]$, for all $r \in R$, $w \in I$. Which gives $I \subseteq Z(R)$. That is, R is commutative, by Theorem 3.3. □

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Nikhil D. Sonone
Department of Mathematics, School of Mathematical Sciences, Kavayitri Bahinabai Chaudhari North Maharashtra University Jalgaon-425 001, India.
Email: nikhilsonone77@gmail.com

Kishor F. Pawar
Department of Mathematics, School of Mathematical Sciences, Kavayitri Bahinabai Chaudhari North Maharashtra University Jalgaon-425 001, India.
Email: kfpawar@gmail.com