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(λ, μ)-ANTI-FUZZY LINEAR SPACES

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Abstract. In this article, we introduce the notion of a (λ, μ)-anti-fuzzy fields and a (λ, μ)-anti-fuzzy linear spaces over a (λ, μ)-anti-fuzzy fields and obtain some fundamental properties. Furthermore, we propose the direct product of a (λ, μ)-anti-fuzzy linear spaces and employ it to describe some important properties.

Key Words: (λ, μ)-anti-fuzzy field, (λ, μ)-anti- fuzzy linear space, (λ, μ)-fuzzy linear space, Homomorphic image, Direct product.

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1. Introduction

Zadeh [13] introduced the notion of a fuzzy set in 1965 and this theory is being widely used by many mathematicians and it is implemented by several branches of Mathematics such as algebraic structure, measure theory, topology and graph theory. Nowadays, many research scholars are applying this concept in medical sciences, artificial intelligence, robotics, pattern recognition, machine theory, control engineering, expert system, decision making, operations research, statistical analysis among many others. In 1986 Nanda [8] first introduced the concept of a fuzzy field, fuzzy linear spaces and these concepts were redefined by Biswas [1]. Gu and Lu [5] continued to study fuzzy field and fuzzy linear space. Nanda [9] introduced the concept of algebra over fuzzy field. Gu and Lu [6] redefined this and studied some fundamental properties. Srinivas and Swamy [10] introduced the notion of fuzzy near algebras over fuzzy

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fields and Srinivas et al [11] discussed anti-fuzzy near algebras over anti-fuzzy fields. Yuan et al [12] introduced the concept of fuzzy subgroup with thresholds. Jun [7] introduced the concept of fuzzy subalgebra with thresholds and obtained the relation between a fuzzy algebra with thersholds and $(\in, \in \wedge q)$ fuzzy subalgebra. Feng and Li [2] have introduced the concept of a (λ, μ)-fuzzy subfields and (λ, μ)-fuzzy linear subspaces. Feng and Yao [3, 4] proposed the concept of a (λ, μ) -anti-fuzzy subfields, (λ, μ)-anti-fuzzy subgroups and investigated some fundamental properties.

By the motivation of a (λ, μ)-fuzzy linear subspaces, we introduce the notion of a (λ, μ)-anti-fuzzy fields, (λ, μ)-anti-fuzzy linear spaces over (λ, μ)-anti-fuzzy field and investigated some fundamental properties. We proposed the direct product of a (λ, μ)-anti-fuzzy linear spaces and studied some properties of direct product of (λ, μ)-anti-fuzzy linear spaces.

2. Preliminaries

In this article, we assume that $0 \leq \lambda < \mu \leq 1$.

Definition 2.1. [12] Let A be a fuzzy subset of group G. A is called a (λ, μ)-fuzzy subgroup of G, if for all $x, y \in G$,

i) $A(xy) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$;
ii) $A(x^{-1}) \vee \lambda \geq A(x) \wedge \mu$, where x^{-1} is the inverse element of x.

Clearly $(0, 1)$ fuzzy subgroup is just a fuzzy subgroup, and thus a (λ, μ) -fuzzy subgroup is a generalization of fuzzy subgroup.

Definition 2.2. [4] Let G be a fuzzy subset of group G'. G is called a (λ, μ)-anti-fuzzy subgroup of G', if for all $x, y \in G'$,

i) $G(xy) \wedge \mu \leq G(x) \vee G(y) \vee \lambda$;
ii) $G(x^{-1}) \wedge \mu \leq G(x) \vee \lambda$, where x^{-1} is the inverse element of x.

Definition 2.3. [6] Let X be a field and F be a fuzzy subset of X. Then F is said to be a fuzzy field of X, if it satisfies the following conditions for all $x, y \in X$;

i) $F(x + y) \geq F(x) \wedge F(y)$;
ii) $F(-x) \geq F(x)$;
iii) $F(xy) \geq F(x) \wedge F(y)$;
iv) $F(x^{-1}) \geq F(x)$, for every $x (\neq 0) \in X$.

A fuzzy field F of X is denoted by (F, X).

Proposition 2.1. [6] *Let (F, X) be a fuzzy field of X, then for all $x \in X$, we have*

i) $F(0) \geq F(x)$;
ii) $F(1) \geq F(x)$, *for any* $x (\neq 0) \in X$;
iii) $F(0) \geq F(1)$.

Definition 2.4. [2] A fuzzy subset F of a field X is said to be a (λ, μ)-fuzzy subfield of X, if for all $x, y \in X$, we have

i) $F(x + y) \vee \lambda \geq F(x) \wedge F(y) \wedge \mu$;
ii) $F(-x) \vee \lambda \geq F(x) \wedge \mu$;
iii) $F(xy) \vee \lambda \geq F(x) \wedge F(y) \wedge \mu$;
iv) $F(x^{-1}) \vee \lambda \geq F(x) \wedge \mu$, for any $x (\neq 0) \in X$.

Definition 2.5. [2] Let F be a (λ, μ)-fuzzy subfield of a field X, Y be a linear space over X and V be a fuzzy subset of Y. (V, Y) is called a (λ, μ)-fuzzy linear subspace of Y over F, if for all $x, y \in Y$, $k \in X$ we have

i) $V(x + y) \vee \lambda \geq V(x) \wedge V(y) \wedge \mu$;
ii) $V(-x) \vee \lambda \geq V(x) \wedge \mu$;
iii) $V(kx) \vee \lambda \geq F(k) \wedge V(x) \wedge \mu$;
iv) $F(1) \vee \lambda \geq V(x) \wedge \mu$.

Definition 2.6. [11] A fuzzy subset A of Y is called an anti-fuzzy near-algebra of Y over an anti-fuzzy field (F, X) if

i) $A(x + y) \leq A(x) \vee A(y)$;
ii) $A(kx) \leq F(k) \vee A(x)$;
iii) $A(xy) \leq A(x) \vee A(y)$;
iv) $F(1) \leq A(x)$, for all $x, y \in Y$ and $k \in X$.

An anti-fuzzy near-algebra A of Y is denoted by (A, Y).

Definition 2.7. Let A and B be two fuzzy subsets of sets X and Y respectively then the direct product of A and B, denoted by $A \times B$ is a fuzzy subset of $X \times Y$ defined as $(A \times B)(x, y) = A(x) \vee B(y), \forall (x, y) \in X \times Y$

3. (λ, μ)-ANTI-FUZZY FIELDS

In this section we define the concept of a (λ, μ)-anti-fuzzy fields and discuss some fundamental properties.

Definition 3.1. A fuzzy subset F of a field X is said to be a (λ, μ)-anti-fuzzy field of X, if for all $x, y \in X$, we have

i) $F(x+y) \wedge \mu \leq F(x) \vee F(y) \vee \lambda$;
ii) $F(-x) \wedge \mu \leq F(x) \vee \lambda$;
iii) $F(xy) \wedge \mu \leq F(x) \vee F(y) \vee \lambda$;
iv) $F(x^{-1}) \wedge \mu \leq F(x) \vee \lambda$, for any $x(\neq 0) \in X$.

An (λ, μ)-anti-fuzzy field of X is denoted by (F, X).

Example 3.1. Consider the field $(Z_5, \oplus_5, \otimes_5)$ with following Cayley table:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

.	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Let $F : Z_5 \rightarrow D[0,1]$ be a fuzzy set defined by

$$F(x) = \begin{cases} 0.1, & \text{if } x = 0 \\ 0.4, & \text{otherwise} \end{cases}, \ \lambda = 0.2 \text{ and } \mu = 0.7$$

Clearly, F is a (λ, μ)-anti-fuzzy field.

Proposition 3.2. *Let F be a (λ, μ)-anti-fuzzy field of X, then for all $x \in X$, we have*

i) $F(0) \wedge \mu \leq F(x) \vee \lambda$;
ii) $F(1) \wedge \mu \leq F(x) \vee \lambda$, *for any* $x(\neq 0) \in X$;
iii) $F(0) \wedge \mu \leq F(1) \vee \lambda$.

Proof. Let (F, X) is a (λ, μ)-anti-fuzzy field of X,

i) For all $x \in X$, we have

$$\begin{aligned} F(0) \wedge \mu &= (F(x + (-x)) \wedge \mu) \wedge \mu \\ &\leq (F(x) \vee F(-x) \vee \lambda) \wedge \mu \\ &= (F(x) \wedge \mu) \vee (F(-x) \wedge \mu) \vee (\lambda \wedge \mu) \\ &\leq F(x) \vee (F(x) \vee \lambda) \vee \lambda \end{aligned}$$

$$= F(x) \vee \lambda.$$

$(F(x) \leq \mu$ then $F(x) \wedge \mu = F(x)$ and $\mu \leq F(x)$ then $F(x) \wedge \mu \leq F(x))$
ii) For all $x(\neq 0) \in X$, we have

$$\begin{aligned} F(1) \wedge \mu &= (F(xx^{-1}) \wedge \mu) \wedge \mu \\ &\leq (F(x) \vee F(x^{-1}) \vee \lambda) \wedge \mu \\ &= (F(x) \wedge \mu) \vee (F(x^{-1}) \wedge \mu) \vee (\lambda \wedge \mu) \\ &\leq F(x) \vee (F(x) \vee \lambda) \vee \lambda = F(x) \vee \lambda. \end{aligned}$$

iii) Proof is a corollary of (i). □

Theorem 3.3. (F, X) *is a* (λ, μ)*-anti-fuzzy field if and only if the following two conditions hold:*

i) $F(x - y) \wedge \mu \leq F(x) \vee F(y) \vee \lambda$, *for all* $x, y \in X$*;*
ii) $F(xy^{-1}) \wedge \mu \leq F(x) \vee F(y) \vee \lambda$, *for all* $x, y \ (\neq 0) \in X$*.*

Proof. Suppose (F, X) is a (λ, μ)-anti fuzzy field. Then, we have
i) For all $x, y \in X$,

$$\begin{aligned} F(x - y) \wedge \mu &= F(x - y) \wedge \mu \wedge \mu \\ &\leq (F(x) \vee F(-y) \vee \lambda) \wedge \mu \\ &= (F(x) \wedge \mu) \vee (F(-y) \wedge \mu) \vee (\lambda \wedge \mu) \\ &\leq F(x) \vee (F(y) \vee \lambda) \vee \lambda \\ &= F(x) \vee F(y) \vee \lambda. \end{aligned}$$

ii) For all $x, y \ (\neq 0) \in X$,

$$\begin{aligned} F(xy^{-1}) \wedge \mu &= (F(xy^{-1}) \wedge \mu) \wedge \mu \\ &\leq (F(x) \vee F(y^{-1}) \vee \lambda) \wedge \mu \\ &= (F(x) \wedge \mu) \vee (F(y^{-1}) \wedge \mu) \vee (\lambda \wedge \mu) \\ &\leq F(x) \vee (F(y) \vee \lambda) \vee \lambda \\ &= F(x) \vee F(y) \vee \lambda. \end{aligned}$$

Conversely, suppose that the two conditions of the hypothesis hold. Then for all $x, y \in X$

$$\begin{aligned} F(-x) \wedge \mu &= F(0 - x) \wedge \mu \wedge \mu \\ &\leq (F(0) \vee F(-x) \vee \lambda) \wedge \mu \\ &= (F(0) \wedge \mu) \vee (F(-x) \wedge \mu) \vee (\lambda \wedge \mu) \end{aligned}$$

$$
\begin{aligned}
&\leq (F(x) \vee \lambda) \vee (F(x) \vee \lambda) \vee \lambda = F(x) \vee \lambda. \\
F(x+y) \wedge \mu &= F(x-(-y)) \wedge \mu \wedge \mu \\
&\leq (F(x) \vee F(-y) \vee \lambda) \wedge \mu \\
&= (F(x) \wedge \mu) \vee (F(-y) \wedge \mu) \vee (\lambda \wedge \mu) \\
&\leq F(x) \vee (F(y) \vee \lambda) \vee \lambda \\
&= F(x) \vee F(y) \vee \lambda. \\
F(x^{-1}) \wedge \mu &= (F(1.x^{-1}) \wedge \mu) \wedge \mu \\
&\leq (F(1) \vee F(x^{-1}) \vee \lambda) \wedge \mu \\
&= (F(1) \wedge \mu) \vee (F(x^{-1}) \wedge \mu) \vee (\lambda \wedge \mu) \\
&\leq \ (F(x) \vee \lambda) \vee (F(x) \vee \lambda) \vee \lambda = F(x) \vee \lambda \\
F(xy) \wedge \mu &= \{F(x((y^{-1})^{-1}))\} \wedge \mu \wedge \mu \\
&\leq (F(x) \vee F(y^{-1}) \vee \lambda) \wedge \mu \\
&= (F(x) \wedge \mu) \vee (F(y^{-1}) \wedge \mu) \vee (\lambda \wedge \mu) \\
&\leq F(x) \vee (F(y) \vee \lambda) \vee \lambda \\
&= F(x) \vee F(y) \vee \lambda.
\end{aligned}
$$

Thus, F is a (λ, μ)-anti-fuzzy field of a field X. □

Theorem 3.4. *(F, X) is a (λ, μ)-anti-fuzzy field if and only if F_α is a subfield of X, for any $\alpha \in (\lambda, \mu]$, where $F_\alpha \neq \emptyset$.*

Proof. Let (F, X) be a (λ, μ)-anti-fuzzy field of a field X, take any $\alpha \in (\lambda, \mu]$ with $F_\alpha \neq \emptyset$, where $F_\alpha = \{x \in X | F(x) < \alpha\}$. We need to show that F_α is a subfield of X.
For all $x, y \in F_\alpha$, $F(x) < \alpha, F(y) < \alpha$ and $\lambda < \alpha \leq \mu$ and (F, X) be a (λ, μ)-anti-fuzzy field of a field X. Then

$$
\begin{aligned}
F(x-y) \wedge \mu \leq F(x) \vee F(y) \vee \lambda < \alpha \\
x - y \in F_\alpha. \\
F(xy^{-1}) \leq F(x) \vee F(y) \vee \lambda < \alpha \\
xy^{-1} \in F_\alpha.
\end{aligned}
$$

Thus, F_α is a subfield of X, for any $\alpha \in (\lambda, \mu]$, where $F_\alpha \neq \emptyset$.
Conversely, let F_α is a subfield of X, for any $\alpha \in (\lambda, \mu]$. If there exist $x, y \in X$ such that $F(x-y) \wedge \mu = \alpha > F(x) \vee F(y) \vee \lambda$ and $F(xy^{-1}) \wedge \mu = \alpha > F(x) \vee F(y) \vee \lambda$. Then $\alpha \in (\lambda, \mu]$, $x, y \in F_\alpha$. But $F(x-y) \wedge \mu > \alpha$

implies $x - y \notin F_\alpha$ and $F(xy^{-1}) \wedge \mu > \alpha$ implies $xy^{-1} \notin F_\alpha$. This is a contradiction to F_α is a subfield of X for any $\alpha \in (\lambda, \mu]$, where $F_\alpha \neq \emptyset$. Thus, (F, X) be a (λ, μ)-anti-fuzzy field. □

4. (λ, μ)-ANTI-FUZZY LINEAR SPACES

In this section we define the concept of a (λ, μ)-anti-fuzzy linear space over (λ, μ)-anti-fuzzy field. Also we discuss some elementary properties.

Definition 4.1. Let F be a (λ, μ)-anti-fuzzy field of X. Y be a linear space over X and V be a fuzzy subset of Y. (V, Y) is called a (λ, μ)-anti-fuzzy linear space over (F, X), if for all $x, y \in Y$, and $k \in X$ we have

i) $V(x + y) \wedge \mu \leq V(x) \vee V(y) \vee \lambda$;
ii) $V(-x) \wedge \mu \leq V(x) \vee \lambda$;
iii) $V(kx) \wedge \mu \leq F(k) \vee V(x) \vee \lambda$;
iv) $F(1) \wedge \mu \leq V(x) \vee \lambda$.

Example 4.1. Consider a linear space $\{Z_3, \oplus_3, \otimes_3\}$ over field $\{Z_5, \oplus_5, \otimes_5\}$ with the following Cayley tables:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Also, the scalar multiplication on Z_3 defined by $kx = \begin{cases} 0, & \text{if } k = 0 \\ x, & \text{otherwise} \end{cases}$
for every $x \in Z_3$ and $k \in Z_5$
Let $F : Z_5 \to D[0, 1]$ be a(λ, μ)-anti-fuzzy field defined by
$F(x) = \begin{cases} 0.1, & \text{if } x = 0 \\ 0.4, & \text{otherwise} \end{cases}$
Let $V : Z_3 \to D[0, 1]$ be a fuzzy set of Z_3 defined by
$V(x) = \begin{cases} 0.4, & \text{if } x = 0 \\ 0.8, & \text{otherwise} \end{cases}, \ \lambda = 0.2 \ \textit{and} \ \mu = 0.7$

Clearly, V is a (λ,μ)-anti-fuzzy linear space of Z_3 over (λ,μ)-anti-fuzzy field (F, Z_5) .

Proposition 4.2. *If (V,Y) is (λ,μ)-anti-fuzzy linear space over (F,X), then*
i) $F(0)\wedge\mu \le V(x)\vee\lambda, x\in Y$;
ii) $V(0)\wedge\mu \le V(x)\vee\lambda, x\in Y$.
iii) $F(0)\wedge\mu \le V(0)\vee\lambda$.

Theorem 4.3. *Let F be a (λ,μ)-anti-fuzzy field of X. Let Y be a linear space over X and V be a fuzzy subset of Y. (V,Y) is a (λ,μ)-anti-fuzzy linear space of Y over F if and only if for all $x,y\in V$, $k,l\in X$, we have*

(1) $V(kx+ly)\wedge\mu \le F(k)\vee V(x)\vee F(l)\vee V(y)\vee\lambda$;
(2) $F(1)\wedge\mu \le V(x)\vee\lambda$.

Proof. Suppose V is a (λ,μ)-anti-fuzzy linear space of Y. Then for any $x,y\in V, k,l\in X$, we have

$$\begin{aligned}V(kx+ly)\wedge\mu &= V(kx+ly)\wedge\mu\wedge\mu\\ &\le (V(kx)\vee V(ly)\vee\lambda)\wedge\mu\\ &= (V(kx)\wedge\mu)\vee(V(ly)\wedge\mu)\vee(\lambda\wedge\mu)\\ &\le F(k)\vee V(x)\vee\lambda\vee F(l)\vee V(y)\vee\lambda\vee\lambda\\ &\le F(k)\vee V(x)\vee F(l)\vee V(y)\vee\lambda.\\ \text{and}\quad F(1)\wedge\mu &\le V(x)\vee\lambda.\end{aligned}$$

Conversely, suppose that conditions (1) and (2) given in the hypothesis hold for all $x,y\in V,k,l\in X$, then

$$\begin{aligned}V(x+y)\wedge\mu &= V(1.x+1.y)\wedge\mu\wedge\mu\\ &\le (F(1)\vee V(x)\vee F(1)\vee V(y)\vee\lambda)\wedge\mu\\ &= (F(1)\wedge\mu)\vee(V(x)\wedge\mu)\vee(F(1)\wedge\mu)\vee(V(y)\wedge\mu)\vee(\lambda\wedge\mu)\\ &\le (V(x)\vee\lambda)\vee V(x)\vee(V(y)\vee\lambda)\vee V(y)\vee\lambda\\ V(x+y)\wedge\mu &\le V(x)\vee V(y)\vee\lambda.\\ V(-x)\wedge\mu &= V(-1.x+0.x)\wedge\mu\wedge\mu\\ &\le (F(-1)\vee V(x)\vee F(0)\vee V(x)\vee\lambda)\wedge\mu\\ &= (F(-1)\wedge\mu)\vee(V(x)\wedge\mu)\vee(F(0)\wedge\mu)\vee(V(x)\wedge\mu)\vee(\lambda\wedge\mu)\\ V(-x)\wedge\mu &\le (V(x)\vee\lambda)\vee V(x)\vee(V(x)\vee\lambda)\vee V(x)\vee\lambda\\ V(-x)\wedge\mu &\le V(x)\vee\lambda.\\ V(kx)\wedge\mu &= V(k.x+0.x)\wedge\mu\wedge\mu\end{aligned}$$

$$
\begin{aligned}
&\leq (F(k) \vee V(x) \vee F(0) \vee V(x) \vee \lambda) \wedge \mu \\
&= (F(k) \wedge \mu) \vee (V(x) \wedge \mu) \vee (F(0) \wedge \mu) \vee (V(x) \wedge \mu) \vee (\lambda \wedge \mu) \\
&\leq F(k) \vee V(x) \vee (V(x) \vee \lambda) \vee \lambda
\end{aligned}
$$

$$V(kx) \wedge \mu \leq F(k) \vee V(x) \vee \lambda.$$

and $F(1) \wedge \mu \leq V(x) \vee \lambda.$

Hence, V is a (λ, μ)-anti-fuzzy linear space of Y over F. □

Theorem 4.4. *Let F be a (λ, μ)-anti-fuzzy field of a field X. Let Y be a linear space over X and V be a fuzzy subset of Y, then (V, Y) is a (λ, μ)-anti-fuzzy linear space over (F, X) if and only if V_α is a linear space over F_α, for any $\alpha \in (\lambda, \mu]$, where $F_\alpha \neq \emptyset$, $V_\alpha \neq \emptyset$.*

Proof. V is a (λ, μ)-anti-fuzzy linear space over F.
For all $\alpha \in (\lambda, \mu]$, $F_\alpha \neq \emptyset$, $V_\alpha \neq \emptyset$, We need to show that V_α is a linear space over F_α, for all $x, y \in V_\alpha$ and for all $k, l \in F_\alpha$, we have

$$
\begin{aligned}
V(kx + ly) \wedge \mu &\leq (F(k) \vee V(x) \vee F(l) \vee V(y) \vee \lambda) = \alpha \\
V(kx + ly) \wedge \mu &\leq \alpha \\
V(kx + ly) &\leq \alpha \\
kx + ly &\leq V_\alpha.
\end{aligned}
$$

Thus, V_α is a linear space over F_α.
Conversely V_α is a linear space over F_α, for all $\alpha \in (\lambda, \mu]$.
If there exist $x, y \in V$, $k, l \in F$ such that $V(kx+ly) \wedge \mu > F(k) \vee V(x) \vee F(l) \vee V(y) \vee \lambda = \alpha$.
Since, $F(k) \leq \alpha$,$V(x) \leq \alpha$,$F(l) \leq \alpha$,$V(y) \leq \alpha$ we conclude that, $V(kx + ly) \wedge \mu > \alpha$ implies $kx + ly \notin V_\alpha$.
This contradiction with that V_α is a linear space over F_α.
Hence, $V(kx + ly) \wedge \mu \leq F(k) \vee V(x) \vee F(l) \vee V(y) \vee \lambda$.
Again, if there exist $x \in V$ such that $F(1) \wedge \mu > V(x) \vee \lambda = \alpha$, then for all $\alpha \in (\lambda, \mu]$, $x \in V_\alpha$ and $1 \notin F_\alpha$. This is a contradiction to that V_α is a linear space over F_α . □

Proposition 4.5. *Intersection of family of a (λ, μ)-anti-fuzzy linear space is an (λ, μ)-anti-fuzzy linear space.*

Proof. Let $\{A_i\}_{i \in I}$ be family of a (λ, μ)-anti-fuzzy linear space over a (λ, μ)-anti-fuzzy field (F, X), where I is a finite set.
Let $A(x) = \bigcap_{i \in I} A_i(x) = \inf_{i \in I} A_i(x) = \bigwedge_{i \in I} A_i(x)$.
For any $x, y \in Y$ and $k, l \in X$, we have

$$A(kx + ly) \wedge \mu = \inf_{i \in I} A_i(kx + ly) \wedge \mu$$

$$
\begin{aligned}
&= \inf_{i\in I}(A_i(kx+ly)\wedge\mu)\\
&\le \inf_{i\in I}\{F(k)\vee A_i(x)\vee F(l)\vee A_i(y)\vee\lambda\}\\
&\le F(k)\vee \inf_{i\in I} A_i(x)\vee F(l)\vee \inf_{i\in I} A_i(y)\vee\lambda\\
&= F(k)\vee A(x)\vee F(l)\vee A(y)\vee\lambda\\
A(kx+ly)\wedge\mu &\le F(k)\vee A(x)\vee F(l)\vee A(y)\vee\lambda.
\end{aligned}
$$

Since, each A_i is a (λ,μ)-anti-fuzzy linear space, for every $x\in Y$ and $i\in I$ we have

$$
\begin{aligned}
A(x)\vee\lambda &= \inf_{i\in I} A_i(x)\vee\lambda\\
&= \inf_{i\in I}(A_i(x)\vee\lambda)\\
&\ge F(1)\wedge\mu\\
F(1)\wedge\mu &\le A(x)\vee\lambda.
\end{aligned}
$$

Hence, (A,Y) is a (λ,μ)-anti-fuzzy linear space is a (λ,μ)-anti-fuzzy field (F,X). □

Proposition 4.6. *Union of family of a (λ,μ)-anti-fuzzy linear space is a (λ,μ)-anti-fuzzy linear space.*

Proof. Let $\{A_i : i\in I\}$ be a family of a (λ,μ)-anti-fuzzy linear space of Y over a (λ,μ)-anti-fuzzy field (F,X) and $A(x)=\bigcup_{i\in I}A_i(x)=\sup_{i\in I}A_i(x)=\bigvee_{i\in I}A_i(x)$.
Let $x,y\in Y$ and $k\in X$. Then,

$$
\begin{aligned}
A(kx+ly)\wedge\mu &= \sup_{i\in I} A_i(kx+ly)\wedge\mu\\
&= \sup_{i\in I}(A_i(kx+ly)\wedge\mu)\\
&\le \sup_{i\in I}\{F(k)\vee A_i(x)\vee F(l)\vee A_i(y)\vee\lambda\}\\
&= F(k)\vee \sup_{i\in I} A_i(x)\vee F(l)\vee \sup_{i\in I} A_i(y)\vee\lambda\\
&= F(k)\vee A(x)\vee F(l)\vee A(y)\vee\lambda.\\
A(x)\vee\lambda &= (\sup_{i\in I} A_i(x))\vee\lambda\\
&= \sup_{i\in I}(A_i(x)\vee\lambda)\\
&\ge \sup_{i\in I}(F(1)\wedge\mu)
\end{aligned}
$$

$$= F(1) \wedge \mu.$$

Hence, $A(x) = \bigcup_{i \in I} A_i(x)$ is a (λ, μ)-anti-fuzzy linear space. □

Example 4.7. Consider an algebra $\{Z_3, \oplus_3, \otimes_3\}$ over field $\{Z_5, \oplus_5, \otimes_5\}$. Also, the scalar multiplication on Z_3 defined by $kx = \begin{cases} 0, & \text{if } k = 0 \\ x, & \text{otherwise} \end{cases}$ for every $x \in Z_3$ and $k \in Z_5$. Let $F : Z_5 \to D[0,1]$ be a(λ, μ)-anti-fuzzy field defined by $F(x) = \begin{cases} 0.1, & \text{if } x = 0 \\ 0.4, & \text{otherwise} \end{cases}$. Let $A, B : Z_3 \to D[0,1]$ be a fuzzy sets of Z_3 defined by $A(x) = \begin{cases} 0.4, & \text{if } x = 0 \\ 0.8, & \text{otherwise} \end{cases}$ *and* $B(x) = \begin{cases} 0.4, & \text{if } x = 0 \\ 0.7, & \text{otherwise} \end{cases}$, $\lambda = 0.2$ *and* $\mu = 0.7$

Clearly, $A \cup B$ and $A \cap B$ are (λ, μ)-anti-fuzzy algebra of Z_3 over (λ, μ)-anti-fuzzy field (F, Z_5) .

Example 4.8. Consider a linear space $\{Z_3, \oplus_3, \otimes_3\}$ over field $\{Z_5, \oplus_5, \otimes_5\}$. Also, the scalar multiplication on Z_3 defined by $kx = \begin{cases} 0, & \text{if } k = 0 \\ x, & \text{otherwise} \end{cases}$ for every $x \in Z_3$ and $k \in Z_5$. Let $F : Z_5 \to D[0,1]$ be a(λ, μ)-anti-fuzzy field defined by $F(x) = \begin{cases} 0.1, & \text{if } x = 0 \\ 0.4, & \text{otherwise} \end{cases}$. Let $A, B : Z_3 \to D[0,1]$ be a fuzzy sets of Z_3 defined by $A(x) = \begin{cases} 0.4, & \text{if } x = 0 \\ 0.8, & \text{otherwise} \end{cases}$ *and* $B(x) = \begin{cases} 0.4, & \text{if } x = 0 \\ 0.7, & \text{otherwise} \end{cases}$, $\lambda = 0.2$ *and* $\mu = 0.7$

Clearly, $A \cup B$ and $A \cap B$ are (λ, μ)-anti-fuzzy linear spaces of Z_3 over (λ, μ)-anti-fuzzy field (F, Z_5) .

Proposition 4.9. *Let Y_1 and Y_2 be two (λ, μ)-anti-fuzzy linear spaces over a field X and f is a linear transformation from Y_1 onto Y_2. If (V_1, Y_1) is a (λ, μ)-anti-fuzzy linear space over (F, X) then $(f(V_1), Y_2)$ is a (λ, μ)-anti-fuzzy linear space over (F, X), where for all $y \in Y_2$,*

$$f(V_1)(y) = \begin{cases} \inf_{t \in Y_1} \{V_1(t) | f(t) = y\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 1, & \text{otherwise.} \end{cases}$$

Proof. If $f^{-1}(y_1) = \emptyset$ or $f^{-1}(y_2) = \emptyset$ for any $y_1, y_2 \in Y_2$, then $f(V_1)(ky_1 + ly_2) \wedge \mu \leq 1 = F(k) \vee f(V_1)(y_1) \vee F(l) \vee f(V_1)(y_2) \vee \lambda$. Suppose that $f^{-1}(y_1) \neq \emptyset$, $f^{-1}(y_2) \neq \emptyset$ for any $y_1, y_2 \in Y_2$, then $f^{-1}(ky_1 + ly_2) = kf^{-1}(y_1) + lf^{-1}(y_1) \neq \emptyset$. So for any $k, l \in F$, we have

$$\begin{aligned}
f(V_1)(ky_1 + ly_2) \wedge \mu &= \inf_{t \in Y_1} \{V_1(t) | f(t) = ky_1 + ly_2\} \wedge \mu \\
&= \inf_{t \in Y_1} \{V_1(t) \wedge \mu | f(t) = ky_1 + ly_2\} \\
&\leq \inf_{x_1, x_2 \in Y_1} \{V_1(kx_1 + lx_2) \wedge \mu | f(x_1) = y_1, f(x_2) = y_2\} \\
&\leq \inf_{x_1, x_2 \in Y_1} \{F(k) \vee V_1(x_1) \vee F(l) \vee V_1(x_2) \vee \lambda | f(x_1) = y_1, f(x_2) = y_2\} \\
&\leq F(k) \vee \{\inf_{x_1 \in Y_1} V_1(x_1) | f(x_1) = y_1\} \vee F(l) \vee \{\inf_{x_2 \in Y_1} V_1(x_2) | f(x_2) = y_2\} \vee \lambda \\
&= F(k) \vee f(V_1)(y_1) \vee F(l) \vee f(V_1)(y_2) \vee \lambda.
\end{aligned}$$

And for all $y \in Y_2$, we have

$$\begin{aligned}
f(V_1)(y) \vee \lambda &= \inf_{t \in Y_1} \{V_1(t) | f(t) = y\} \vee \lambda \\
&= \inf_{x \in Y_1} \{V_1(x) \vee \lambda | f(x) = y\} \\
&\geq \inf_{x \in Y_1} \{F(1) \wedge \mu | f(x) = y\} \\
&= F(1) \wedge \mu. \\
F(1) \wedge \mu &\leq f(V_1)(y) \vee \lambda.
\end{aligned}$$

Hence, $f(V_1)$ is a (λ, μ)-anti-fuzzy linear space of Y_2 over fuzzy field (F, X). □

Proposition 4.10. *Let Y_1 and Y_2 be two (λ, μ)-anti-fuzzy linear spaces over a field X and f is a linear transformation of Y_1 onto Y_2. If (V_2, Y_2) be a (λ, μ)-anti-fuzzy linear space over fuzzy field (F, X) then $f^{-1}(V_2)$ is a (λ, μ)-anti-fuzzy linear space of Y_1 over fuzzy field (F, X), where* $f^{-1}(V_2)(x) = V_2(f(x)), \quad \forall x \in V_1$.

Proof. For any $x_1, x_2 \in Y_1, k, l \in F$, we have

$$\begin{aligned}
f^{-1}(V_2)(kx_1 + lx_2) \wedge \mu &= V_2(f(kx_1 + lx_2)) \wedge \mu \\
&= V_2(f(kx_1) + f(lx_2)) \wedge \mu \\
&= V_2(kf(x_1) + lf(x_2)) \wedge \mu \\
&\leq F(k) \vee V_2(f(x_1)) \vee F(l) \vee V_2(f(x_2)) \vee \lambda
\end{aligned}$$

$$= F(k) \vee f^{-1}(V_2)(x_1) \vee F(l) \vee f^{-1}(V_2)(x_2) \vee \lambda.$$

And for all $x_1 \in Y_1$, we have
$f^{-1}(V_2)(x) \vee \lambda = V_2(f(x)) \vee \lambda \geq F(1) \wedge \mu.$
$F(1) \wedge \mu \leq f^{-1}(V_2)(x)$
Hence, $f^{-1}(V_2)$ is a (λ, μ)-anti-fuzzy linear space of Y_1 over fuzzy field (F, X). □

5. Direct product of (λ, μ)-anti-fuzzy linear space

In this section we discuss direct product of (λ, μ)-anti-fuzzy linear spaces and some fundamental properties of them.

Proposition 5.1. *Let Y and Y' be two linear spaces over a field X. If A and B are two (λ, μ)-anti-fuzzy linear spaces of Y and Y' respectively over (λ, μ)-anti-fuzzy field (F, X), then $A \times B$ is a (λ, μ)-anti-fuzzy linear space of $Y \times Y'$.*

Proof. Let A and B be two (λ, μ)-anti-fuzzy linear spaces of Y and Y' respectively.
For any $x = (x_1, y_1), y = (x_2, y_2) \in Y \times Y'$ and $k, l \in X$, we have

$$\begin{aligned}
(A \times B)(kx + ly) \wedge \mu &= (A \times B)(k(x_1, y_1) + l(x_2, y_2)) \wedge \mu \wedge \mu \\
&= (A \times B)((kx_1, ky_1) + (lx_2, ly_2)) \wedge \mu \wedge \mu \\
&= \{(A \times B)(kx_1 + lx_2, ky_1 + ly_2) \wedge \mu\} \wedge \mu \\
&= (A(kx_1 + lx_2) \vee B(ky_1 + ly_2) \vee \lambda) \wedge \mu \\
&= (A(kx_1 + lx_2) \wedge \mu) \vee (B(ky_1 + ly_2) \wedge \mu) \vee (\lambda \wedge \mu) \\
&\leq F(k) \vee A(x_1) \vee F(l) \vee A(x_2) \vee \lambda \vee F(k) \vee B(y_1) \vee F(l) \vee B(y_2) \vee \lambda \\
&= F(k) \vee (A(x_1) \vee B(y_1)) \vee F(l) \vee (A(x_2) \vee B(y_2)) \vee \lambda \\
&= F(k) \vee (A \times B)(x_1, y_1) \vee F(l) \vee (A \times B)(x_2, y_2) \vee \lambda.
\end{aligned}$$

And for all $(x, y) \in Y \times Y'$, we have

$$\begin{aligned}
F(1) \wedge \mu &= (F(1) \wedge \mu) \vee (F(1) \wedge \mu) \\
&\leq (A(x_1) \vee \lambda) \vee (B(y_1) \vee \lambda) \\
&= A(x_1) \vee B(y_1) \vee \lambda \\
&= (A \times B)(x) \vee \lambda.
\end{aligned}$$

Hence, $A \times B$ is a (λ, μ)-anti-fuzzy linear space of $Y \times Y'$. □

Proposition 5.2. *Let Y and Y' be two linear spaces over a field X and let A and B be (λ, μ)-anti-fuzzy linear spaces of Y and Y' respectively. If $(A \times B)$ is a (λ, μ)-anti-fuzzy linear space of $(Y \times Y')$, then $(A \times B)(0, 0') \wedge \mu \leq (A \times B)(x, y) \vee \lambda$, for all $x \in Y,\ y \in Y'$.*

Proof. Proof is straightforward and hence omitted. □

Proposition 5.3. *Let Y and Y' be linear spaces over a field X and let A and B be two fuzzy subsets of Y and Y' respectively. Suppose that 0 and $0'$ are the zero elements of Y and Y' respectively. If $A \times B$ is (λ, μ)-anti-fuzzy linear space of $Y \times Y'$, then at least one of the following statements must holds:*

a) $B(0') \wedge \mu \leq A(x) \vee \lambda, \quad \forall x \in Y,$
b) $A(0) \wedge \mu \leq B(y) \vee \lambda, \quad \forall y \in Y'.$

Proof. Let $A \times B$ is a (λ, μ)-anti-fuzzy linear space of $Y \times Y'$.
By contraposition, suppose that none of the statements (a) and (b) holds. Then we can find x in Y and y in Y' such that $A(x) \vee \lambda < B(0') \wedge \mu$ and $B(y) \vee \lambda < A(0) \wedge \mu$, we have

$$\begin{aligned}
(A \times B)(x, y) \vee \lambda &= (A(x) \vee B(y)) \vee \lambda \\
&= (A(x) \vee \lambda) \vee (B(y) \vee \lambda) \\
&< (B(0') \wedge \mu) \vee (A(0) \wedge \mu) \\
&= (A(0) \vee B(0')) \wedge \mu \\
&= (A \times B)(0, 0') \wedge \mu.
\end{aligned}$$

Thus, $A \times B$ is a (λ, μ)-anti-fuzzy linear space of $Y \times Y'$ satisfying $(A \times B)(x, y) \vee \lambda < (A \times B)(0, 0') \wedge \mu$. This contradicts with $(0,\ 0')$ is the identity of $Y \times Y'$. □

Proposition 5.4. *Let the direct product $A \times B$ be a (λ, μ)-anti-fuzzy linear spaces of $Y \times Y'$, where Y and Y' be linear spaces over a field X. Let A and B be two fuzzy subsets of Y and Y' respectively such that $B(0') \wedge \mu \leq A(x) \vee \lambda$ for any $x \in Y$, $0'$ being the zero element of Y', then A is a (λ, μ)-anti-fuzzy linear space of Y'.*

Proof. Let the direct product $A \times B$ be a (λ, μ)-anti-fuzzy linear space of $Y \times Y'$ and $x, y \in Y$. Then $(x, 0'), (y, 0')$ are in $Y \times Y'$. Now using $B(0') \wedge \mu \leq A(x) \vee \lambda$ for any $x \in Y$, we get

$$\begin{aligned}
A(x + y) \wedge \mu &\leq \{A(x + y) \wedge \mu\} \vee \lambda \\
&= \{A(x + y) \vee \lambda\} \wedge \mu
\end{aligned}$$

$$
\begin{aligned}
&= \{(A(x+y) \vee \lambda) \vee (B(0') \wedge \mu)\} \wedge \mu \\
&= \{(A(x+y) \vee B(0')) \wedge \mu\} \vee \lambda \\
&= \{(A(x+y) \vee B(0'+0')) \wedge \mu\} \vee \lambda \\
&= \{(A \times B)(x+y, 0'+0') \wedge \mu\} \vee \lambda \\
&= \{(A \times B)((x, 0') + (y, 0')) \wedge \mu\} \vee \lambda \\
&\leq \{(A \times B)(x, 0') \vee (A \times B)(y, 0') \vee \lambda\} \vee \lambda \\
&= A(x) \vee B(0') \vee A(y) \vee B(0') \vee \lambda \\
A(x+y) \wedge \mu \wedge \mu &= \{A(x) \vee B(0') \vee A(y) \vee B(0') \vee \lambda\} \wedge \mu \\
&= (A(x) \wedge \mu) \vee (B(0') \wedge \mu) \vee (A(y) \wedge \mu) \vee (B(0') \wedge \mu) \vee (\lambda \wedge \mu) \\
&\leq A(x) \vee (A(x) \vee \lambda) \vee A(y) \vee (A(y) \vee \lambda) \vee \lambda \\
&= A(x) \vee A(y) \vee \lambda \\
A(x+y) \wedge \mu &\leq A(x) \vee A(y) \vee \lambda. \\
A(-x) \wedge \mu &\leq \{A(-x) \wedge \mu\} \vee \lambda \\
&= \{A(-x) \vee \lambda\} \wedge \mu \\
&= \{(A(-x) \vee \lambda) \vee (B(0') \wedge \mu)\} \wedge \mu \\
&= \{(A(-x) \wedge \mu) \vee (B(0') \wedge \mu)\} \vee \lambda \\
&= \{(A(-x) \wedge \mu) \vee (B(0') \wedge \mu)\} \vee \lambda \\
&= \{(A(-x) \vee B(0')) \wedge \mu\} \vee \lambda \\
&= \{(A \times B)(-x, 0') \wedge \mu\} \vee \lambda \\
&\leq ((A \times B)(x, 0') \vee \lambda) \vee \lambda \\
&= A(x) \vee B(0') \vee \lambda \\
A(-x) \wedge \mu \wedge \mu &= \{A(x) \vee B(0') \vee \lambda\} \wedge \mu \\
&= (A(x) \wedge \mu) \vee (B(0') \wedge \mu) \vee (\lambda \wedge \mu) \\
&\leq A(x) \vee (A(x) \vee \lambda) \vee \lambda \\
&= A(x) \vee \lambda. \\
A(kx) \wedge \mu &\leq (A(kx) \wedge \mu) \vee \lambda \\
&= (A(kx) \vee \lambda) \wedge \mu \\
&= \{(A(kx) \vee \lambda) \vee (B(0') \wedge \mu)\} \wedge \mu \\
&= \{(A(kx) \vee \lambda) \wedge \mu\} \vee (B(0') \wedge \mu)
\end{aligned}
$$

$$
\begin{aligned}
&= (A(kx) \wedge \mu) \vee (\lambda \wedge \mu) \vee (B(0') \wedge \mu)\\
&= \{A(kx) \vee B(0') \wedge \mu\} \vee \lambda\\
&= \{(A \times B)(kx, 0') \wedge \mu\} \vee \lambda\\
&= \{(A \times B)(k(x, 0')) \wedge \mu\} \vee \lambda\\
&\leq \{F(k) \vee (A \times B)(x, 0') \vee \lambda\} \vee \lambda\\
&= \{F(k) \vee A(x) \vee B(0') \vee \lambda\}\\
A(kx) \wedge \mu \wedge \mu &\leq (F(k) \wedge \mu) \vee (A(x) \wedge \mu) \vee (B(0') \wedge \mu) \vee (\lambda \wedge \mu)\\
&\leq F(k) \vee A(x) \vee \lambda.\\
A(x) \vee \lambda &\geq (A(x) \vee \lambda) \wedge \mu\\
&= \{(A(x) \vee \lambda) \vee (B(0') \wedge \mu)\} \wedge \mu\\
&= \{(A(x) \wedge \mu) \vee (B(0') \wedge \mu)\} \vee \lambda\\
&= \{(A(x) \vee B(0')) \wedge \mu\} \vee \lambda\\
&= \{(A \times B)(x, 0') \vee \lambda\} \wedge \mu\\
&\geq F(1) \wedge \mu \wedge \mu = F(1) \wedge \mu.
\end{aligned}
$$

Hence, A is a (λ, μ)-anti-fuzzy linear space of Y. □

Proposition 5.5. *Let the direct product $A \times B$ be a (λ, μ)-anti-fuzzy linear spaces of $Y \times Y'$, where Y and Y' be two (λ, μ)-anti-fuzzy linear spaces over a field X. let A and B be two fuzzy subsets of Y and Y' such that $A(0) \wedge \mu \leq B(x) \vee \lambda$, for any $x \in Y'$, 0 being the zero element of Y, then B is a (λ, μ)-anti-fuzzy linear space of Y.*

Proof. Let the direct product $A \times B$ be a (λ, μ)-anti-fuzzy linear space of $Y \times Y'$ and $x, y \in Y$. Then $(0, x), (0, y)$ are in $Y \times Y'$. Now using $A(0) \wedge \mu \leq B(x) \vee \lambda$ for any $x \in Y'$, we get

$$
\begin{aligned}
B(x + y) \wedge \mu &\leq \{B(x + y) \wedge \mu\} \vee \lambda\\
&= \{B(x + y) \vee \lambda\} \wedge \mu\\
&= \{(B(x + y) \vee \lambda) \vee (A(0) \wedge \mu)\} \wedge \mu\\
&= \{(B(x + y) \vee A(0)) \wedge \mu\} \vee \lambda\\
&= \{(B(x + y) \vee A(0 + 0)) \wedge \mu\} \vee \lambda\\
&= \{(A \times B)(0 + 0, x + y) \wedge \mu\} \vee \lambda\\
&= \{(A \times B)((0, x) + (0, y)) \wedge \mu\} \vee \lambda\\
&\leq \{(A \times B)(0, x) \vee (A \times B)(0, y) \vee \lambda\} \vee \lambda
\end{aligned}
$$

$$
\begin{aligned}
&= A(0) \vee B(x) \vee A(0) \vee B(y) \vee \lambda \\
B(x+y) \wedge \mu \wedge \mu &= \{A(0) \vee B(x) \vee A(0) \vee B(y) \vee \lambda\} \wedge \mu \\
&= (A(0) \wedge \mu) \vee (B(x) \wedge \mu) \vee (A(0) \wedge \mu) \vee (B(y) \wedge \mu) \vee (\lambda \wedge \mu) \\
&\leq B(x) \vee \lambda \vee B(x) \vee B(y) \vee \lambda \vee B(y) \vee \lambda \\
&= B(x) \vee B(y) \vee \lambda \\
B(x+y) \wedge \mu &\leq B(x) \vee B(y) \vee \lambda. \\
B(-x) \wedge \mu &\leq \{B(-x) \wedge \mu\} \vee \lambda \\
&= \{B(-x) \vee \lambda\} \wedge \mu \\
&= \{(B(-x) \vee \lambda) \vee (A(0) \wedge \mu)\} \wedge \mu \\
&= \{(B(-x) \wedge \mu) \vee (A(0) \wedge \mu)\} \vee \lambda \\
&= \{(B(-x) \wedge \mu) \vee (A(0) \wedge \mu)\} \vee \lambda \\
&= \{(B(-x) \vee A(0)) \wedge \mu\} \vee \lambda \\
&= \{(A \times B)(0, -x) \wedge \mu\} \vee \lambda \\
&\leq ((A \times B)(0, x) \vee \lambda) \vee \lambda \\
&= A(0) \vee B(x) \vee \lambda \\
B(-x) \wedge \mu \wedge \mu &= \{A(0) \vee B(x) \vee \lambda\} \wedge \mu \\
&= (A(0) \wedge \mu) \vee (B(x) \wedge \mu) \vee (\lambda \wedge \mu) \\
&\leq B(x) \vee \lambda B(x) \vee \lambda \\
&= B(x) \vee \lambda. \\
B(kx) \wedge \mu &\leq (B(kx) \wedge \mu) \vee \lambda \\
&= (B(kx) \vee \lambda) \wedge \mu \\
&= \{(B(kx) \vee \lambda) \vee (A(0) \wedge \mu)\} \wedge \mu \\
&= \{(B(kx) \vee \lambda) \wedge \mu\} \vee (A(0) \wedge \mu) \\
&= (B(kx) \wedge \mu) \vee (\lambda \wedge \mu) \vee (A(0) \wedge \mu) \\
&= \{B(kx) \vee A(0) \wedge \mu\} \vee \lambda \\
&= \{(A \times B)(0, kx) \wedge \mu\} \vee \lambda \\
&= \{(A \times B)(k(0, x)) \wedge \mu\} \vee \lambda \\
&\leq \{F(k) \vee (A \times B)(0, x) \vee \lambda\} \vee \lambda \\
&= \{F(k) \vee A(0) \vee B(x) \vee \lambda\} \\
B(kx) \wedge \mu \wedge \mu &\leq (F(k) \wedge \mu) \vee (A(0) \wedge \mu) \vee (B(x) \wedge \mu) \vee (\lambda \wedge \mu)
\end{aligned}
$$

$$\begin{aligned}
&\le F(k) \vee B(x) \vee \lambda. \\
B(x) \vee \lambda &\ge (B(x) \vee \lambda) \wedge \mu \\
&= \{(B(x) \vee \lambda) \vee (A(0) \wedge \mu)\} \wedge \mu \\
&= \{(B(x) \wedge \mu) \vee (A(0) \wedge \mu)\} \vee \lambda \\
&= \{(B(x) \vee A(0)) \wedge \mu\} \vee \lambda \\
&= \{(A \times B)(0, x) \vee \lambda\} \wedge \mu \\
&\ge F(1) \wedge \mu \wedge \mu = F(1) \wedge \mu.
\end{aligned}$$

Hence, B is a (λ, μ)-anti-fuzzy linear space of Y. □

From the Proposition 5.4 and 5.5, we have the following corollary.

Corollary 5.6. *Let Y and Y' be two linear spaces over a field X and let A and B be two fuzzy subsets of Y and Y' respectively. If $A \times B$ is a (λ, μ)-anti-fuzzy linear space of $Y \times Y'$ then either A is (λ, μ)-anti-fuzzy linear space of Y or B is (λ, μ)-anti-fuzzy linear space of Y'.*

6. Conclusion

In this article, we developed a theory of a (λ, μ)-anti-fuzzy linear spaces over (λ, μ)-anti-fuzzy field. We proved that the notion of (λ, μ)-anti-fuzzy linear spaces have nice level characterizations and algebraic properties. In the future, we will research on (λ, μ)-anti-fuzzy algebra over a (λ, μ)-anti-fuzzy fields and discuss fuzzy ideals and fuzzy prime ideals in a (λ, μ)-anti-fuzzy algebras.

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