Journal of Hyperstructures 11 *(1)* (*2022*)*, 130-142.*
ISSN: 2322-1666 print/2251-8436 online

SOME SUZUKI-TYPE BEST PROXIMITY POINT RESULTS ON METRIC SPACES ENDOWED A GRAPH

SOOMIEH KHALEGHIZADEH

Abstract. In this paper, the researcher proved the best proximity point theorem for Suzuki type mappings in the setting of metric spaces endowed a graph. In particular, some earlier results in the literature on both best proximity theory and metric fixed point theory were enriched, extended, and at last generalized.

Key Words: Fixed point, Metric space, Best proximity point, Generalized ψ-Geraghty contractions.
2010 Mathematics Subject Classification: Primary: 13A15; Secondary: 13F30, 13G05.

1. Introduction

In the last decade, the answers of the following question turns into one of the core subjects of applied mathematics and nonlinear functional analysis: Is there a point x_0 in a metric space (X, d) such that $d(x_0, Tx_0) = dist(A, B)$ where A, B are nonempty subsets of a metric space X and $T : A \to B$ is a non-self-mapping with $dist(A, B) = inf\{d(x, y) : x \in A, y \in B\}$?. Here, the point $x_0 \in X$ is called the best proximity point. The object of best proximity theory is to determine to the minimal conditions on a non-self-mapping T to guarantee the existence and uniqueness of the best proximal point. The setting of best proximity point is richer and more general than the theory of metric fixed point in two sense. Firstly, usually the mappings considered in fixed point theory are self-mappings which are not necessary in the theory of best proximity. Secondly, if one takes $A = B$ in the above setting,

Received: 28 November 2021, Accepted: 24 January 2022. Communicated by Nasrin Eghbali;
∗Address correspondence to Soomieh Khaleghizadeh; E-mail: Skhaleghizadeh @yahoo.com.

the best proximity point becomes fixed point. It is also well known that fixed point theory combine various disciplines of mathematics, such as topology, operator theory, geometry to show the existence (and usually uniqueness) of a solution of certain equation under proper conditions. Moreover, best proximity theory gives chance to handle some problems in which fixed point theory techniques are not adequate. Consequently, best proximity point has a huge application potential due to the richness applications of fixed point theory, for more details see e.g. [1-3,7-17,20-25]. Desides, some essential notations, required definitions and primary results to coherence with the literature were recollected.

Suppose that A and B are two non-empty subsets of metric space (X, d). Throughout the paper, we use the notation $d(A, B)$ instead of $dist(A, B)$ and further

$$\begin{aligned} d(a, B) &:= inf\{d(a, b) : b \in B\}, \ a \in A, \\ A_0 &:= \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\}, \\ B_0 &:= \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}. \end{aligned}$$

Under the assumption of $A_0 \neq \emptyset$, we say that the pair (A, B) has the P-property [13] if the following condition holds:

$$\begin{matrix} d(x_1, y_1) = d(A, B), \\ d(x_2, y_2) = d(A, B), \end{matrix} \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

for all $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Recently, Zhang et al. [28] introduced the notion of weak P-property as follows.

Under the assumption of $A_0 \neq \emptyset$, we say that the pair (A, B) has the weak P-property [13] if the following condition holds:

$$\begin{matrix} d(x_1, y_1) = d(A, B), \\ d(x_2, y_2) = d(A, B), \end{matrix} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2),$$

for all $x_1,\ x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Consistent with Jachymski [9], let (X, d) be a metric space and Δ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [[4], p. 309]) by assigning to each edge the distance between its vertices. If x

and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \cdots, N$. A graph G is connected if there is a path between any two vertices. G is weaklyconnected if $\tilde{G}$ is connected(see for details [6,9].

Recently, some results have appeared providing sufficient conditions for a mapping to be a Picard Operator if (X, d) is endowed with a graph. The first result in this direction was given by Jachymski [9].

Definition 1.1 (9)**.** We say that a mapping $T : X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G, i.e.:

$$\forall x, y \in X((x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)),$$

and T decreases weights of edges of G in the following way:

$$\exists \alpha \in (0,1), \forall x, y \in X((x, y) \in E(G) \Rightarrow d(T(x), T(y)) \leq \alpha d(x, y)).$$

2. Main Results

We start to this section with the following definition.

A function $\psi : [0, \infty) \to [0, \infty)$ is called Bianchini-Grandolfi gauge function [5,18,19] if the following conditions holds:

(i) ψ is nondecreasing;

(ii) There exist $k_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that:

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k,$$

for $k \geq k_0$ and any $t \in \mathbb{R}^+$.

In some sources, Bianchini-Grandolfi gauge function is known as (c)-comparison functions (see e.g. [4]). We denote by Ψ the family of Bianchini-Grandolfi gauge functions. The following lemma illustrate the substance of these functions.

Lemma 2.1. *(See [4]) If $\psi \in \Psi$, then the followings hold:*

(i) $(\psi^n(t))_{n \in N}$ converges to 0 as $n \to \infty$ for all $t \in R^+$;

(ii) $\psi(t) < t$, for any $t \in (0, \infty)$;

(iii) ψ is continuous at 0;

(iv) The series $\sum_{k=1}^{\infty} \psi^k(t)$ converges for any $t \in \mathbb{R}^+$.

Assume that A and B are two non-empty subsets of metric space (X, d). Let Ξ denote the set of all functions $\xi : [d(A, B), \infty)^4 \to R^+$ satisfying:

: (Ξ_1) ξ is continuous;
: (Ξ_2) $\xi(t_1, t_2, t_3, d(A, B) = 0$ for all $t_1, t_2, t_3 \in [d(A, B), \infty)$

Example 2.2. The following functions belong to Ξ

(i) $\xi(t_1, t_2, t_3, t_4) = L \min\{t_1, t_2, t_3, t_4 - d(A, B)\}$ where $L \geq 0$;
(ii) $\xi(t_1, t_2, t_3, t_4) = Lt_1t_2t_3(t4 - d(A, B))$ where $L \geq 0$;
(iii) $\xi(t_1, t_2, t_3, t_4) = \ln(1 + Lt_1t_2t_3(t_4 - d(A, B))$ where $L \geq 0$;
(iv) $\xi(t_1, t_2, t_3, t_4) = e^{Lt_1t_2t_3(t_4 - d(A,B))} - 1$ where $L \geq 0$.

Definition 2.3. Let (X, d) be a metric space endowed with a graph G. Also suppose that A and B are two non-empty subsets of metric space (X, d). A non-self-mapping $T : A \to B$ is said to be a Suzuki type $G - (\xi, \psi)$-proximal contraction, if:

$$\begin{cases} (x, y) \in E(G), \\ d(u, Tx) = d(A, B), & \Rightarrow (u, v) \in E(G) \\ d(v, Ty) = d(A, B) \end{cases}$$

for some x, y, u, $v \in A$ and there exists $r \in (0, 1)$ such that:

$$\frac{1}{2} d^*(x, Tx) \leq d(x, y)$$

$$\Rightarrow d(Tx, Ty) \leq \psi(d(x, y)) + \xi(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)),$$

for all x, $y \in A$ with $(x, y) \in E(G)$ where $d^*(x, y) = d(x, y) - d(A, B)$, $\xi \in \Xi$ and $\psi \in \Psi$.

The following is the first result of this section.

Theorem 2.4. *Let (X, d) be a complete metric space endowed with a graph G. Also suppose that A and B are two non-empty subsets of metric space (X, d). Let $T : A \to B$ be a Suzukitype $G - (\xi, \psi)$-proximal contraction satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;*
(ii) *There exist elements x_0 and x_1 in A_0 such that:*

$$d(x_1, Tx_0) = d(A, B) \text{ and } (x_0, x_1) \in E(G),$$

(iii) *T is a continuous mapping.*

Then T has a best proximity point.

Proof. Due to assumption (iii), there exist elements x_0 and x_1 in A_0 such that:

$$d(x_1, Tx_0) = d(A, B) \text{ and } (x_0, x_1) \in E(G).$$

Owing to the fact that $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that:

$$d(x_2, Tx_1) = d(A, B).$$

Since T is Suzuki type $G-(\xi, \psi)$-proximal contraction, we have $(x_1, x_2) \in E(G)$. Again, by using the fact that $T(A_0) \subseteq B_0$, we guarantee that there exists $x_3 \in A_0$ such that:

$$d(x_3, Tx_2) = d(A, B).$$

So we conclude that:

$$d(x_2, Tx_1) = d(A, B), d(x_3, Tx_2) = d(A, B), (x_1, x_2) \in E(G).$$

Again by recalling that fact that the map T is Suzuki type $G-(\xi, \psi)$-proximal contraction, we derive that $(x_2, x_3) \in E(G)$, that is:

$$d(x_3, Tx_2) = d(A, B), (x_2, x_3) \in E(G).$$

By repeating this process, we observe that:

$$d(x_{n+1}, Tx_n) = d(A, B), (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N}\{0\}. \tag{2.1}$$

By triangle inequality, we have:

$$d(x_{n-1}, Tx_{n-1}) \leq d(x_n, x_{n-1}) + d(x_n, Tx_{n-1}) = d(x_n, x_{n-1}) + d(A, B),$$

which implies:

$$\frac{1}{2} d^*(x_{n-1}, Tx_{n-1}) \leq d^*(x_{n-1}, Tx_{n-1}) \leq d(x_n, x_{n-1}).$$

From (11) and (iv), we derive that:

$$\begin{aligned} d(Tx_{n-1}, Tx_n) &\leq \psi(d(x_{n-1}, x_n)) + \xi d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}) \\ &\leq \psi(d(x_{n-1}, x_n)) + \xi d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(A, B) \\ &\leq \psi(d(x_{n-1}, x_n)) + 0 = \psi(d(x_{n-1}, x_n)). \end{aligned} \tag{2.2}$$

Due to the fact that the pair (A, B) has the weak P-property together with (2), we conclude that:

$$d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n) \text{ for all } n \in \mathbb{N}.$$

Consequently, from (3), we obtain:

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then, the proof is completed. Indeed, (2) yields that:

$$d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B),$$

that is, x_{n_0} is a best proximity point of T. Hence, we assume that:

$$d(x_{n+1}, x_n) > 0 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.4}$$

We find that:

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_1, x_0)) \text{ for all } n \in \mathbb{N} \cup \{0\},$$

by using the fact that ψ is nondecreasing together with the assumption (11), inductively. Fix $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$\sum_{n \geq N} \psi^n(d(x_0, x_1)) < \epsilon \quad \text{for all } n \in \mathbb{N}.$$

Let m, $n \in \mathbb{N}$ with $m > n \geq N$. On the other hand, by the triangular inequality, we have:

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k(d(x_0, x_1)) < \sum_{n \geq N} \psi^n(d(x_0, x_1)) < \epsilon,$$

which yields that $\lim_{m,n,\to+\infty} d(x_n, xm) = 0$. Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is $z \in X$ such that $x_n \to z$. By the continuity of T, we derive that $Tx_n \to Tz$ as $n \to \infty$. Hence, we get the desired result:

$$d(A, B) = \lim_{n \to \infty} d(x_{n+1}, Tx_n) = d(z, Tz).$$

□

In the following theorem, we get an analog of Theorem 2.1, by removing the continuity condition on Suzuki type $G-(\xi, \psi)$-proximal contraction mapping.

Theorem 2.5. *Let (X, d) be a complete metric space endowed with a graph G. Also suppose that A and B are two non-empty subsets of metric space (X, d). Let $T : A \to B$ be a Suzuki type $G-(\xi, \psi)$-proximal contraction satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;
(ii) There exist elements x_0 and x_1 in A_0 such that:

$$d(x_1, Tx_0) = d(A, B) \text{ and } (x_0, x_1) \in E(G),$$

(iii) If $\{x_n\}$ is a sequence in A such that $(x_n, x_{n+1}) \in E(G)$ with $x_n \to x \in A$ as $n \to \infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

Then T has a unique best proximity point.

Proof. Following the lines in the proof of Theorem 2.1, we conclude the sequence $\{x_n\}$ is Cauchy sequence, and there is $z \in X$ such that $x_n \in z$ since X is complete.

Suppose that the condition (v) holds, that is, $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$. From (4) and (5) we obtain that:

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n),$$

for all $n \in \mathbb{N}$. By using (2), we have:

$$\begin{aligned} d^*(x_n, Tx_n) &= d(x_n, Tx_n) - d(A, B) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) - d(A, B) \\ &= d(x_n, x_{n+1}), \end{aligned} \tag{2.5}$$

and:

$$\begin{aligned} &d^*(x_{n+1}, Tx_{n+1}) \\ &= d(x_{n+1}, Tx_{n+1}) - d(A, B) \leq d(x_{n+1}, x_{n+2}) + d(x_{n+2}, Tx_{n+1}) - d(A, B) \\ &\qquad = d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}). \end{aligned}$$

Hence, (6) and (7) implies that:

$$\frac{1}{2}[d^*(x_n, Tx_n) + d^*(x_{n+1}, Tx_{n+1})] < d(x_n, x_{n+1}). \tag{2.6}$$

holds for all $n \in \mathbb{N}$. We suppose that there exists $n_0 \in \mathbb{N}$ such that the following inequalities hold:

$$\frac{1}{2}d^*(x_{n_0}, Tx_{n_0}) > d(x_{n_0}, z),$$

and:

$$\frac{1}{2}d^*(x_{n_0+1}, Tx_{n_0+1}) > d(x_{n_0+1}, z).$$

Hence, by using (8) we can write:

$$\begin{aligned} &d(x_{n_0}, x_{n_0+1}) \leq d(x_{n_0}, z) + d(x_{n_0+1}, z) \\ &\qquad < \frac{1}{2}[d^*(x_{n_0}, Tx_{n_0}) + d^*(x_{n_0+1}, Tx_{n_0+1})] < d(x_{n_0}, x_{n_0+1}), \end{aligned}$$

which is a contradictions. Hence, for all $n \in \mathbb{N}$, we have either:

$$\frac{1}{2}d^*(x_n, Tx_n) \leq d(x_n, z),$$

or:

$$\frac{1}{2}d^*(x_{n+1}, Tx_{n+1}) \leq d(x_{n+1}, z).$$

Regarding (11) we obtain either:

(2.7)
$$d(Tx_n, Tz) \leq \psi(d(x_n, z)) + \xi d(x_n, Tx_n), d(z, Tz), d(x_n, Tz), d(z, Tx_n),$$

or:

(2.8)
$$d(Tx_{n+1}, Tz) \leq \psi(d(x_{n+1}, z)) + \xi d(x_{n+1}, Tx_{n+1}), d(z, Tz), d(x_{n+1}, Tz), d(z, Tx_{n+1}).$$

On the other hand we know that:

$$d(A, B) \leq \lim_{n\to\infty} d(z, Tx_{n+1}) \leq \lim_{n\to\infty} d(z, x_{n+2}) + \lim_{n\to\infty} d(x_{n+2}, Tx_{n+1}) = d(A, B).$$

That is, $\lim_{n\to\infty} d(z, Tx_{n+1}) = d(A, B)$. Now, if we take the limit as $n \to +\infty$ in each of these inequalities (i.e., (9) or (10)), then we have:

$$Tx_n \to Tz \text{ or } Tx_{n+1} \to Tz \text{ as } n \to \infty.$$

Consequently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \to T_z$ as $x_{n_k} \to z$.

Therefore:

$$d(A, B) = \lim_{k\to\infty} d(x_{n_k+1}, Tx_{n_k}) = d(z, Tz).$$

Therefore, T has a best proximity point. □

Example 2.6. Let $X = \mathbb{R}$ endowed with metric $d(x, y) = |x - y|$. Let $A = [-4, 4]$ and $B = (-\infty, -5] \cup [5, \infty)$. Define, $T : A \to B$ by:

$$Tx = \begin{cases} -5, & \text{if } x \in [-4, -3) \\ x^7 + 3x^5 + 2x^3 - 8, & \text{if } x \in [-3, -2) \\ \frac{x^2}{1+|\sin x|} + 10 & \text{if } x \in [-2, -1) \\ x^4 + 2x^2 + 14 & \text{if } x \in [-1, 0) \\ 5, & \text{if } x \in [0, 4] \end{cases}$$

Also define the graph G by $E(G) = [0, 4] \times [0, 4]$.

Clearly, $d(A, B) = 1$, $A_0 = \{-4, 4\}$, $B_0 = \{-5, 5\}$ and $TA_0 \subseteq B_0$. Further, define:

$$\xi : [d(A, B) = 1, \infty)^4 \to R^+,$$

by $\xi(t_1, t_2, t_3, t_4) = t_1t_2t_3(t_4 - 1)$. Let $d(x_1, y_1) = d(A, B) = 1$ and $d(x_2, y_2) = d(A, B) = 1$. Then:

$$(x_1, y_1), (x_2, y_2) \in \{(-4, -5), (4, 5)\}.$$

Now if $(x_1, y_1) \neq (x^2, y^2)$ then $(x_1, y_1) = (-4, -5)$ and $(x_2, y_2) = (4, 5)$ or $(x_1, y_1) = (4, 5)$ and $(x_2, y_2) = (-4, -5)$ which implies:

$$d(x_1, x_2) = d(-4, 4) = d(4, -4) = 8 \leq 10 = d(-5, 5) = d(5, -5) = d(y_1, y_2).$$

Moreover, if $(x_1, y_1) = (x_2, y_2)$ then $d(x_1, x_2) = 0 = d(y_1, y_2)$. Therefore, the pair (A, B) satisfiesthe weak P-property. Let:

$$\begin{cases} (x, y) \in E(G) = [0, 4] \times [0, 4] \\ d(u, Tx) = d(A, B) = 1 \\ d(v, Ty) = d(A, B) = 1 \end{cases}$$

then, $u = v = 4$. That is, $(u, v) \in E(G)$. Also, suppose that, $(x, y) \in E(G)$. Hence, $x, y \in [0, 4]$, which implies:

$$d(Tx, Ty) = 0 \leq \psi(d(x, y)) + \xi(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)).$$

Therefore, T is a Suzuki type $G - (\xi, \psi)$-proximal contraction mapping. Let, $\{x_n\}$ be a sequence in A such that $(x_n, x_{n+1}) \in E(G)$ with $x_n \to x \in A$ as $n \to \infty$. Then, $\{x_n\} \subseteq [0, 4]$. This implies, $x \in [0, 4]$. Thus, $(x_n, x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$. Hence, all conditions of Theorem 2.2 holds and T has a best proximity point. Here, $x = 4$ is best proximity point of T.

3. Best proximity points of Suzuki type (ξ, ψ)-proximal contraction inpartially ordered metric spaces

Definition 3.1. ([22]). Let $(X, d, \preceq)$ be a partially ordered metric space. We say that a nonself-mapping $T : A \to B$ is a proximally ordered-preserving if and only if, for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{cases} x_1 \preceq x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{cases} \Rightarrow u_1 \preceq u_2.$$

Definition 3.2. Let $(X, d, \preceq)$ be a partially ordered metric space. Also suppose that A and B are two non-empty subsets of metric space (X, d). A non-self-mapping $T : A \to B$ is said to be a Suzuki type (ξ, ψ)-proximal contraction, if there exists $r \in (0, 1)$ such that:

$$\frac{1}{2} d^*(x, Tx) \leq d(x, y)$$

$$\Rightarrow d(Tx, Ty) \leq \psi(d(x, y)) + \xi(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)),$$

for all $x, y \in A$ with $x \preceq y$ where $d^*(x, y) = d(x, y) - d(A, B), \xi \in \Xi$ and $\psi \in \Psi$.

Let $(X, d, \preceq)$ be a partially ordered metric space. Define the graph G by:

$$E(G) := \{(x, y) \in X \times X : x \preceq y\}. \tag{3.1}$$

By applying Theorem 2.1, 2.2 and the above graph we have the following results.

Theorem 3.3. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Also suppose that A and B are two non-empty subsets of metric space (X, d). Let $T : A \to B$ be a Suzuki type (ξ, ψ)-proximal contraction satisfying the following conditions:*

(i) T is proximally ordered-preserving, $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;

(ii) There exist elements x_0 and x_1 in A_0 such that:

$$d(x_1, Tx_0) = d(A, B) \text{ and } x_0 \preceq x_1,$$

(iii) T is a continuous mapping.

Then T has a best proximity point.

Theorem 3.4. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Also suppose that A and B are two non-empty subsets of metric space (X, d). Let $T : A \to B$ be a Suzuki type (ξ, ψ)-proximal contraction satisfying the following conditions:*

(i) T is proximally ordered-preserving, $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;

(ii) There exists elements x_0 and x_1 in A_0 such that:

$$d(x_1, Tx_0) = d(A, B) \text{ and } x_0 \preceq x_1,$$

(iii) If $\{x_n\}$ is an increasing sequence in A such that $x_n \to x \in A$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then T has a best proximity point.

4. Application to fixed point theory

If in definition of the function ξ we take $A = B = X$ then we have the following class of functions:

Let Υ denote the set of all functions $\eta : [0, \infty)^4 \to R^+$ satisfying:

: $(\Upsilon 1)$ η is continuous;

: $(\Upsilon 2)$ $\eta(t_1, t_2, t_3, 0) = 0$ for all $t_1, t_2, t_3 \in [0, \infty)$.

Definition 4.1. Let (X, d) be a metric space endowed with a graph G. A self-mapping $T : X \to X$ is said to be a Suzuki type $G-(\eta, \psi)$-contraction, if:

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G),$$

for some $x, y \in X$ and there exists $r \in (0, 1)$ such that:

$\frac{1}{2}d(x, Tx) \leq d(x, y)$

$\Rightarrow d(Tx, Ty) \leq \psi(d(x, y)) + \eta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx),$

for all $x, y \in X$ with $(x, y) \in E(G)$ where $\eta \in \Upsilon$ and $\psi \in \Psi$.

Definition 4.2. Let $(X, d, \preceq)$ be a partially ordered metric space. Also suppose that A and B are two non-empty subsets of metric space (X, d). A self-mapping $T : X \to X$ is said to be a Suzuki type (η, ψ)-contraction, if there exists $r \in (0, 1)$ such that:

$\frac{1}{2}d(x, Tx) \leq d(x, y)$

$\Rightarrow d(Tx, Ty) \leq \psi(d(x, y)) + \eta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)),$

for all $x, y \in X$ with $x \preceq y$ where $\eta \in \Upsilon$ and $\psi \in \Psi$.

If in Theorem 2.1, 2.2, 3.1 and 3.2 we take $A = B = X$ then we deduce the following fixed point results.

Theorem 4.3. *Let (X, d) be a complete metric space endowed with a graph G. Let T be a continuous Suzuki type (η, ψ)-contraction mapping. If there exist element x_0 in X such that, $(x_0, Tx_0) \in E(G)$ then T has a fixed point.*

Theorem 4.4. *Let (X, d) be a complete metric space endowed with a graph G. Let T be a Suzuki type (η, ψ)-contraction. Also suppose that the following assertions holds:*

(i) There exists element x_0 in X such that, $(x_0, Tx_0) \in E(G)$,

(ii) If $\{x_n\}$ is a sequence in A such that $(x_n, x_{n+1}) \in E(G)$ with $x_n \to x \in X$ as $n \to \infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

Then T has a unique best proximity point.

Theorem 4.5. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let T be a continuous increasing Suzuki type (η, ψ)-contraction. If there exist element x_0 in X such that, $x_0 \preceq Tx_0$,then T has a fixed point.*

Theorem 4.6. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let T be a increasing Suzuki type (η, ψ)-contraction. Also suppose that the following assertions holds:*

(i) There exists element x_0 in X such that, $x_0 \preceq Tx_0$

(ii) If $\{x_n\}$ is an increasing sequence in X such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in N$.

Then T has a fixed point.

5. Preparation of manuscript

The manuscript (written in English) with wide margins and double spaced should be submitted in a PDF format to the journal via online submissions. For more information about the preparation of the paper, please see the author guidlines on the website of the journal.

Upon acceptance authors are requested to transmit the $\LaTeX\,2_\varepsilon$ file of the manuscript via e-mail to `managing-editor@jhs-uma.com`, after all revisions have been incorporated and the manuscript has been accepted for publication.

The corresponding author receives proofs, which should be corrected and returned within 48 hours of receipt.

References

[1] M.A. Al-Thagafi and N. Shahzad, Convergence and existence results for best proximity points, *Nonlinear Anal.*, **70**, pp. 3665-3671 (2009).

[2] J. Anuradha and P. Veeramani, Proximal pointwise contraction, *Topol. Appl.*, **156**, pp. 2942-2948 (2009).

[3] S.S. Basha and P. Veeramani, "Best proximity pair theorems for multifunctions with open fibres, *J. Approx. Theory*, **103**, pp. 119-129 (2000).

[4] V. Berinde, *Iterative Approximation of Fixed Points*, Editura Efemeride, Baia Mare (2002).

[5] R.M. Bianchini, M. Grandolfi, Transformazioni di tipo contracttivo gen- eralizzato in uno spazio metrico, *Atti Acad. Naz. Lincei, VII. Ser., Rend., Cl. Sci. Fis. Mat. Natur.*, **45**, pp. 212-216 (1968).

[6] F. Bojor, Fixed point theorems for Reich type contraction on metric spaces with a graph, *Nonlinear Anal.*, **75**, pp. 3895-3901 (2012).

[7] J. Caballero, J. Harjani and K. Sadarangani, A best proximity point theorem for Geraghty-contractions, *Fixed Point Theory and Applications*, **2012**, 231 (2012).

[8] A.A. Eldred and P. Veeramani Existence and convergence of best proximity points, *J. Math. Anal. Appl.*, **323**, pp. 1001-1006 (2006).

[9] J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Amer. Math. Soc.*, **1**(136) pp. 13591373 (2008).

[10] M. De la Sen, Fixed point and best proximity theorems under two classes of integral-type contractive conditions in uniform metric spaces, *Fixed Point Theor. Appl.*, 12 (Art. ID 510974) (2010).

[11] E. Karapınar, Best proximity points of cyclic mappings, *Appl. Math. Lett.*, **25**(11), pp. 1761-1766 (2012).

[12] E. Karapınar, On best proximity point of ψ-Geraghty contractions, preprint.
[13] E. Karapınar and I.M. Erhan, Best proximity point on different type contractions, *Appl. Math. Inf. Sci.*, **3**(3), pp. 342-353 (2011).
[14] E. Karapınar, Best proximity points of Kannan type cylic weak phi-contractions in ordered metric spaces, *Analele Stiintifice ale Universitatii Ovidius Constanta*, **20**(3), pp. 51-64 (2012).
[15] W.A. Kirk, S. Reich and P. Veeramani, Proximinal retracts and best proximity pair theorems, *Numer. Funct. Anal. Optim.*, **24**, pp. 851-862 (2003).
[16] J. Markin, N. Shahzad, Best approximation theorems for nonexpansive and condensing mappings in hyperconvex spaces, *Nonlinear Anal.*, **70**, pp. 2435-2441 (2009).
[17] V. Pragadeeswarar, and M. Marudai, Best proximity points: approximation and optimization in partially ordered metric spaces, *Optim. Lett.*, DOI 10.1007/s11590-012-0529-x.
[18] P.D. Proinov, A generalization of the Banach contraction principle with high order of convergence ofsuccessive approximations, *Nonlinear Analysis: Theory, Methods & Applications*, **67**, pp. 2361-2369 (2007).
[19] P.D. Proinov, New general convergence theory for iterative processes and its applications to NewtonKantorovich type theorems, *Journal of Complexity*, **26**, pp. 3-42 (2010).
[20] V.S. Raj, P. Veeramani, Best proximity pair theorems for relatively nonexpansive mappings, *Appl. Gen.Topol.*, **10**, pp. 21-28 (2009).
[21] V.S. Raj, A best proximity theorem for weakly contractive non-self mappings, *Nonlinear Anal.*, **74**, pp. 4804-4808 (2011).
[22] S. Sadiq Basha, P. Veeramani, Best proximity point theorem on partially ordered sets, *Optim. Lett.*, (2012). doi: 10.1007/s11590-012-0489-1
[23] B. Samet, Some results on best proximity points, *J. Optim. Theory Appl.*, DOI 10.1007/s10957-013-0269-9.
[24] N. Shahzad, S.S. Basha, R. Jeyaraj, Common best proximity points: global optimal solutions, *J. Optim.Theor. Appl.*, **148**, pp. 69-78 (2011).
[25] P.S. Srinivasan, Best proximity pair theorems, *Acta Sci. Math. (Szeged)*, **67**, pp. 421—429 (2001).
[26] K. Suzuki, A new type of fixed point theorem in metric spaces, *Nonlinear Analysis: Theory, Methods &Applications*, **71**(11), pp. 5313-5317 (2009).
[27] K. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, *Proc. Amer.Math. Soc.*, **136**, 18611869 (2008).
[28] J. Zhang, Y. Su and Q. Cheng, A note on A best proximity point theorem for Geraghty-contractions, *Fixed Point Theory and Appl.*, **2013**, 99 (2013)

Soomieh Khaleghizadeh
Department of Management, Payame Noor University (PNU), P.O.Box19395-4697, Tehran, Iran.
Email: `Skhaleghizadeh@yahoo.com`