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ANTI FUZZY IDEAL EXTENSION OF SEMIGROUPS

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Abstract. In this paper the concepts of anti fuzzy l-prime ideals, anti fuzzy semiprime ideals and anti fuzzy ideal extensions in a semigroup have been introduced. They are found to satisfy characteristic function criterion and anti level subset criterion. Here some properties of anti fuzzy ideal extension of a semigroup have been investigated in terms of anti fuzzy l-prime and anti fuzzy semiprime ideals. Among other results we obtain characterization of l-prime ideals of a semigroup in terms of anti fuzzy ideal extension.

Key Words: Semigroup, Anti level subset, Anti fuzzy l-prime ideal, Anti fuzzy semiprime ideal, Anti fuzzy ideal extension.

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1. Introduction

Uncertainty is an attribute of information and uncertain data are presented in various domains. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [12]. In our daily life, we usually want to seek opinions from professional persons with the best qualifications, for examples, the best medical doctors can

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provide the best diagnostics, the best pilots can provide the best navigation suggestions for airplanes, etc. It is therefore desirable to incorporate the knowledge of these experts into some automatic systems so that it would become helpful for other people to make appropriate decisions which are (almost) as good as the decisions made by the top experts. With this aim in mind, our task is to design a system that would provide the best advice from the best experts in the field. However, one of the main hurdles of this incorporation is that the experts are usually unable to describe their knowledge by using precise and exact terms. For example, in order to describe the size of certain type of a tumor, a medical doctor would rarely use the exact numbers. Instead he would say something like the size is between 1.4 and 1.6 cm. Also, an expert would usually use some words from a natural language, e.g., the size of the tumor is approximately 1.5 cm, with an error of about 0.1 cm. Thus, under such circumstances, the way to formalize the statements given by an expert is one of the main objectives of fuzzy logic.

Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation[2]. The formal study of semigroups began in the early 20th century. The topic of investigations about fuzzy semigroups belongs to the theoretical soft computing (fuzzy structures). Indeed, it is well known that semigroups are basic structures in many applicative branches like automata and formal languages, coding theory, finite state machines and others. Due to these possibilities of applications, semigroups and related structures are presently extensively investigated in fuzzy settings. Azirel Rosenfeld [9] used the idea of fuzzy set to introduce the notions of fuzzy subgroups. Nobuaki Kuroki [5, 6, 7] is the pioneer of fuzzy ideal theory of semigroups. The idea of fuzzy subsemigroup was also introduced by Kuroki [5, 7]. In [6], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. Others who worked on fuzzy semigroup theory, such as X.Y. Xie [11], Y.B. Jun [3], are mentioned in the bibliography. X.Y. Xie [11] introduced the idea of extensions of fuzzy ideals in semigroups.

In 2009 M. Shabir and Y. Nawaz [10], M. Khan and T. Asif [4] introduced the concept of anti fuzzy ideals in semigroups and characterized different classes of semigroups by the properties of their anti fuzzy ideals. The purpose of this paper is as stated in the abstract.

2. Preliminaries

Unless or otherwise mentioned throughout this paper S stands for semigroup. We recall the following which will be required in the sequel.

Definition 2.1. [1] Let S be a semigroup. A nonempty subset A of S is called a left (resp., right) ideal of S if $SA \subseteq A$ (resp., $AS \subseteq A$). A two-sided ideal (or simply an ideal) of S is a subset of S which is both a left and right ideal of S.

Definition 2.2. [1] Let S be a semigroup. An ideal P of S is said to be prime if, for any two ideals A and B of $S, AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

Definition 2.3. [1] Let S be a semigroup. An ideal P of S is said to be completely prime if, for any two elements $x, y \in S$, $xy \in P$ implies that $x \in P$ or $y \in P$. An ideal P of S is said to be semiprime if, for an element $x \in S$, $x^2 \in P$ implies that $x \in P$.

Definition 2.4. [12] A function μ from a non-empty set S to the unit interval $[0, 1]$ is called a fuzzy subset of S.

Definition 2.5. [4] Let μ be a fuzzy subset of a semigroup S and let $t \in [0, 1]$. Then the set $\mu_t := \{x \in S : \mu(x) \leq t\}$ is called the anti level subset of μ.

Definition 2.6. [4] Let S be a semigroup and μ be a non-empty fuzzy subset of S. Then μ is called an anti fuzzy left ideal(anti fuzzy right ideal) of S if $\mu(xy) \leq \mu(y)$(resp. $\mu(xy) \leq \mu(x)$) for all $x, y \in S$.

Definition 2.7. [4] A non-empty fuzzy subset μ of a semigroup S is called an anti fuzzy ideal of S if it is both an anti fuzzy left and an anti fuzzy right ideal of S.

Theorem 2.8. [4] *Let I be a non-empty subset of a semigroup S and χ be a fuzzy subset of S such that*

$$\chi(x) = \begin{cases} 0 & \text{if } x \in I \\ 1 & \text{if } x \notin I \end{cases}$$

Then I is a left ideal(right ideal, ideal) of S if and only if χ is an anti fuzzy left ideal(resp. anti fuzzy right ideal, anti fuzzy ideal) of S.

Theorem 2.9. [4] *Let S be a semigroup and μ be a non-empty fuzzy subset of S. Then μ is an anti fuzzy left ideal(anti fuzzy right ideal, anti fuzzy ideal) of S if and only if μ_t's are left ideals(resp. right ideals, ideals) of S for all $t \in Im(\mu)$, where $\mu_t = \{x \in S : \mu(x) \leq t\}$.*

3. Anti Fuzzy l-Prime and Anti Fuzzy Semiprime Ideal

Definition 3.1. Let S be a semigroup and μ be a non-empty fuzzy subset of S. Then μ is called an anti fuzzy l-prime ideal if $\mu(xy) = \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$.

Note 1. If any non-empty fuzzy subset μ of a semigroup S satisfies the property $\mu(xy) = \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$, then μ is an anti fuzzy ideal of S.

Theorem 3.2. *Let I be a non-empty subset of a semigroup S and χ be a fuzzy subset of S such that*

$$\chi(x) = \begin{cases} 0 & \text{if } x \in I \\ 1 & \text{if } x \notin I \end{cases}.$$

Then I is a completely prime ideal of S if and only if χ is an anti fuzzy l-prime ideal of S.

Proof. Let I be a completely prime ideal of S and χ be the characteristic function of I. Let $x, y \in S$. Suppose $xy \in I$. Then $\chi(xy) = 0$. Now I being completely prime, $x \in I$ or $y \in I$. Hence $\chi(x) = 0$ or $\chi(y) = 0$ which gives $\min\{\chi(x), \chi(y)\} = 0$. Thus we see that $\chi(xy) = \min\{\chi(x), \chi(y)\}$. Now suppose that $xy \notin I$. Then $\chi(xy) = 1$. Now since I is a completely prime ideal of S, $x \notin I$ and $y \notin I$. Hence $\chi(x) = 1$ and $\chi(y) = 1$. Consequently, $\min\{\chi(x), \chi(y)\} = 1$. Thus we see that in this case also $\chi(xy) = \min\{\chi(x), \chi(y)\}$.

Conversely, let χ be an anti fuzzy l-prime ideal of S. Then χ is an anti fuzzy ideal of S. So by Theorem 2.8, I is an ideal of S. Let $x, y \in S$ such that $xy \in I$. Then $\chi(xy) = 0$. Let $x \notin I$ and $y \notin I$. Then $\chi(x) = 1 = \chi(y)$, which means $\min\{\chi(x), \chi(y)\} = 1$. This implies that $\chi(xy) = 1$. Thus we get a contradiction. Hence $x \in I$ or $y \in I$. Thus we see that I is a completely prime ideal of S(cf. Definition 2.3). □

Theorem 3.3. *Let S be a semigroup and μ be a non-empty fuzzy subset of S. Then μ is an anti fuzzy l-prime ideal of S if and only if μ_t's are completely prime ideals of S for all $t \in Im(\mu)$, where $\mu_t = \{x \in S : \mu(x) \leq t\}$.*

Proof. Let μ be an anti fuzzy l-prime ideal of S. Then by Note 1 and Theorem 2.9 μ_t's are ideals of S for all $t \in Im(\mu)$. Let $t \in Im(\mu)$. Let $x, y \in S$ such that $xy \in \mu_t$. Then $\mu(xy) \leq t$. Hence $\min\{\mu(x), \mu(y)\} \leq t$. So $\mu(x) \leq t$ or $\mu(y) \leq t$. Hence $x \in \mu_t$ or $y \in \mu_t$. So μ_t is a completely prime ideal of S(cf. Definition 2.3).

Conversely, let μ_t's be completely prime ideals of S for each $t \in Im(\mu)$. Let $x, y \in S$. Then $xy \in \mu_t$ where $t = \mu(xy)$. Since μ_t is a completely prime ideal of S, $x \in \mu_t$ or $y \in \mu_t$(cf. Definition 2.3). So $\mu(x) \leq t$ or $\mu(y) \leq t$. So $\min\{\mu(x), \mu(y)\} \leq t, i.e., \min\{\mu(x), \mu(y)\} \leq \mu(xy)$......$(A)$. Again by Theorem 2.9, μ is an anti fuzzy ideal of S. So $\mu(xy) \leq \mu(x)$ and $\mu(xy) \leq \mu(y)$. Hence $\mu(xy) \leq \min\{\mu(x), \mu(y)\}$......$(B)$. Combining (A) and (B), we obtain $\mu(xy) = \min\{\mu(x), \mu(y)\}$. Hence μ is an anti fuzzy l-prime ideal of S. □

Definition 3.4. Let S be a semigroup and μ be an anti fuzzy ideal of S. Then μ is called an anti fuzzy semiprime ideal if $\mu(x) \leq \mu(x^2)$ for all $x \in S$.

Theorem 3.5. *Let I be a non-empty subset of a semigroup S and χ be a fuzzy subset of S such that*

$$\chi(x) = \begin{cases} 0 & \text{if } x \in I \\ 1 & \text{if } x \notin I \end{cases}.$$

Then I is a semiprime ideal of S if and only if χ is an anti fuzzy semiprime ideal of S.

Proof. Let I be a semiprime ideal of S. Then as I is an ideal of S, by Theorem 2.8 χ is an anti fuzzy ideal of S. Let $x \in S$ such that $x^2 \in I$. Then $\chi(x^2) = 0$. Now I being semiprime, $x \in I(cf.$ Definition 2.3). Hence $\chi(x) = 0$. Thus we see that $\chi(x) = \chi(x^2)$. Now suppose $x^2 \notin I$. Then $\chi(x^2) = 1$. Now since I is a semiprime ideal of S, $x \notin I$. Hence $\chi(x) = 1$. Thus we see that in this case also $\chi(x) = \chi(x^2)$. Hence χ is an anti fuzzy semiprime ideal of S.

Conversely, let χ be an anti fuzzy semiprime ideal of S. Then χ is an anti fuzzy ideal of S. So by Theorem 2.8, I is an ideal of S. Let $x \in S$ such that $x^2 \in I$. Then $\chi(x^2) = 0$. Let $x \notin I$. Then $\chi(x) = 1 > \chi(x^2)$. Thus we get a contradiction. Hence $x \in I$. Thus we see that I is a semiprime ideal of $S(cf.$ Definition 2.3). □

Theorem 3.6. *Let S be a semigroup and μ be a non-empty fuzzy subset of S. Then μ is an anti fuzzy semiprime ideal of S if and only if μ_t's are semiprime ideals of S for all $t \in Im(\mu)$, where $\mu_t = \{x \in S : \mu(x) \leq t\}$.*

Proof. Let μ be an anti fuzzy semiprime ideal of S. Then by Definition 3.4 and Theorem 2.9, μ_t's are ideals of S for all $t \in Im(\mu)$. Let $t \in Im(\mu)$. Let $x \in S$ such that $x^2 \in \mu_t$. Then $\mu(x^2) \leq t$. Since μ is an anti fuzzy semiprime ideal of S, it follows that $\mu(x) \leq t$. Hence $x \in \mu_t$. So μ_t is a semiprime ideal of $S(cf.$ Definition 2.3).

Conversely, let μ_t's be semiprime ideals of S for all $t \in Im(\mu)$. Then by Definition 2.3 and Theorem 2.9, μ is an anti fuzzy ideal of S. Let $x \in S$. Let $\mu(x^2) = t$. Then $\mu(x^2) \leq t$. So $x^2 \in \mu_t$. Since μ_t is semiprime ideal of S, $x \in \mu_t(cf.$ Definition 2.3). So $\mu(x) \leq t$, $i.e., \mu(x) \leq \mu(x^2)$. Hence μ is an anti fuzzy semiprime ideal of S. □

Theorem 3.7. *Let S be a semigroup and $\{\mu_i\}_{i\in I}$ be a family of anti fuzzy semiprime ideals of S such that $\bigcap_{i\in I}\mu_i$ is non-empty. Then $\bigcap_{i\in I}\mu_i$ is an anti fuzzy semiprime ideal of S.*

Proof. Clearly $\bigcap_{i\in I}\mu_i$ is an anti fuzzy ideal of S. Now since each μ_i is an anti fuzzy semiprime ideal, $\mu_i(a) \leq \mu_i(a^2)$ for all $a \in S$. So $\bigcap_{i\in I}\mu_i(a) = \inf\{\mu_i(a) : i \in I\} \leq \inf\{\mu_i(a^2) : i \in I\} = \bigcap_{i\in I}\mu_i(a^2)$ for all $a \in S$. Hence $\bigcap_{i\in I}\mu_i$ is an anti fuzzy semiprime ideal of S. □

Theorem 3.8. *Let S be a semigroup and let μ be an anti fuzzy ideal of S. Then the following are equivalent:* (1) *μ is an anti fuzzy semiprime ideal and* (2) *$\mu(a) = \mu(a^2)$ for all $a \in S$.*

Proof. It is clear that (2) implies (1). Let us assume that (1) holds. Let $a \in S$. Then, since μ is an anti fuzzy semiprime ideal of S, we have $\mu(a) \leq \mu(a^2)$. Again since μ is an anti fuzzy ideal of S, $\mu(a^2) \leq \mu(a)$. Consequently, $\mu(a) = \mu(a^2)$ for all $a \in S$. Hence the proof. □

We can prove the following theorem by routine calculation.

Theorem 3.9. *Let S be a semigroup and μ be a fuzzy subset of S. Then μ is an anti fuzzy l-prime ideal(anti fuzzy semiprime ideal) of S if and only if μ^C(where $\mu^C(x) = 1 - \mu(x)$ for all $x \in S$) is a fuzzy l-prime ideal(resp. fuzzy semiprime ideal) of S.*

4. Anti Fuzzy Ideal Extension

Definition 4.1. Let S be a semigroup, μ be a fuzzy subset of S and $x \in S$. Then the fuzzy subset $< x, \mu >: S \to [0, 1]$ defined by $< x, \mu > (y) = \mu(xy)$, for all $y \in S$, is called the extension of μ by x.

Proposition 4.2. *Let S be a commutative semigroup, μ be an anti fuzzy ideal of S and $x \in S$. Then $< x, \mu >$ is an anti fuzzy ideal of S.*

Proof. Let $p, q \in S$. Then $< x, \mu > (pq) = \mu(xpq) \leq \mu(xp)$(since μ is an anti fuzzy ideal of S) $=< x, \mu > (p)$. Again using commutativity of S, $< x, \mu > (pq) = \mu(xpq) = \mu(xqp) \leq \mu(xq) =< x, \mu > (q)$. Hence $< x, \mu >$ is an anti fuzzy ideal of S. □

Remark 1. Commutativity of semigroup S is not required to prove that $< x, \mu >$ is an anti fuzzy right ideal of S when μ is an anti fuzzy right ideal of S.

Proposition 4.3. *Let S be a semigroup and μ be an anti fuzzy l-prime ideal of S. Then $< x, \mu >$ is an anti fuzzy l-prime ideal of S for all $x \in S$.*

Proof. Let μ be an anti fuzzy l-prime ideal of S. Then

$$\begin{aligned}
< x, \mu > (yz) &= \mu(xyz)(cf.\text{ Definition 4.1})\\
&= \min\{\mu(x), \mu(yz)\}(cf.\text{ Definition 3.1})\\
&= \min[\mu(x), \min\{\mu(y), \mu(z)\}](cf.\text{ Definition 3.1})\\
&= \min[\min\{\mu(x), \mu(y)\}, \min\{\mu(x), \mu(z)\}]\\
&= \min\{\mu(xy), \mu(xz)\}(cf.\text{ Definition 3.1})\\
&= \min\{< x, \mu > (y), < x, \mu > (z)\}(cf.\text{ Definition 4.1}).
\end{aligned}$$

Hence by Definition 4.1, $< x, \mu >$ is an anti fuzzy l-prime ideal of S. □

Definition 4.4. Suppose S is a semigroup and μ is a fuzzy subset of S. Then we define *Anti Supp* $\mu = \{x \in S : \mu(x) < 1\}$.

Proposition 4.5. *Let S be a semigroup, μ be an anti fuzzy ideal of S and $x \in S$. Then we have the following:*

(1) $\mu \supseteq < x, \mu >$,

(2) $< x^n, \mu > \supseteq < x^{n+1}, \mu >$ *for all* $n \in N$,

(3) *If* $\mu(x) < 1$ *then Anti Supp* $< x, \mu >= S$.

Proof. (1) Let $y \in S$. Then

$< x, \mu > (y) = \mu(xy) \leq \mu(y)$(as μ is an anti fuzzy ideal of S). Hence $\mu \supseteq < x, \mu >$.

(2) $< x^{n+1}, \mu > (y) = \mu(x^{n+1}y) = \mu(xx^n y) \leq \mu(x^n y)$(since μ is an anti fuzzy ideal of S) $=< x^n, \mu > (y)$. Hence $< x^n, \mu > \supseteq < x^{n+1}, \mu >$.

(3) Since $< x, \mu >$ is a fuzzy subset of S, by definition, *Anti Supp* $< x, \mu > \subseteq S$. Let $y \in S$. Since μ is an anti fuzzy ideal of S, we have, $< x, \mu > (y) = \mu(xy) \leq \mu(x) < 1$. Then $< x, \mu > (y) < 1$ and consequently, $y \in$ *Anti Supp* $< x, \mu >$. Thus $S \subseteq$ *Anti Supp* $< x, \mu >$. Hence *Anti Supp* $< x, \mu >= S$. □

By routine calculation we can prove the following.

Corollary 4.6. *Let S be a commutative semigroup and μ be an anti fuzzy semiprime ideal of S. Then $< x, \mu >=< x^2, \mu >$ for all $x \in S$.*

Definition 4.7. [11] Suppose S is a semigroup, $A \subseteq S$ and $x \in S$. We define $< x, A >= \{y \in S : xy \in A\}$.

Proposition 4.8. [11] *Let S be a semigroup and $\phi \neq A \subseteq S$. Then $< x, \mu_A >= \mu_{<x.A>}$ for every $x \in S$, where μ_A denotes the characteristic function of A.*

Proposition 4.9. *Let S be a semigroup and μ be a non-empty fuzzy subset of S. Then for any $t \in \mathrm{Im}(\mu)$, $< x, \mu_t >=< x, \mu >_t$ for all $x \in S$.*

Proof. Let $y \in< x, \mu >_t$. Then $< x, \mu > (y) \leq t$. So $\mu(xy) \leq t$ and hence $xy \in \mu_t$. Consequently, $y \in< x, \mu_t >$. It follows that $< x, \mu >_t \subseteq < x, \mu_t >$. Reversing the above argument we can deduce that $< x, \mu_t > \subseteq < x, \mu >_t$. Hence $< x, \mu >_t =< x, \mu_t >$. □

Proposition 4.10. [11] *Let S be a commutative semigroup and μ be a fuzzy subset of S such that $< x, \mu >= \mu$ for every $x \in S$. Then μ is a constant function.*

Corollary 4.11. *Let μ be an anti fuzzy l-prime ideal of a commutative semigroup S. If μ is not constant, then μ is not a minimal anti fuzzy l-prime ideal of S.*

Proof. Let μ be an anti fuzzy l-prime ideal of S. Then by Proposition 4.3, for each $x \in S$, $< x, \mu >$ is an anti fuzzy l-prime ideal of S. Since μ is an anti fuzzy l-prime ideal of S, μ is an anti fuzzy ideal of S. Now by Proposition 4.5(1), $\mu \supseteq < x, \mu >$ for all $x \in S$. If $\mu =< x, \mu >$ for all $x \in S$ then by Proposition 4.10, μ is constant which is not the case by hypothesis. Hence there exists $x \in S$ such that $\mu \supsetneq < x, \mu >$. This completes the proof. □

Proposition 4.12. *Let S be a commutative semigroup and μ be an anti fuzzy semiprime ideal of S. Then $< x, \mu >$ is an anti fuzzy semiprime ideal of S for all $x \in S$.*

Proof. Let μ be an anti fuzzy semiprime ideal of S and $x, y \in S$. Then

$$\begin{aligned} < x, \mu > (y^2) &= \mu(xy^2) \\ &\geq \mu(xy^2x)(\text{since } \mu \text{ is an anti fuzzy ideal of } S) \\ &= \mu(xyyx) = \mu(xyxy)(\text{since } S \text{ is commutative}) \\ &= \mu((xy)^2) \\ &\geq \mu(xy)(\text{since } \mu \text{ is anti fuzzy semiprime ideal of } S) \\ &=< x, \mu > (y). \end{aligned}$$

Again by Proposition 4.2, $< x, \mu >$ is an anti fuzzy ideal of S. Consequently, $< x, \mu >$ is an anti fuzzy semiprime ideal of S for all $x \in S$. □

Using Theorem 3.7 and Proposition 4.12 we can prove the following corollary.

Corollary 4.13. *Let S be a commutative semigroup, $\{\mu_i\}_{i \in I}$ be a non-empty family of anti fuzzy semiprime ideals of S and let $\mu = \inf\{\mu_i : i \in I\}$. Then for any $x \in S, < x, \mu >$ is an anti fuzzy semiprime ideal of S.*

Remark 2. From the proof of the above proposition it is clear that in a semigroup the intersection of arbitrary family of anti fuzzy semiprime ideals is an anti fuzzy semiprime ideal.

Corollary 4.14. *Let S be a commutative semigroup, $\{S_i\}_{i\in I}$ a non-empty family of semiprime ideals of S and $A := \cap_{i\in I} S_i \neq \phi$. Then $< x, \mu_A >$ is an anti fuzzy semiprime ideal of S for all $x \in S$, where μ_A is the characteristic function of A.*

Proof. By supposition $A \neq \phi$. Then for an element $x \in S$, $x^2 \in A$ implies that $x^2 \in S_i$ for all $i \in I$. Since each S_i is a semiprime ideal of S, $x \in S_i$ for all $i \in I(cf.$ Definition 2.3). So $x \in \cap_{i\in I} S_i = A$. Hence A is a semiprime ideal of $S(cf.$ Definition 2.3). So the characteristic function μ_A of A is an anti fuzzy semiprime ideal of $S(cf.$ Theorem 3.5). Hence by Proposition 4.12, for all $x \in S, < x, \mu_A >$ is an anti fuzzy semiprime ideal of S.

Alternative Proof: $A = \cap_{i\in I} S_i \neq \phi$(by the given condition). Hence $\mu_A \neq \phi$. Let $x \in S$. Then $x \in A$ or $x \notin A$. If $x \in A$ then $\mu_A(x) = 0$ and $x \in S_i$ for all $i \in I$. Hence

$$\inf\{\mu_{S_i} : i \in I\}(x) = \inf_{i\in I}\{\mu_{S_i}(x)\} = 0 = \mu_A(x).$$

If $x \notin A$ then $\mu_A(x) = 1$ and for all $i \in I$, $x \notin S_i$. It follows that $\mu_{S_i}(x) = 1$. Hence

$$\inf\{\mu_{S_i} : i \in I\}(x) = \inf_{i\in I}\{\mu_{S_i}(x)\} = 1 = \mu_A(x).$$

Thus we see that $\mu_A = \inf\{\mu_{S_i} : i \in I\}$. Again μ_{S_i} is an anti fuzzy semiprime ideal of S for all $i \in I(cf.$ Theorem 3.5). Consequently by Proposition 4.12, for all $x \in S, < x, \mu_A >$ is an anti fuzzy semiprime ideal of S. □

Theorem 4.15. *Let S be a semigroup and μ an anti fuzzy l-prime ideal of S such that $\mu(x) = \sup_{y\in S}\mu(y)$, where $x \in S$. Then $< x, \mu >= \mu$.*

Proof. Let $z \in S$. Then $\mu(x) \geq \mu(z)$. Hence $\min\{\mu(x), \mu(z)\} = \mu(z)$. Since μ is an anti fuzzy l-prime ideal of S, $\mu(xz) = \min\{\mu(x), \mu(z)\}$. So $\mu(xz) = \mu(z)$. Hence $< x, \mu > (z) = \mu(xz) = \mu(z)$. Consequently, $< x, \mu >= \mu$. □

Theorem 4.16. *Suppose S is a semigroup and μ is an anti fuzzy ideal of S. If for $y \in S$, $\mu(y)$ is not minimal in $\mu(S)$ implies that $< y, \mu >= \mu$ then μ is anti fuzzy l-prime ideal.*

Proof. Let $x_1, x_2 \in S$. Then μ being an anti fuzzy ideal of S, $\mu(x_1x_2) \leq \mu(x_1)$ and $\mu(x_1x_2) \leq \mu(x_2)$. Now two cases may arise viz. *Case* (1) : Either $\mu(x_1)$ or $\mu(x_2)$ is minimal in $\mu(S)$. *Case* (2) : Neither $\mu(x_1)$ nor $\mu(x_2)$ is minimal in $\mu(S)$. *Case* (1) : Without loss of generality, let $\mu(x_1)$ be minimal in $\mu(S)$. Then $\mu(x_1x_2) \leq \mu(x_1)$. Consequently $\mu(x_1x_2) = \mu(x_1) = \min\{\mu(x_1), \mu(x_2)\}$. *Case* (2) : By the hypothesis $< x_1, \mu >= \mu$ and $< x_2, \mu >= \mu$. Hence

$$< x_1, \mu > (x_2) = \mu(x_2) \Rightarrow \mu(x_1x_2) = \mu(x_2)$$
$$= \min\{\mu(x_1), \mu(x_2)\}(\text{since } \mu \text{ is an anti fuzzy ideal }).$$

Thus we conclude that μ is an anti fuzzy l-prime ideal of S. □

To conclude the paper we obtain the following characterization of a completely prime ideal of a semigroup which follows as a corollary to the above theorem.

Corollary 4.17. *Let S be a semigroup and I be an ideal of S. Then I is completely prime if and only if for $x \in S$ with $x \notin I, < x, \mu_I >= \mu_I$, where μ_I is the characteristic function of I.*

Proof. Let I be a completely prime ideal of S. Then, by Theorem 3.2, μ_I is an anti fuzzy l-prime ideal of S. For $x \in S$ such that $x \notin I$, we have $\mu_I(x) = 1 = \sup\limits_{y \in S} \mu_I(y)$. Then by Theorem 4.15, $< x, \mu_I >= \mu_I$.

Conversely, let $< x, \mu_I >= \mu_I$ for all x in S with $x \notin I$. Now $\mu(I)$ is an anti fuzzy ideal of $S(cf.$ Theorem 2.8). Let $y \in S$ be such that $\mu_I(y)$ is not minimal in $\mu_I(S)$. Then $\mu_I(y) = 1$ and so $y \notin I$. So $< y, \mu_I >= \mu_I$. So by the Theorem 4.16, μ_I is an anti fuzzy l-prime ideal of S. So I is a completely prime ideal of $S(cf.$ Theorem 3.2). □

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