Journal of Hyperstructures 2 *(1)* (*2013*), *8-17.*
ISSN: 2251-8436 print/2322-1666 online

FUZZY SMALL RIGHT IDEALS OF RINGS

P. DHEENA AND G. MOHANRAJ

Abstract. We introduce the notion of fuzzy small right ideal,fuzzy small right prime ideal and fuzzy maximal small right ideal in a ring. We have obtained necessary and sufficient condition for a fuzzy small right ideal to be fuzzy small prime right ideal. We have also shown that fuzzy Jacobson radical is the sum of fuzzy small right ideals.

Key Words: Fuzzy algebra, fuzzy small right ideals,maximal small right ideal,small prime right ideal.
2010 Mathematics Subject Classification: Primary: 08A72; Secondary: 13A15.

1. Introduction

Liu [7] introduced the notion of fuzzy ideals in ring.We introduce the notion of a fuzzy small right ideal,fuzzy small prime right ideal and fuzzy maximal small right ideal in a ring.We have shown that the image of fuzzy small prime ideal will contain only two elements.We have obtained necessary and sufficient condition for a fuzzy small right ideal to be fuzzy small prime right ideal.We have shown that every maximal small right ideal is a small prime right ideal.We have also shown that fuzzy Jacobson radical is the sum of fuzzy small right ideals.

2. Preliminaries

Let R be a ring with identity. $(A, +)$, a subgroup of $(R, +)$ is said to be a right (left) ideal if $xr \in A$ $(rx \in A)$ for all $r \in R$ and $x \in A$. A subgroup $(A, +)$ of R is called an ideal if it is both right and left ideal.

Received: 1 April 2012, Accepted: 21 August 2013. Communicated by A. Yousefian Darani
∗Address correspondence to G. Mohanraj; E-mail: gmohanraaj@gmail.com

An ideal A is called small right ideal [3] if for all right ideals B of R, $A + B = R$ implies $B = R$. Through out this paper, R is a ring with identity unless otherwise specified.

Definition 2.1.. [4] *The* **Jacobson radical** $\mathfrak{J}(R)$ *of a ring* R *is defined as follows:*

$$\mathfrak{J}(R) = \{a|a \in R, aR \text{ is right quasi regular}\}$$

Definition 2.2.. [4] *A right ideal* A *in a ring* R *is said to be a* **modular** *right ideal if there exists an element* e *of* R *such that* $er - r \in A$ *for every element* r *of* R.

Theorem 2.3.. [4] *Let* R *be a ring such that* $\mathfrak{J}(R) \neq R$, *and let* $A_i, i \in \mathfrak{U}$, *be all the modular maximal right ideals in* R. *Then*

$$\mathfrak{J}(R) = \bigcap_{i \in \mathfrak{U}} A_i.$$

Remark 2.4..
(1) [4] *If* R *has identity (or just a left identity for that matter), then every right ideal in* R *is modular.*
(2) *If* R *has identity , then* $\mathfrak{J}(R) \neq R$.
(3) *If* R *has identity , then by Theorem 2.3.,* $\mathfrak{J}(R) = \bigcap_{M \in \mathfrak{M}} M$ *where* $\mathfrak{M}$ *is a set of maximal right ideals in* R.

Lemma 2.5.. [3] *If* R *is a ring with identity, then* $\mathfrak{J}(R) = \mathbb{M} = \sum_{A \in \mathfrak{S}} A$ *where* $\mathfrak{S}$ *is a set of all small right ideals in* R *and* $\mathbb{M} = \bigcap_{M \in \mathfrak{M}} M$.

Theorem 2.6.. *If* B *is an ideal and* A *is a small right ideal of* R *such that* $B \subseteq A$, *then* B *is a small right ideal of* R.

Lemma 2.7.. *If* A *is a small right ideal and* M *is a maximal right ideal of* R, *then* $A \subseteq M$.

Definition 2.8.. *An ideal* M *is said to be a maximal small right ideal if for any small ideal* $A \nsubseteq M$, $A + M = R$.

Definition 2.9.. *An ideal* P *is called small prime right ideal if* $A.B \subseteq P$ *implies* $A \subseteq P$ *or* $B \subseteq P$ *for all small right ideals* A, B *of* R.

Example 2.10.. *Every maximal small ideal need not be a maximal ideal and every small prime need not be a prime ideal. Let* $R = \mathbb{Z}_4 \times \mathbb{Z}_4$. *Let* $I = \{0, 2\}$. *Clearly* $I \times I$ *is a maximal small ideal. Since* $I \times I \subset \mathbb{Z}_4 \times I \subset R$, $I \times I$ *is not a maximal ideal.Clearly* $I \times I$ *is a small prime ideal but* *not a prime ideal since* $(\mathbb{Z}_4 \times I).(I \times \mathbb{Z}_4) \subseteq I \times I$ *with* $I \times \mathbb{Z}_4 \nsubseteq I \times I$ *and* $\mathbb{Z}_4 \times I \nsubseteq I \times I$

Theorem 2.11.. *Every maximal small right ideal of* R *is a small prime right ideal of* R.

Definition 2.12.. *A mapping* $f : X \to [0, 1]$, *is called a fuzzy subset in* X *where* X *is nonempty set.*

Definition 2.13.. *A fuzzy subset* f *of a ring* R *is called fuzzy right (left) ideal of* R *if*

(1) $f(x - y) \geq f(x) \wedge f(y)$
(2) $f(xy) \geq f(x)$, $(f(xy) \geq f(y))$ *for all* $x, y \in R$.

Definition 2.14.. *A level set of a fuzzy set* f *denoted by* f_t *is defined as* $f_t = \{x \in R | f(x) \geq t\}$ *where* $t \in [0, 1]$.

Theorem 2.15.. [5] *A fuzzy set* f *of a ring* R *is a fuzzy ideal if and only if for all* $t \in (0, 1]$, f_t *is an ideal of* R *whenever non-empty*

3. Fuzzy Small Right Ideal

Definition 3.1.. *A fuzzy ideal* f *is called fuzzy small right ideal if* f_t *is a small right ideal for all* $t \in [0, 1]$ *whenever* f_t *is non empty and* $f_t \neq R$.

Theorem 3.2.. *Let* f *be a fuzzy small right ideal of a ring* R. *If* g *is a fuzzy right ideal such that* $g \subseteq f$ *and* $\inf\{Im\ f\} = \inf\{Im\ g\}$, *then,* g *is a fuzzy small right ideal.*

Proof: Let f be a fuzzy small right ideal.Let $t \in [0, 1]$ such that g_t is non empty and $g_t \neq R$. If $f_t \neq R$, then f_t is a small ideal in R. Since $g \subseteq f, g_t \subseteq f_t$ for all$t \in [0, 1]$. If $f_t = R$, then $t \leq \inf\{Im\ f\} = \inf\{Im\ g\}$. Then $g_t = R$ which is a contradiction. Thus $g_t \subset f_t \neq R$. By Theorem 2.6. g_t is a small right ideal of R. Thus g is a fuzzy small right ideal. ■

Example 3.3.. *Let* R *be a set of* 2×2 *matrices over* $\mathbb{Z}_6$.

$$f(x) = \begin{cases} 0.8\ if\ x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ 0.5\ 0therwise \end{cases}$$

$$g(x)=\begin{cases} 0.7\ if\ x=\begin{pmatrix}0&0\\0&0\end{pmatrix} \\ 0.4\ if\ x=\begin{pmatrix}a&b\\0&0\end{pmatrix}\ where\ a,b\in\{2,4\} \\ 0\ 0therwise \end{cases}$$

Clearly f is a fuzzy small right ideal and $g \subset f$. Since $g_{0.4} \neq R$ is not a small right ideal, g is not a fuzzy small right ideal. Here $\inf\{Im\ f\} = 0.5 \neq \inf\{Im\ g\} = 0$.

Theorem 3.4.. *If f and g are fuzzy small right ideals, then $f \cap g$ is a fuzzy small right ideal.*

Proof: Let f and g be fuzzy small right ideals of R and $t \in (0,1]$. Now, $x \in (f \cap g)_t$ implies $\min\{f(x), g(x)\} \geq t$. Then $x \in f_t \cap g_t$. Thus $(f \cap g)_t \subseteq f_t \cap g_t$. Now, $x \in f_t \cap g_t$ implies $\min\{f(x), g(x)\} \geq t$. Then $x \in (f \cap g)_t$. Thus $f_t \cap g_t \subseteq (f \cap g)_t$. Therefore $f_t \cap g_t = (f \cap g)_t$. If $(f \cap g)_t \neq R$, then either $f_t \neq R$ or $g_t \neq R$. Then either $f_t \neq R$ or $g_t \neq R$ is a small right ideal of R. Since $(f \cap g)_t \subseteq f_t$ and $(f \cap g)_t \subseteq g_t$, $(f \cap g)_t$ is a small right ideal of R. Hence $f \cap g$ is a fuzzy small right ideal of R.■

Theorem 3.5.. *χ_A is a fuzzy small right ideal of a ring R if and only if A is a small right ideal in R.*

Proof: Let χ_A be a fuzzy small right ideal. Then by definition,$(\chi_A)_1 = A$ is a small right ideal. On the other hand, $(\chi_A)_t = A$ for all $t > 0$ is a small right ideal and $(\chi_A)_0 = R$. Thus χ_A is a fuzzy small right ideal.■

Definition 3.6.. *A homomorphism $\phi : R \to R^*$ is called small right ideal preserving homomorphism if I is a small right ideal in R implies $\phi(I)$ is a small right ideal in R^*.*

Example 3.7.. *A mapping $\phi : \mathbb{Z}_8 \to \mathbb{Z}_4$ is defined by $\phi(x) = x(mod4)$. Clearly ϕ is a homomorphism.$\phi(\{0,2,4,6\}) = \{0,2\}$ and $\phi(\{0,4\}) = \{0\}$ are small right ideals in $\mathbb{Z}_4$. Hence ϕ is a small right ideal preserving homomorphism.*
A mapping $\psi : \mathbb{Z}_8 \to \mathbb{Z}_6$ is defined by $\psi(x) = x(mod6)$. {0,2,4,6} is a small ideal in $\mathbb{Z}_8$. But $\psi(\{0,2,4,6\}) = \{0,2,4\}$ is not a small ideal in$\mathbb{Z}_6$, so ψ is not a small right ideal preserving homomorphism.

Theorem 3.8.. *Let $\phi : R \to R^*$ be a onto small right ideal preserving homomorphism . If f is a fuzzy small right ideal of R then $\phi(f)$ is a fuzzy small right ideal of R^* where $(\phi(f))(x) = \sup\{f(y) | y \in \phi^{-1}(x)\}$.*

Proof: Let $\phi : R \to R^*$ be a small right ideal preserving homomorphism and f be a fuzzy small right ideal of R. We assert that $\phi(f_t) = [\phi(f)]_t$. Let $x \in [\phi(f)]_t$. Then there is a $y \in R$ such that $\phi(y) = x$ and $f(y) \geq t$. Thus $x \in \phi(f_t)$. If $x \in \phi(f_t)$, then there is a $y \in R$ such that $\phi(y) = x$ and $f(y) \geq t$. Therefore $x \in [\phi(f)]_t$. Hence $\phi(f_t) = [\phi(f)]_t$. If $[\phi(f)]_t \neq R^*$ for some $t \in [0,1]$, then $f_t \neq R$. Then f_t is a small ideal of R. Since ϕ is a small right ideal preserving homomorphism, $[\phi(f)]_t$ is a small right ideal in R^*. Hence $\phi(f)$ is a fuzzy small right ideal of R^*. ∎

4. Fuzzy small prime ideal

Definition 4.1.. *Let f, g be fuzzy sets of a ring R. The product of fuzzy sets f and g is defined as follows*

$$(f.g)(x) = \begin{cases} & \sup\limits_{x=yz} \min\{f(y), g(z)\} \\ 0 & otherwise \end{cases}$$

Definition 4.2.. *Let f, g be fuzzy sets of a ring R. The sum of fuzzy sets f and g is defined as follows*

$$(f+g)(x) = \begin{cases} & \sup\limits_{x=y+z} \min\{f(y), g(z)\} \\ 0 & otherwise \end{cases}$$

Definition 4.3.. *A fuzzy ideal h is called fuzzy prime ideal of a ring R if $f.g \subseteq h$ implies $f \subseteq h$ or $g \subseteq h$ for all fuzzy ideals f, g of R*

Definition 4.4.. *A fuzzy ideal h is called fuzzy small prime right ideal of a ring R if $f.g \subseteq h$ implies $f \subseteq h$ or $g \subseteq h$ for all fuzzy small right ideals f, g of R*

Note: Every fuzzy prime ideal is fuzzy small prime ideal.But fuzzy small prime ideal need not be a fuzzy prime ideal as shown by the following example

Example 4.5.. *Consider $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ is a ring. Let $I = \{0, 2\}$.*

$$h(x) = \begin{cases} 1 & if\ x \in I \times I \\ 0.3 & otherwise \end{cases} \qquad f(x) = \begin{cases} 0.8 & if\ x \in I \times \mathbb{Z}_4 \\ 0 & otherwise \end{cases}$$

$$g(x) = \begin{cases} 0.9 & if\ x \in \mathbb{Z}_4 \times I \\ 0.3 & otherwise \end{cases}$$

Clearly h is a fuzzy small prime right ideal and f, g are fuzzy ideals of R. $f.g \subseteq h$ but $f[(2,3)] = 0.8 > h[(2,3)] = 0.3$ and $g[(1,0)] = 0.9 > h[(1,0)] = 0.3$ Hence h is not a fuzzy prime ideal.

Lemma 4.6.. *If h is a fuzzy small prime right ideal of a ring R then, $Im\ h$ contains two elements.*

Proof: Let h be a fuzzy small prime right ideal of a ring R. If $Im\ h = \{t_1, t_2, t_3\}$ where $1 > t_1 > t_2 > t_3 \geq 0$, then h_{t_1}, h_{t_2} are small ideals in R since $h_{t_1} \neq R$ and $h_{t_2} \neq R$. Then $h_{t_1} \subset h_{t_2}$. Choose $s_1 \in [0,1]$ such that $t_1 > s_1 > t_2$.

$$f(x) = \begin{cases} 1 & \text{if } x \in h_{t_1} \\ 0 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} s_1 & \text{if } x \in h_{t_2} \\ 0 & \text{otherwise} \end{cases}$$

$$f.g(x) = \begin{cases} s_1 & \text{if } x \in (h_{t_1}).(h_{t_2}) = h_{t_1} \\ 0 & \text{otherwise} \end{cases}$$

Clearly f and g are fuzzy small right ideals.Therefore $f.g \subseteq h$ but $f \nsubseteq h$ and $g \nsubseteq h$. This contradicts that h is a fuzzy small prime ideal. Hence $Im\ h$ contains two elements. ■

Lemma 4.7.. *If h is a fuzzy small prime right ideal of a ring R then, $h(0) = 1$*

Proof: Let h be a fuzzy small prime right ideal of a ring$R$. If $h(0) \neq 1$, then by Lemma 4.6. $Im\ h = \{t_1, t_2\}$ where $1 > t_1 > t_2 \geq 0$.

$$f(x) = \begin{cases} 1 & \text{if } x \in h_{t_1} \\ 0 & \text{otherwise} \end{cases} \qquad g(x) = s \ for\ all\ x \in R$$

where $t_1 > s > t_2$. Clearly f is fuzzy small right ideal and since constant map is always fuzzy small right ideal,g is also fuzzy small right ideal. Then $f.g \subseteq h$ but $f \nsubseteq h$ and $g \nsubseteq h$. This contradicts that h is a fuzzy small prime ideal. Therefore $h(0) = 1$. ■

Lemma 4.8.. *If h is a fuzzy small prime right ideal of a ring R then, $\{x|h(x) = h(0)\}$ is a small prime right ideal.*

Proof: Let h be a fuzzy small prime ideal of a ring$R$. Then by Lemmas 4.6.and 4.7. $Im\ h = \{1, t_1\}$ where $1 > t_1 \geq 0$. Let $h_1 = \{x|h(x) = h(0)\}$. If there exits small right ideals I_1, I_2 in R such that $I_1.I_2 \subseteq h_{t_1}$

with $I_1 \nsubseteq h_{t_1}$ and $I_2 \nsubseteq h_{t_1}$. Then there is $x \in I_1$ *but* $x \notin h_{t_1}$ and $y \in I_2$ *but* $y \notin h_{t_1}$.

$$f(x) = \begin{cases} t_1 & \text{if } x \in I_1 \\ 0 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} t_1 & \text{if } x \in I_2 \\ 0 & \text{otherwise} \end{cases}$$

Clearly f and g are fuzzy small right ideals.Then $f.g \subseteq h$ but $f \nsubseteq h$ and $g \nsubseteq h$ which contradicts that h is a fuzzy small prime ideal. Hence $\{x|h(x) = h(0)\}$ is a small prime right ideal. ■

Theorem 4.9.. *If h is a fuzzy ideal of a ring R, then h is a fuzzy small prime right ideal if and only if*

(1) *$Im\ h = \{1, t\}$ where $1 > t \geq 0$.*
(2) *$h_1 = \{x|h(x) = h(0) = 1\}$ is a small prime right ideal.*

Proof: Let h be a fuzzy small prime right ideal. Then (1) and (2) follows from Lemmas 4.6.,4.8. and 4.7.

On the other hand, if there exits fuzzy small right ideals f and g of R such that $f \cdot g \subseteq h$ with $f \nsubseteq h$ and $g \nsubseteq h$. Then there exits $x, y \in R$ such that $f(x) = t_1 > t = h(x)$ and $g(y) = s_1 > t = h(y)$ for $t_1, s_1 \in (0, 1]$. Thus $x \in f_{t_1}$ and $y \in g_{s_1}$ with $x \notin h_1$ and $y \notin h_1$. Let $a \in f_{t_1}$ and $b \in g_{s_1}$. Then $ab \in f_{t_1} \cdot g_{s_1}$ and $f(a) \geq t_1$ and $g(b) \geq s_1$. Thus $(f \cdot g)(ab) \geq t_1 \wedge s_1 > t$. Therefore $f \cdot g \subseteq h$ implies $h(ab) = 1$. Hence $f_{t_1} \cdot g_{s_1} \subseteq h_1$. Now,if $f_{t_1} = R$ and $g_{s_1} = R$, then $f_{t_1} \cdot g_{s_1} \subseteq h_1$ implies $R \cdot R = R \subseteq h_1$. Thus h is constant which is a contradiction.If $f_{t_1} \neq R$ and $g_{s_1} = R$, then $x \cdot 1 = x \in f_{t_1} \cdot R = f_{t_1} \cdot g_{s_1} \subseteq h_1$ is a contradiction.If $f_{t_1} = R$ and $g_{s_1} \neq R$, then $1 \cdot y = y \in R \cdot g_{s_1} = f_{t_1} \cdot g_{s_1} \subseteq h_1$ is a contradiction.If $f_{t_1} \neq R$ and $g_{s_1} \neq R$, then f_{t_1} and g_{s_1} are small right ideals. Then $f_{t_1} \cdot g_{s_1} \subseteq h_1$ implies either $f_{t_1} \subseteq h_1$ or $g_{s_1} \subseteq h_1$ which is a contradiction. Therefore h is a fuzzy small prime right ideal. ■

Corollary 4.10.. *If h is a fuzzy ideal of a ring R, then h is a fuzzy prime right ideal if and only if*

(1) *$Im\ h = \{1, t\}$ where $1 > t \geq 0$.*
(2) *$h_1 = \{x|h(x) = h(0) = 1\}$ is a prime right ideal.*

Proof: The proof follows from Theorem 4.9. ■

Theorem 4.11.. *χ_P is a fuzzy small prime ideal of a ring R if and only if P is a small prime ideal of a ring R.*

Proof: Let χ_P be a fuzzy small prime right ideal of a ring R. Then by Theorem 4.9. P is a small prime right ideal of a ring R. Conversely, by Theorem 4.9. χ_P is a fuzzy small prime right ideal of a ring R. ■

5. Fuzzy maximal small ideal

Definition 5.1.. *A fuzzy right ideal g is said to be fuzzy maximal right ideal if $Im\ g = \{1, t\}$ where $1 > t \geq 0$ and $\{x \in R | g(x) = 1 = g(0)\}$ is a maximal right ideal in R.*

Definition 5.2.. *A fuzzy ideal g is said to be fuzzy maximal small right ideal if $Im\ g = \{1, t\}$ where $1 > t \geq 0$ and $\{x \in R | g(x) = 1\}$ is a maximal small right ideal in R.*

Fuzzy maximal small right ideal need not be a fuzzy maximal ideal as shown by the following Example 5.3..

Example 5.3.. *Consider the ring as in the Example 4.5.*

$$g(x) = \begin{cases} 1 & \text{if } x \in I \times I \\ 0.3 & \text{otherwise} \end{cases} \qquad h(x) = \begin{cases} 1 & \text{if } x \in I \times \mathbb{Z}_4 \\ 0 & \text{otherwise} \end{cases}$$

Clearly g is a fuzzy maximal small right ideal but not fuzzy maximal ideal since $I \times I$is not a maximal ideal. h is a fuzzy maximal ideal and it is not a fuzzy small right ideal.

Definition 5.4.. *Let R be a ring.The fuzzy Jacobson radical denoted by $\mathfrak{J}_f(R)$ is defined as follows:*

$$\mathfrak{J}_f(R) = \bigcap \{h|\ h \text{ is a fuzzy maximal right ideal of } R\}$$

Theorem 5.5.. *If g is a fuzzy maximal small right ideal, then g is fuzzy small prime ideal.*

Proof: Let g be a fuzzy maximal small right ideal of R. Then $Im\ g = \{1, t\}$ and g_1 is a maximal small right ideal. By Theorem 2.11., g_1 is a small prime ideal. Then by Theorem 4.9., g is a fuzzy small prime ideal of R. ■

Lemma 5.6.. *An ideal M is a maximal small right ideal if and only if χ_M is a fuzzy maximal small right ideal in R.*

Proof: The proof is straightforward.

Theorem 5.7.. *The fuzzy Jacobson radical $\mathfrak{J}_f(R)$ is given by the equation as follows:*

$$(\mathfrak{J}_f(R))(x) = \begin{cases} 1 & \text{if } x \in \mathbb{M} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbb{M} = \bigcap \{M | M \text{is a maximal ideal of } R\}$. Moreover $\mathfrak{J}_f(R) = \chi_{\mathbb{M}}$.

Proof: Let $x \in R$. If $(\mathfrak{J}_f(R))(x) = 1$, then $h(x) = 1$ for all fuzzy maximal right ideal h of R. Thus $x \in h_1$ for all fuzzy maximal right ideal h of R. Therefore $x \in M$ for all maximal right ideal M of R. Hence $x \in \mathbb{M}$. If $x \in \mathbb{M}$, then $x \in h_1$ for all fuzzy maximal right ideal h of R. Therefore $h(x) = 1$ for all fuzzy maximal right ideal h of R. Hence $(\mathfrak{J}_f(R))(x) = 1$. If $(\mathfrak{J}_f(R))(x) = 0$, then by above argument $x \notin \mathbb{M}$. If $x \notin \mathbb{M}$, then there is a maximal right ideal M_i of R such that $x \notin M_i$. By Lemma 5.6.χ_{M_i} is a fuzzy maximal right ideal in R. Thus $(\mathfrak{J}_f(R))(x) \leq \chi_{M_i} = 0$ implies $(\mathfrak{J}_f(R))(x) = 0$. Hence $\mathfrak{J}_f(R) = \chi_{\mathbb{M}}$. ■

Lemma 5.8.. *If A and B are subsets of a nonempty set X, then $\chi_A + \chi_B = \chi_{A+B}$.*

Proof: Let $x \in X$. If $\chi_{A+B}(x) = 1$, then $x = a+b$, for some $a \in A, b \in B$. Then $(\chi_A + \chi_B)(x) \geq \min\{\chi_A(a), \chi_B(b)\} = 1$. Thus $(\chi_A + \chi_B)(x) = 1$. If x can not be expressible as $x = a + b$, for all $a \in A, b \in B$, then $\chi_{A+B}(x) = 0$. Then $(\chi_A + \chi_B)(x) = 0$. If $x = y + z$ for some $y \in A, z \notin B$ or $y \notin A, z \in B$ then $\chi_{A+B}(x) = 0 = (\chi_A + \chi_B)(x)$. Therefore $\chi_A + \chi_B = \chi_{A+B}$.

Lemma 5.9.. *If A and B are small right ideals of a ring R, then $\chi_A + \chi_B = \chi_{A+B}$. Moreover $\chi_{\sum A_i} = \sum \chi_{A_i}$ for all small right ideals A_i in R.* ■

Proof: The result follows from Lemma 5.8. ■

Theorem 5.10.. *If $\mathfrak{S}$ is a set of all small right ideals in R, then*

$$\mathfrak{J}_f(R) = \sum_{A \in \mathfrak{S}} \chi_A.$$

Proof: Let $\mathfrak{S}$ be a set of all small right ideals in R. By Theorem 5.7. we have $\mathfrak{J}_f(R) = \chi_{\mathbb{M}}$ where $\mathbb{M} = \bigcap\{M | M \text{is a maximal right ideal of } R\}$. Then by Lemma 2.5. $\mathfrak{J}_f(R) = \chi_{\mathbb{M}} = \chi_{\sum\limits_{A \in \mathfrak{S}} A}$. By Lemma 5.9. we have $\mathfrak{J}_f(R) = \sum \chi_A, A \in \mathfrak{S}$. ■

References

[1] A. W. Goldie,*Semiprime Rings with maximum condition,* Proc. London Math. Soc.**10** (3)(1960), 201–220.

[2] F. A. M. Aldosray and S. A. A. Bashsmmakh, *Rings with chain conditions on small right ideals,* Punjab University Journal of Mathematics, **37** (2005), 1–10.

[3] F. W. Anderson and K. R. Fuller, *Rings and Categories,* Springer-verlag, New york, (1974).

[4] N. H. McCoy ,*The theory of rings,* The Macmillan company, New York, (1964).
[5] R. Kumar, *Fuzzy Algebra,* Publication division, University of Delhi, **1** (1993).
[6] V. P. Camilo, *Commutative Rings whose quotients are Goldie,* Glasgow Math. J., **16** (1975), 32–33.
[7] W. J. Liu, *Fuzzy invariant subgroups and fuzzy ideals,* Fuzzy Sets and Systems, **8** (1982), 133–215.
[8] Z. Wang, F. Dai and Y. Yu, *Radicals of TL-ideals,* Fuzzy Sets and Systems1, **21** (2001), 301–314.

P. Dheena
Department of Mathematics, Annamalai University, Annamalai nagar, India
Email: dheenap@yahoo.com

G. Mohanraj
Department of Mathematics, Annamalai University, Annamalai nagar, India
Email: gmohanraaj@gmail.com