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BOLZA TYPE PROBLEMS IN INFINITE DIMENSIONAL DISCRETE TIME

RAHMA SAHRAOUI

Abstract. In this paper we study a discrete time version of deterministic models in optimization in infinite dimensional. The functionals are assumed to be merely lower semi continuous. We obtain optimality conditions which are always necessary and which are also sufficient in the convex case whenever the given problem satisfies a qualification condition.

Key Words: Nonsmooth analysis, Subdifferential, Qualification condition, Normal compacity, Epi-Lipschitz, Prox-regular set.

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1. Introduction

Our aim here is to treat optimization problems of Bolza type in the context of infinite dimensional Banach spaces which can't be consequences of the results in the finite dimensional cases (see Sahraoui-Thibault (2008)). Let X a Banach space and $l : X \times X \to \mathbb{R} \cup \{+\infty\}$, $L_t : X \times X \to \mathbb{R} \cup \{+\infty\}$, for all $t = 1, \cdots, T$ which is supposed to be lower semi continuous (briefly, l.s.c). For each vector $x = (x_0, \cdots, x_T) \in X^{T+1}$, we associate the differences $\Delta x_t = x_t - x_{t-1}$ and also the problem which consist to minimize the function

$$j(x) = l(x_0, x_1) + \sum_{t=1}^{T} L_t(x_{t-1}, \Delta x_t)$$

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∗Address correspondence to Rahma Sahraoui; E-mail: Sahraouimosta@yahoo.fr.

over the space of all the vectors $x \in X^{T+1}$. We note that the function j is l.s.c over X^{T+1}. This is the problem of Bolza type on the Banach space X and at discrete time.

In this problem of Bolza type, which is noted $(\mathcal{P}_{det})$, the constraints are implicit in the inequality $j(x) < \infty$. Consider now a *closed* subset C of $X \times X$ and, for all $t = 1, \cdots, T$, let $F_t : X \rightrightarrows X$ be a multifunction with a closed graph. For the functions l and L_t with finite values and locally Lipschitzian, we consider the problem with explicit constraints $\mathcal{P}_{C,F}(l, L)$ described above but satisfying the constraints

$$(x_0, x_T) \in C \quad \text{and} \quad \Delta x_t \in F_t(x_{t-1}),\ \forall\, t = 1, \cdots, T. \tag{1.1}$$

Implicitly in the dynamic constraint $\Delta x_t \in F_t(x_{t-1})$ is the state constraint $x_{t-1} \in Z_t$ for all $t = 1, \cdots, T$ or $Z_t = \{z_t \in \mathbb{R}^n | F_t(Z_t) \neq \emptyset\}$.

We will establish the optimality necessary conditions of those problems in two contexts, the first one is when the Banach space X is an Asplund space, and in the second context X is an arbitrary Banach space. We have already remind the concept of the limiting subdifferential and a tot of basic elements in the context of Asplund space (see Sahraoui and Thibault (2008)). Outside the Asplund spaces, the limiting subdifferential, for all locally Lipschitzian functions, also can be empty at all point of the domain. For the spaces which are not Asplund spaces, the limiting subdifferential has not the calculus rules. So in this case we will use the Clarke subdifferential.

2. Definitions and preliminaries

To introduce the concept of Clarke's subdifferential, firstly we need to define the concept of tangent cone of Clarke. In all the rest X is a real Banach space. Let C be a nonempty closed subset of X and $\bar{x}$ a point of C. We say that a vector $v \in X$ is in the tangent cone of Clarke to C at the point $\bar{x}$ see Clarke, Stern and Wolenski(1995), and we write $v \in T_c(C, \bar{x})$, when there exist some sequences $t_n \downarrow 0$, $x_n \xrightarrow{C} \bar{x}$ and $v_n \to v$ such that, for all $n \in \mathbb{N}$, we have $x_n + t_n v_n \in C$. The normal cone of Clarke to C at the point $\bar{x} \in C$ is given by the negative polar cone of tangent cone, which means

$$N_c(C, \bar{x}) = (T_c(C, \bar{x}))^o = \{x^* \in X^* \mid \langle x^*, v\rangle \leq 0,\ \forall v \in T_c(C, \bar{x})\}.$$

Let now $f : X \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c function and $\bar{x}$ a point where f is finite. considering the normal cone to the epigraph $\operatorname{epi} f$ of f we can define the Clarke subdifferential of f at $\bar{x}$ by

$$\partial_c f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N_c(\operatorname{epi} f, (\bar{x}, f(\bar{x})))\}.$$

We define also the singular subdifferential of Clarke by

$$\partial_c^\infty f(\bar{x}) := \{x^* \in X^* \mid (x^*, 0) \in N_c(\operatorname{epi} f, (\bar{x}, f(\bar{x})))\}.$$

As usual, we put $\partial_c f(\bar{x}) = \emptyset = \partial_c^\infty f(\bar{x})$ when $f(\bar{x})$ is not finite. Contrary to limiting subdifferential, the Clarke subdifferential can be found through its suport function see Rockafellar(1980) and Clarke(1983). For each vector $v \in X$ and $\bar{x} \in \operatorname{dom} f$, we consider the general directional derivative of Rockafellar at the direction v defined by

$$d^\uparrow(\bar{x}; v) := \limsup_{x \to_f \bar{x}} \inf_{v' \to v} t^{-1}[f(x + tv') - f(x)],$$

with

$$\limsup_{x \to_f \bar{x}} \inf_{v' \to v} \psi(x, v') = \inf_{\varepsilon > 0} \sup_{\eta > 0} \inf_{\substack{\|x - \bar{x}\| < \varepsilon \\ |f(x) - f(\bar{x})| < \varepsilon}} \sup_{\|v' - v\| < \eta} \psi(x, v').$$

Rockafellar(1979) had proved that this directional derivative is a function of v which is sublinear and l.s.c and it was also proved that the Clarke subdifferential is characterize by

$$\partial_c f(\bar{x}) = \{x^* \in X^* \mid \langle x^*, v\rangle \le d^\uparrow f(\bar{x}; v) \ \forall v \in X\}.$$

Concerning the calculus rules, we start by considering the case of the composition with a linear continuous surjective mapping. Let $A : Z \to X$ a linear continuous surjective mapping and $z \in Z$ with $Az \in \operatorname{dom} f$. Then

$$\partial_c(f \circ A)(z) = A^* \partial_c f(Az). \tag{2.1}$$

2.1. **Around the Borwein property.** We need also to remind the *Borwein property* for the closed sets of a Banach space. So the closed subset C satisfies the Borwein property at $\bar{x} \in C$ (see Borwein(1982)) when there exist two reals numbers $r, s \in]0, +\infty[$, a closed convex subset W which the polar W^o is weakly locally compact and a vector $v \in X$ such that

$$C \cap B(\bar{x}, r) + [0, s](v + W) \subset C,$$

when $W^o := \{x^* \in X^* \mid \langle x^*, x\rangle \le 0, \ \forall x \in W\}$. If the dimension of X is finite, obviously all closed subset has the Borwein property at each its point. Although if C has the Borwein property at $x \in C$, then C is compactly epi-Lipschitzian at x see Borwein(1987). Thus, and if X is an Asplund space and if the set C has the Borwein property at x, then this later is normally compact at the point x.

We can extended this notion for the functions by using their epigraph. A function l.s.c $f : X \to \overline{\mathbb{R}}$ is said to have the *Borwein property* at a point $\bar{x}$ when $f(\bar{x})$ is finite if its epigraph epi f has the Borwein property at the point $(\bar{x}, f(\bar{x}))$. This can be also translated by the fact that there exist three reals $\beta \in \mathbb{R}$ and $r, s \in]0, +\infty[$, a closed convex subset W with the polar W^o is weakly locally compact and a vector $v \in X$ such that

$$\sup_{w \in W} t^{-1}[f(x + tv + tw) - f(x)] \leq \beta,$$

for all $t \in]0, s]$ et $x \in B(\bar{x}, r)$ with $|f(x) - f(\bar{x})| < r$.

3. NECESSARY OPTIMALITY CONDITIONS

Through the functions satisfying the Borwein property we can state the following result of Jourani and Thibault (1996).

Theorem 3.1. *Let X be an arbitrary Banach space and let f_i be l.s.c functions around a point $\bar{x}$ where they are all finite, for $i = 1, \cdots, n$. We suppose that all these functions satisfy the Borwein property at $\bar{x}$ possibly excepted one of them and assume that we have the following qualification condition*

$$[x_i^* \in \partial_c^\infty f_i, i = 1, \cdots, n \,/\, x_1^* + \ldots + x_n^* = 0] \Rightarrow x_1^* = \ldots = x_n^* = 0. \tag{3.1}$$

Then

$$\partial_c(f_1 + \ldots + f_n)(\bar{x}) \subset (\partial_c f_1 + \ldots + \partial_c f_n)(\bar{x}). \tag{3.2}$$

We can now start the study of the problem of Bolza type in infinite dimensional. Firstly, we establish the following fundamental result.

3.1. Basic theorem.

Theorem 3.2. *Let X be an Asplund space and let $\bar{x} \in X^{T+1}$ be a solution of $(\mathcal{P}_{det})$ with l and L_t l.s.c. Assume that the function l is normally compact at $(\bar{x}_0, \bar{x}_T)$ and that L_t is normally compact at $(\bar{x}_{t-1}, \Delta\bar{x}_t)$ for all $t \in 1, \cdots, T$. We also suppose that the following qualification condition*

$Q(\bar{x})$ holds:

The only vector $y^* = (y_0^*, \cdots, y_T^*) \in (X^*)^{T+1}$ *for which* $(y_0^*, -y_T^*) \in \partial^\infty l(\bar{x}_0, \bar{x}_T)$ *and* $(\Delta y_t^*, y_t^*) \in \partial^\infty L_t(\bar{x}_{t-1}, \Delta \bar{x}_t)$, $\forall\, t = 1, \cdots, T$ *is the zero vector in* $(X^*)^{T+1}$.

Then there exists a vector $p^* = (p_0^*, \cdots, p_T^*) \in (\mathbb{X}^*)^{T+1}$ *such that:*
a) $(p_0^*, -p_T^*) \in \partial l(\bar{x}_0, \bar{x}_T)$
b) $(\Delta p_t^*, p_t^*) \in \partial L_t(\bar{x}_{t-1}, \Delta \bar{x}_t)$ *for all* $t = 1, \cdots, T$.

Proof. We follow the same procedures of the finite dimensional case as in Sahraoui-Thibault (2008) introducing good number of modifications and some new arguments. We will also make call at new results.

Step 1. Consider the function $\varphi : X^{T+1} \longrightarrow \bar{\mathbb{R}}$

$$x \mapsto \varphi(x) := l(x_0, x_T) + \sum_{t=1}^{T} L_t(x_{t-1}, \Delta x_t),$$

and put

$$\varphi_0(x) := l(x_0, x_t) = (l \circ A_0)(x),$$

and

$$\varphi_t(x) := L_t(x_{t-1}, \Delta x_t) = (L_t \circ A_t)(x), \quad \text{for } t = 1, \cdots, T,$$

where $A_0, A_t : X^{T+1} \longrightarrow X^2$ are the linear continuous and surjective mappings defined by:

$$A_0 x := (x_0, x_T) \text{ and } A_t x := (x_{t-1}, \Delta x_t) \text{ for all } t = 1, \cdots, T.$$

As $\bar{x}$ is a solution of the minimization problem $(\mathcal{P}_{det})$, the point $\bar{x}$ is a minimum of the function φ and so

$$0 \in \partial \varphi(\bar{x}) = \partial(\varphi_0 + \sum_{t=1}^{T} \varphi_t)(\bar{x}).$$

Step 2. Prove now the qualification condition in terms of singular subdifferentials holds for the functions $\varphi_0, \varphi_1, \ldots, \varphi_T$.
In effect we will prove that, for all $y^* = (y_0^*, \ldots, y_T^*) \in (X^*)^{T+1}$, such that

$$\sum_{t=0}^{T} y_t^* = 0 \text{ with } y_t^* \in \partial^\infty \varphi_t(\bar{x}) \text{ for all } t = 0, \cdots, T,$$

we have necessary $y^* = 0$.

First we fix an arbitrary y^*. As the linear continuous mappings A_t are surjective for all $t = 0, \cdots, T$, we have

$$y_0^* \in \partial^\infty \varphi_0(\bar{x}) = \partial^\infty (l \circ A_0)(\bar{x}) \subset A_0^* \partial^\infty l(A_0 \bar{x}),$$

and

$$y_t^* \in \partial^\infty \varphi_t(\bar{x}) = \partial^\infty (L_t \circ A_t)(\bar{x}) \subset A_t^* \partial^\infty L_t(A_t \bar{x}), \quad \forall\, t = 1, \cdots, T,$$

so there exists some:

$$z_0^* = (z_0^1, z_0^2) \in \partial^\infty l(\bar{x}_0, \bar{x}_T) \,\text{such that}\, y_0^* = A_0^* z_0^* \tag{3.3}$$

and some

(3.4)

$$z_t^* = (z_t^1, z_T^2) \in \partial^\infty L_t(\bar{x}_{t-1}, \Delta\bar{x}_t) \,\text{such that}\, y_t^* = A_t^* z_t^* \quad \text{for } t = 1, \cdots, T.$$

Now we must calculate A_0^* et A_t^* for all $t = 1, \cdots, T$.
We have $A_t^* : (X^*)^2 \longrightarrow (X^*)^{T+1},\ t = 1, \cdots, T$ and

$$\begin{aligned}\langle A_0^*(z_1^*, z_2^*), h\rangle_{(X^*)^{T+1}} &= \langle (z_1^*, z_2^*), A_0 h\rangle_{(X^*)^2} = \langle (z_1^*, z_2^*), (h_0, h_T)\rangle_{(X^*)^2} \\ &= \langle (z_1^*, 0, \ldots, 0, z_2^*), (h_0, h_1, \cdots, h_T)\rangle_{(X^*)^{T+1}} .\end{aligned}$$

Then

$$A_0^*(z_1^*, z_2^*) = (z_1^*, 0, \cdots, 0, z_2^*) \quad \text{for all } (z_1^*, z_2^*) \in (X^*)^2$$

Following the same procedures for A_1^* we have

$$\begin{aligned}\langle A_1^*(z_1^*, z_2^*), h\rangle_{(X^*)^{T+1}} &= \langle (z_1^*, z_2^*), A_1 h\rangle_{(X^*)^2} = \langle (z_1^*, z_2^*), (h_0, h_1 - h_0)\rangle_{(X^*)^2} \\ &= \langle (z_1^* - z_2^*, z_2^*, 0, \cdots, 0), (h_0, h_1, \cdots, h_T)\rangle_{(X^*)^{T+1}} ,\end{aligned}$$

so

$$A_1^*(z_1^*, z_2^*) = (z_1^* - z_2^*, z_2^*, 0, \cdots, 0) \,\text{for all}\, (z_1^*, z_2^*) \in (X^*)^2.$$

And we can write:

$$A_t^*(z_1^*, z_2^*) = (0, \cdots, 0, z_1^* - z_2^*, z_2^*, 0, \cdots, 0) \,\text{for all}\, t = 1, \cdots, T.$$

Consequently, by the relations (3.3) and (3.4) we have

$$\begin{aligned} y_0^* &= (z_{0,1}^*, 0, \cdots, 0, z_{0,2}^*) \\ y_1^* &= (z_{1,1}^* - z_{1,2}^*, z_{1,2}^*, 0, \cdots, 0) \\ &\vdots \\ y_t^* &= (0, \cdots, 0, z_{t,1}^* - z_{t,2}^*, z_{t,2}^*, 0, \cdots, 0) \\ y_T^* &= (0, \cdots, 0, z_{T,1}^* - z_{T,2}^*, z_{T,2}^*). \end{aligned}$$

As $\sum_{t=0}^T y_t^* = 0$, we obtain

$$\begin{aligned} &(a) \quad z^*0, 1 + z_{1,1}^* - z_{1,2}^* = 0 \\ &(b) \quad z_{t-1,2}^* + z_{t,1}^* - z_{t,2}^* = 0 \, for \, t = 2, \cdots, T-1 \\ &(c) \quad z_{0,2}^* + z_{T,2}^* = 0. \end{aligned}$$

We put: $q_0^* = z_{0,1}^*$ and $q_t^* = z_{t,2}^*$ for all $t = 1, \cdots, T$. Then for all $t = 2, \cdots, T-1$ we have $\Delta q_t^* = q_t^* - q_{t-1}^* = z_{t,2}^* - z_{t-1,2}^*$ and so from the equality (b) we have $\Delta q_t^* = z_{t,1}^*$, from the equality (a) we have $\Delta q_1^* = q_1^* - q_0^* = z_{1,2}^* - z_{0,1}^* = z_{1,1}^*$ and from the equality (c) we also have $q_T^* = z_{T,2}^* = -z_{0,2}^*$. If we substitute in the relations (3.3) and (3.4), we obtain

$$(q_0^*, -q_T^*) \in \partial^\infty l(\bar{x}_0, \bar{x}_T) \text{ and } (\Delta q_t^*, q_t^*) \in \partial^\infty L_t(\bar{x}_{t-1}, \Delta\bar{x}_t),\ \forall t = 1, \cdots, T.$$

According to the qualification conditions $Q(\bar{x})$ that we have assumed, we see that $q_0^* = q_1^* = \cdots = q_T^* = 0$, and then $0 = q_0^* = z_{0,1}^*; 0 = q_t^* = z_{t,2}^*; 0 = \Delta q_t^* = z_{t,1}^* \,\text{for all}\, t = 1, \cdots, T; 0 = q_T^* = -z_{0,2}^*$. This yields $z_{0,1}^* = z_{0,2}^* = z_{t,1}^* = z_{t,2}^* = 0$ for all $t = 1, \cdots, T$ and hence

$$y_0^* = A^* z_0^* = 0 \text{ and } y_t^* = A^* z_t^* = 0 \text{ for all } t = 1, \cdots, T,$$

which means

$$y^* = (y_0^*, y_1^*, \cdots, y_T^*) = 0.$$

Etape 3 Through the surjectively of the linear continuous mapping A_t and the normally compact property of l and L_t, we verify that the functions φ_t are normally compact at $\bar{x}$. As these functions are l.s.c over the Asplund space X^{T+1} for all $t = 0, \cdots, T$ and as the qualification condition of the Step 2 above holds, we have by the calculus rules of subdifferential of sum as in Mordukhovich-Shao (1996) and Sahraoui-Thibault (2008)

$$\partial(\varphi_0 + \sum_{t=1}^{T} \varphi_t)(\bar{x}) \subset \partial\varphi_0(\bar{x}) + \sum_{t=1}^{T} \partial\varphi_t(\bar{x}),$$

which gives

$$0 \in \partial(l \circ A_0)(\bar{x}) + \sum_{t=1}^{T} \partial(L_t \circ A_t)(\bar{x}).$$

This ensures the existence of $\xi_0 \in \partial(l \circ A_0)(\bar{x})$ and $\xi_t \in \partial(L_t \circ A_t)(\bar{x})$ for all $t = 1, \cdots, T$ such that $\sum_{t=0}^{T} \xi_t = 0$. As the linear mappings A_t are continuous and surjective and that the functions l and L_t are l.s.c for all $t = 1, \cdots, T$, according to the calculus rule of subdifferential of composition function as in Mordukhovich-Shao (1996) and Sahraoui-Thibault (2008) we have

$$\partial(l \circ A_0)(\bar{x}) \subset A_0^* \partial l(A_0 \bar{x}) \text{ and } \partial(L_t \circ A_t)(\bar{x}) \subset A_t^* \partial L_t(A_t \bar{x}),\ \forall t = 1, \cdots, T.$$

Then $\xi_0 \in A_0^*\partial l(\bar{x}_0, \bar{x}_T)$ and $\xi_t \in A_t^*\partial l(\bar{x}_{t-1}, \Delta\bar{x}_t)$ for all $t = 1, \cdots, T$, which ensures the existence of some $u_0^* \in \partial l(\bar{x}_0, \bar{x}_T)$ such that $\xi_0 = A_0^* u_0^*$ and some $u_t^* \in \partial L_t(\bar{x}_{t-1}, \Delta\bar{x}_t)$ such that $\xi_t = A_t^* u_t^*$ for all $t = 1, \cdots, T$. This can be translated in the form

$$u_0^* = (u_{0,1}^*, u_{0,2}^*) \in \partial l(\bar{x}_0, \bar{x}_T) \text{ and } \xi_0 = (u_{0,1}^*, 0, \cdots, 0, u_{0,2}^*) \tag{3.5}$$

and for all $t = 1, \cdots, T$

$$u_t^* = (u_{t,1}^*, u_{t,2}^*) \in \partial L_t(\bar{x}_{t-1}, \Delta\bar{x}_t) \text{ and } \xi_t = (0, \cdots, 0, u_{t,1}^* - u_{t,2}^*, u_{t,2}^*, 0, \cdots, 0). \tag{3.6}$$

Putting $p_0^* = u_{0,1}^*$ and $p_t^* = u_{t,2}^*$ for all $t = 1, \cdots, T$. We see that

$$\begin{aligned} 0 = \textstyle\sum_{t=0}^{T} \xi_t = (\quad & u_{0,1}^* + u_{1,1}^* - u_{1,2}^*, u_{1,2}^* + u_{2,1}^* - u_{2,2}^*, \cdots, \\ & u_{t-1,2}^* + u_{t,1}^* - u_{t,2}^*, \cdots, u_{T-1,2}^* + u_{T,1}^* - u_{T,2}^*, u_{0,2}^* + u_{T,2}^*), \end{aligned}$$

which gives $u_{0,1}^* + u_{1,1}^* - u_{1,2}^* = 0$ for the first component, $t = 2, \cdots, T$ and $u_{t-1,2}^* + u_{t,1}^* - u_{t,2}^* = 0$, finally $u_{0,2}^* + u_{T,2}^* = 0$. Then

$$\Delta p_t^* = p_t^* - p_{t-1}^* = u_{t,2}^* - u_{t-1,2}^* = u_{t,1}^*, \ \forall t = 2, \cdots, T$$

and for $t = 1$ we have $\Delta p_1^* = p_1^* - p_0^* = u_{1,2}^* - u_{0,1}^* = u_{1,1}^*$. Also we have $p_T^* = u_T^2 = -u_{0,2}^*$, then $u_{0,2}^* = -p_T^*$. Finally, if we replace in (3.5) and (3.6) we obtain the existence of some vector $p^* = (p_0^*, p_1^*, \cdots, p_T^*) \in (X^*)^{T+1}$ such that $(p_0^*, -p_T^*) \in \partial l(\bar{x}_0, \bar{x}_T)$ and

$$(\Delta p_t^*, p_t^*) \in \partial L_T(\bar{x}_{t-1}, \Delta\bar{x}_t), \quad \forall\, t = 1, \cdots, T.$$

This completes the proof of the theorem. □

In the case of an arbitrary Banach space, using the results recalled above for the subdifferential of Clarke and take again the appropriate modifications in the procedure of the above theorem, we have the following result.

Theorem 3.3. *Let X be an arbitrary Banach space and let $\bar{x} \in X^{T+1}$ be a solution of $(\mathcal{P}_{det})$ with l and L_t l.s.c.. Assume that l satisfy the Borwein property at $(\bar{x}_0, \bar{x}_T)$ and that L_t satisfy the Borwein property at $(\bar{x}_{t-1}, \Delta\bar{x}_t)$ for all $t \in 1, \cdots, T$. Also we suppose that the following qualification condition*
$Q(\bar{x})$ holds:

The only vector $y^* = (y_0^*, \cdots, y_T^*) \in (X^*)^{T+1}$ for which $(y_0^*, -y_T^*) \in \partial_c^\infty l(\bar{x}_0, \bar{x}_T)$ and $(\Delta y_t^*, y_t^*) \in \partial_c^\infty L_t(\bar{x}_{t-1}, \Delta\bar{x}_t)$, $\forall\, t = 1, \cdots, T$ is the zero vector in $(X^*)^{T+1}$.

Then there exists some vector $p^* = (p_0^*, \cdots, p_T^*) \in (X^*)^{T+1}$ *such that:*
a) $(p_0^*, -p_T^*) \in \partial_c l(\bar{x}_0, \bar{x}_T)$
b) $(\Delta p_t^*, p_t^*) \in \partial_c L_t(\bar{x}_{t-1}, \Delta\bar{x}_t)$ *for all* $t = 1, \cdots, T$.

Now we take again the discrete $(\mathcal{P}_{C,F}(l, L))$ which consist to suppose that the functions l and L_t are locally Lipschitzian and to minimise the above function j under the explicit constraints

$$(x_O, x_T) \in C \quad \text{and } \Delta x_t \in F_t(x_{t-1}),\ \forall\, t = 1, \cdots, T,$$

where C is a nonempty closed subset of $X \times X$ and any F_t is a multifunction of X into X has a closed graph of $X \times X$.

Corollary 3.4. *Let* $\bar{x} \in X^{T+1}$ *be a solution of the problem* $(\mathcal{P}_{C,F}(l, L)$. *We assume that the functions* l *and* L_t *are Lipschitzian respectively around* $(\bar{x}_0, \bar{x}_T)$ *and* $(\bar{x}_{t-1}, \Delta\bar{x}_t)$ *for all* $t = 1, \cdots, T$, *that the subsets* C *and* $\operatorname{gph} F_t$ *are closed in* $X \times X$, *and that the following qualification condition* $\tilde{Q}(\bar{x})$ *holds:*

The only vector $y^* = (y_0^*, \cdots, y_T^*) \in (X^*)^{T+1}$ *for which* $(y_0^*, -y_T^*) \in N_C(\bar{x}_0, \bar{x}_T)$ *and* $(\Delta y_t^*, y_t^*) \in N_{gphF_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)$, $\forall\, t = 1, \cdots, T$ *is the zero vector in* $(X^*)^{T+1}$.

Then there exists some vector $p^* = (p_0^*, \cdots, p_T^*) \in (X^*)^{T+1}$ *such that:*
a)$(p_0^*, -p_T^*) \in \partial l(\bar{x}_0, \bar{x}_T) + N_c(\bar{x}_0, \bar{x}_T)$
b)$(\Delta p_t^*, p_t^*) \in \partial L_t(\bar{x}_{t-1}, \Delta\bar{x}_t) + N_{gphF_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)$ *for all* $t = 1, \cdots, T$.

Proof. Putting $S_t = gphF_t, \forall\, t = 1, \cdots, T$, we consider the functions

$$\tilde{l}(x_0, x_T) = l(x_0\, x_T) + \delta_C(x_0, x_T)$$

and

$$\tilde{L}_t(x_{t-1}\, \Delta x_t) = L_t(x_{t-1}\, \Delta x_t) + \delta_{S_t}(x_{t-1}, \Delta x_t),$$

where $S_t := gphF_t$ for all $t = 1, \cdots, T$, and we remark that they are l.s.c. Also we can verify that, by our hypothesis, they are normally compact at $(\bar{x}_0, \bar{x}_T)$ and $(\bar{x}_{t-1}, \Delta\bar{x}_t)$ respectively.

Now we show that the following qualification condition $Q(\bar{x})$ of the Theorem 3.2 holds for the functions $\tilde{l}$ et $\tilde{L}_t$ for all $t = 1, \cdots, T$. So let a vector $y^* \in (X^*)^{T+1}$ such that

$$(y_0^*, -y_T^*) \in \partial^\infty \tilde{l}(\bar{x}_0, \bar{x}_T) \text{ and } (\Delta y_t^*, y_t^*) \in \partial^\infty \tilde{L}_t(\bar{x}_{t-1}, \Delta\bar{x}_t),\ \forall\, t = 1, \cdots, T.$$

As l and L_t are locally Lipschitzian around $(\bar{x}_0, \bar{x}_T)$ and $(\bar{x}_{t-1}, \Delta\bar{x}_t)$ for all $t = 1, \cdots, T$, we have on the one hand

$$(y_0^*, -y_T^*) \in \partial^\infty \tilde{l}(\bar{x}_0, \bar{x}_T) \subset \partial^\infty \delta_C(\bar{x}_0, \bar{x}_T)$$

and on the other hand

$$(\Delta y_t^*, y_t^*) \in \partial^\infty \tilde{L}_t(\bar{x}_{t-1}, \Delta\bar{x}_t) \subset \partial^\infty \delta_{\operatorname{gph} F_t}(\bar{x}_{t-1}, \Delta\bar{x}_t), \forall\, t = 1, \cdots, T.$$

By the qualification condition $\tilde{Q}(\bar{x})$ we have $y_0^* = y_1^* = \cdots = y_T^* = 0$. As $\tilde{l}$ et $\tilde{L}_t$ are l.s.c. for all $t = 1, \cdots, T$ and normally compact at the necessary points and as the qualification condition $Q(\bar{x})$ relative at the problem associated with l and L_t holds, we may apply Theorem 3.2 to obtain some vector $p^* = (p_0^*, \cdots, p_T^*) \in (X^*)^{T+1}$ such that

$$(p_0^*, -p_T^*) \in \partial\tilde{l}(\bar{x}_0, \bar{x}_T) \text{ and } (\Delta p_t^*, p_t^*) \in \partial\tilde{L}_t(\bar{x}_{t-1}, \Delta\bar{x}_t),\ \forall\, t = 1, \cdots, T.$$

As l, L_t are locally Lipschitzian functions satisfying the above property for all $t = 1, \cdots, T$, and according to the calculus rule of subdifferential of sum functions see Mordukhovich-Shao (1996) and Sahraoui-Thibault (2008).

$$\partial\tilde{l}(\bar{x_0}, \bar{x_T}) \subset \partial l(\bar{x_0}, \bar{x_T}) + \partial\delta_C(\bar{x_0}, \bar{x_T})$$

and

$$\partial\tilde{L}_t(x_{\bar{t-1}}, \Delta\bar{x}_t) \subset \partial L_t(x_{\bar{t-1}}, \Delta\bar{x}_t) + \partial\delta_{S_t}(x_{\bar{t-1}}, \Delta\bar{x}_t),\ \forall\, t = 1, \cdots, T.$$

So we conclude that
a) $(p_0^*, -p_T^*) \in \partial l(\bar{x}_0, \bar{x}_T) + N_C(\bar{x}_0, \bar{x}_T)$ and
b) $(\Delta p_t^*, p_t^*) \in \partial L_t(\bar{x}_{t-1}, \Delta\bar{x}_t) + N_{\operatorname{gph} F_t}(\bar{x}_{t-1}, \Delta\bar{x}_t),\ \forall\, t = 1, \cdots, T.$ □

Remark 3.5. A similar corollary has also place in the context of an arbitrary Banach space. We leave the care to the reader to formulate it.

Now we study the case where the function l can be dissociated in a locally Lipschitzian function of x_T only through a constraint on x_0. So we consider a nonempty closed subset C_0 of X and the minimization problem $(\mathcal{P}_{C_0,F}(g, L))$ where the objective is to minimize the function

$$x \mapsto g(x_T) + \sum_{t=1}^{T} L_t(x_{t-1}, \Delta x_t)$$

under the initial constraint $x_0 \in C_0$ and the inclusion constraints $\Delta x_t \in F_t(x_{t-1})$ for $t = 1, \cdots, T$.

Corollary 3.6. *Let X be an Asplund space and let $\bar{x} \in X^{T+1}$ be a solution of the problem $(\mathcal{P}_{C_0,F}(g,L))$. We assume that the functions g and L_t are locally Lipschitzian around $\bar{x}_0$ and $(\bar{x}_{t-1}, \Delta\bar{x}_t)$ respectively for all $t = 1, \cdots, T$, that the subset C_0 is closed in X, normally compact at $\bar{x}_0$ and that the subsets* $\operatorname{gph} F_t$ *are closed in $X \times X$ and normally compacts at $(\bar{x}_{t-1}, \Delta\bar{x}_t)$. We also suppose that the following qualification condition $\widehat{Q}(\bar{x})$ holds:*

The only vector $y^* = (y_0^*, \cdots, y_T^*) \in (X^*)^{T+1}$ for which $y_0^* \in N_{C_0}(\bar{x}_0)$, $y_T^* = 0$ and $(\Delta y_t^*, y_t^*) \in N_{gphF_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)\, \forall\, t = 1, \cdots, T$ is the zero vector in $(X^*)^{T+1}$.

Then there exists a vector $p* = (p_0^*, \cdots, p_T^*) \in (X^*)^{T+1}$ such that:
a) $p_0^* \in N_{C_0}(\bar{x}_0)$, $p_T^* \in \partial g(\bar{x}_T)$
b) $(\Delta p_t^*, p_t^* \in \partial L_t(\bar{x}_{t-1}, \Delta\bar{x}_t) + N_{gphF_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)$ for all $t = 1, \cdots, T$.

Proof. Put $l(x_0, x_T) := g(x_T)$ and $C := C_0 \times X$. Then the normal cone to C is given by $N_C(\bar{x}_0, \bar{x}_T) = \mathbb{N}_{C_0}(\bar{x}_0) \times \{0\}$ and the function l is obviously Lipschitzian around $(\bar{x}_0, \bar{x}_T)$ with the equality $\partial l(\bar{x}_0, \bar{x}_T) = \{0\} \times \partial g(\bar{x}_T)$. Further, it is easy to see that the qualification condition $Q(\bar{x})$ holds. Thus, the result is a consequence of the precedent corollary. □

We also can take again the frame work relative at the case when the images of the set-valued mappings F_t are prox regular which is not studied in this paper because we estimate that the methods used in the proofs above are made the reader to know the changes which are introduce by the arguments evoked in the context of finite dimensional. We restrict our study to the problems with convex data.

Corollary 3.7. *Let X be an arbitrary Banach space. Assume that the functions l and L_t are convex (non necessarily l.s.c.) for all $i = 1, \cdots, T$ and assume also that there exists some vector $z = (z_0, \cdots, z_T) \in X^{T+1}$ such that $(z_0, z_T) \in \operatorname{dom} l$ and such that the functions L_t are continuous at $(z_{t-1}, \Delta z_t)$ for all $t \in \{1, \ldots, T\}$. Then a point $\bar{x} \in X^{T+1}$ is a solution of the problem $(\mathcal{P}_{det})$ if and only if there exists a vector $p^* = (p_0^*, \cdots, p_T^*) \in (X^*)^{T+1}$ satisfying the two relations (a) and (b) of Theorem 3.2.*

Proof. Assume that $\bar{x}$ is a minimum of the problem $(\mathcal{P}_{det})$ and consider the linear mappings A_0, A_t and the functions φ_0, φ_t of the first step of the proof of Theorem 3.2. We see that these linear mappings are continuous and surjective and that the functions φ_t, for $t = 0, 1, \cdots, T$, are convex. Since $\bar{x}$ is a minimum, we have

$$0 \in \partial(\varphi_0 + \sum_{t=1}^{T} \varphi_t)(\bar{x}).$$

Following the procedure used in the finite case as in Sahraoui and Thibault(2008), we obtain that

$$\operatorname{dom} \varphi_0 \cap (\cap_{t=1}^{T} \operatorname{cont} \varphi_t) \neq \emptyset,$$

such that $\operatorname{cont} \varphi_t$ means the set of point of X^{T+1} when the function φ_t is continuous.

This ensures that the subdifferential of the above sum is equal to the sum of subdifferential and so we can follow in the proof of the finite dimensional see Sahraoui and Thibault (2008). ☐

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References

[1] V. Barbu and T. Precupanu, *Convexity and optimazating in Banach space,* Editura Academies Sythoff-Noordhoff (1987).

[2] J. M. Borwein *Continuity and differentiability properties of convex operators,* Research partially supported by N.S.E.R.C, Proc. London. Math.Soc., **44** (1982), 420–444.

[3] J. M. Borwein, *Epi-Lipschitz-like sets in Banach spaces:Theorems and Examples,* Nonlinear Anal., **11** (1987), 1207–1217.

[4] F, Bernard, L. Thibault and Zlateva, *Characterizations of prox-regular sets in uniformly convex Banach spaces,* J. Convex Analysis, **13** (2006), 525–559.

[5] M. Bounkhel and L. Thibault, *On various notions of regularity of sets in nonsmooth analysis,* Nonlinear Anal., **48** (2002), 223–246.

[6] F. H. Clarke, *Optimization and nonsmooth analysis,* A. Wiley-Interscience Publication (1983).

[7] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern and P.R. Wolenski, *Nonsmooth Analysis and Control Theory,* Springer, New York (1997).

[8] F. H. Clarke, R. J. Stern and P. R. Wolenski, *Proximal smoothness and the lower-C^2 property,* J. Convex Analysis, **2** (1995), 117–144.

[9] A. Jourani and L. Thibault, *Extensions of subdifferential calculus rules in Banach space,* Canad. J. Math. **48** (1996), 834–848.

[10] B. S. Mordukhovich, *Discrete approximations and refined Euler-Lagrange conditions for nonconvex differential inclusions,* SIAM J. Control Optim., **33** (1995), 882–915.

[11] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation I and II,* Springer, New-York (2006).

[12] B. S. Mordukhovich, *Variational and Nonsmooth analysis,* Departement of Mathematics Wayne State University, Presented at the Summer School of the First ICCOPT (2004).

[13] B. S. Mordukhovich and Y. Shao, *Nonsmooth sequential analysis in Asplund spaces,* Trans. Amer. Math. Soc., **348** (1996), 1235–1280.

[14] R. A. Poliquin and R. T. Rockafellar, *Prox-regular functions in variational analysis,* Trans. Amer. Math. Soc., **348** (1996), 1805–1838.

[15] R. A. Poliquin, R. T. Rockafellar and L. Thibault, *Local differentiability of distance functions,* Trans. Amer. Math. Soc., **352** (2000), 5231–5249.

[16] R. T. Rockafellar, *Convex Analysis,* Princeton University Press, Princeton, New Jersey (1970).

[17] R. T. Rockafellar, *Directionally Lipschitzian functions and subdifferential calculus,* Proc. London Math. Soc., **39** (1979), 331–355.

[18] R. T. Rockafellar, *Generalized directional derivatives and subgradients of nonconvex functions,* Canad. J. Math., **32** (1980), 257–280.

[19] R. T. Rockafellar and J. B. Wets, *Deterministic and stochastic optimization problems of Bolza type in discrete time,* Stochastics, **10** (1983), 273–312.

[20] R. T. Rockafellar and J. B. Wets, *Variational Analysis,* Springer-Verlag, New York (2004).

[21] R. Sahraoui and L. Thibault, *Bolza type problem in discrete time,* Taiwanese J. Math., **12**(6) (2008), 1385–1400.

[22] L. Thibault, *Calcul sous-differentiel et calcul des variations en dimension infinie,* Bull. Soc. Math. France Mémoire, **60** (1979), 161–175.

[23] L. Thibault, *Sweeping process with regular and nonregular sets,* J. Differential Equations, **193** (2003), 1–26.

Rahma Sahraoui
Laboratory of Pure and Applied Mathematics,, Department of Mathematics, UMAB
University of Mostaganem, City, Algeria
Email: `Sahraouimosta@yahoo.fr`