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IDEALISTIC SOFT $\Gamma-$RINGS

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Abstract. The concept of idealistic soft $\Gamma-$rings is introduced, some properties of idealistic soft $\Gamma-$rings are given. In particular, three basic $\Gamma-$isomorphism theorems are established. Finally, the definitions of prime idealistic soft $\Gamma-$rings and fuzzy ideals of idealistic soft $\Gamma-$rings are proposed, then some theory of them is considered.

Key Words: Idealistic soft $\Gamma-$rings, $\Gamma-$homomorphisms, $\Gamma-$isomorphism theorems, Prime idealistic soft $\Gamma-$rings, Fuzzy ideals.

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1. Introduction

After the concept of soft sets has been introduced by Molodtsov [19] in 1999, soft sets theory has been extensively studied by many authors. Soft set theory has been applied to many different fields, such as function smootheness, Riemann integration, Perron integration, measurement theory, game theory, decision making. Maji et al. [17, 18] pointed out several directions for the applications of soft sets, they also studied several operations on the theory of soft sets. Chen et al. [2] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. Aktas et al. [1] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing some examples to clarify their differences.

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The algebraic structure of set theories dealing with uncertainties has been studied by some authors. Aktas et al. [1] applied the notion of soft sets to the theory of groups. Jun [7] introduced the notions of soft BCK/BCI-algebras, and then investigated their basic properties [8]. We also noticed that Feng et al. [6] have already investigated the structure of soft semirings. Ma et al. [14, 15, 16] discussed some characterizations of Γ-hemirings and Γ-rings. In [10], we have proposed the definition of soft rings, given some properties of soft rings, and established three isomorphism theorems. Recently, Öztürk, et al. [21] discussed a new view of fuzzy Γ−rings.

It is well known that the concept of fuzzy sets, introduced by Zadeh [23], has been extensively applied to many scientific fields. In 1971, Rosenfeld [22] applied the concept to the theory of groupoids and groups. In 1982, Liu [9] defined and studied fuzzy subrings as well as fuzzy ideals. Since then many papers concerning various fuzzy algebraic structures have appeared in the literature. The various constructions of fuzzy quotients rings and fuzzy isomorphisms have been investigated respectively by several researchers (see e.g. [5, 11, 12]). Recent research on hemirings and Γ−rings can be found in [3, 4, 13, 24].

In this paper, we attempt to study Γ−ring theory by using soft sets and fuzzy sets. We first introduce idealistic soft Γ−rings generated by soft sets, give some properties of idealistic soft Γ−rings and establish three basic Γ−isomorphism theorems. Consequently, we propose definitions of prime idealistic soft Γ−rings and fuzzy ideals of idealistic soft Γ−rings, then consider and derive some theory of them, respectively.

2. Preliminaries

To begin with, we let U be an initial universe and E a set of parameters. Denote the power set of U by $P(U)$ and consider $A \subset E$. Then we formulate the following definition.

Definition 2.1. [19] A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.2. [19] Let (F, A) and (G, B) be two soft sets over U. Then (F, A) is said to be a soft subset of (G, B) if

(1) $A \subset B$ and

(2) for all $x \in A$, $F(x)$ and $G(x)$ are identical approximations.

We now denote the above inclusion relationship by $(F, A)\widetilde{\subset}(G, B)$. Similarly, (F, A) is called a soft superset of (G, B) if (G, B) is a soft subset of (F, A). Denoted the above relationship by $(F, A)\widetilde{\supset}(G, B)$.

Definition 2.3. [1] The intersection of two soft sets (F, A) and (G, B) over U is the soft set (H, C), where $C = A \bigcap B$ and for all $x \in C$, either $H(x) = F(x)$ or $H(x) = G(x)$. This intersection is denoted by $(F, A)\widetilde{\bigcap}(G, B) = (H, C)$.

Definition 2.4. [1] Let (F, A) and (G, B) be two soft sets. Then we denote (F, A) and (G, B) by $(F, A)\widetilde{\bigwedge}(G, B)$. The soft set $(F, A)\widetilde{\bigwedge}(G, B)$ is defined by $(H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \bigcap H(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Definition 2.5. [1] The union of two soft sets (F, A) and (G, B) over U is the soft set (H, C), where $C = A \bigcup B$ and for all $x \in C$, we define

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B, \\ G(x) & \text{if } x \in B - A, \\ F(x) \bigcup G(x) & \text{otherwise.} \end{cases}$$

The above relationship is denoted by $(F, A)\widetilde{\bigcup}(G, B) = (H, C)$.

Definition 2.6. [3] Let S and Γ be two additive abelian groups. S is called a $\Gamma-$ring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ by $(a, \alpha, b) \mapsto a\alpha b$ satisfying the following conditions:

(1) $(a + b)\alpha c = a\alpha c + b\alpha c$,
(2) $a\alpha(b + c) = a\alpha b + a\alpha c$,
(3) $a(\alpha + \beta)b = a\alpha b + a\beta b$,
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$,

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

A left (resp., right) ideal of a $\Gamma-$ring S is a subset A of S which is an additive subgroup of S and $S\Gamma A \subset A$ (resp.,$A\Gamma S \subset A$), where $S\Gamma A = \{x\alpha y \mid x \in S, y \in A, \alpha \in \Gamma\}$. If A is both a left and a right ideal, then A is called an ideal of S.

Let S be a $\Gamma-$ring and I an ideal of S, $x + I = \{y \in S \mid x - y \in I\}$ and $S/I = \{x + I \mid x \in S\}$, define two operations by

$$(x + I) + (y + I) = (x + y) + I,$$
$$(x + I)\alpha(y + I) = (x\alpha y) + I$$

for all $x, y \in S$ and for all $\alpha \in \Gamma$.

The following propositions are similar to rings, and we omit the proofs.

Proposition 2.7. *Let S be a $\Gamma-$ring and I an ideal of S, then S/I is also a $\Gamma-$ring.*

Proposition 2.8. *Let S be a $\Gamma-$ring, I and J two ideals of S with $I \subset J$, then J/I is an ideal of S/I.*

Definition 2.9. [4] Let S and K be two $\Gamma-$rings, and f a mapping of S into K. Then f is called a $\Gamma-$homomorphism if $f(a+b) = f(a)+f(b)$ and $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in S, \alpha \in \Gamma$.

Definition 2.10. [3] A fuzzy set μ in a $\Gamma-$ring S is called a fuzzy ideal of S if, for all $x, y \in S$ and $\alpha \in \Gamma$, the following requirements are met:
(1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, and
(2) $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$.

3. Basic properties of idealistic soft $\Gamma-$rings

Throughout this section, S is a $\Gamma-$ring and A a nonempty set. Now, we use ρ to refer to an arbitrary binary relation between an element of A and an element of S. Thus, a set-valued function $F : A \to P(S)$ can be defined by $F(x) = \{y \in S \mid (x, y) \in \rho, x \in A \text{ and } y \in S\}$. Next, we introduce the concept of idealistic soft $\Gamma-$rings and derive some of their basic properties.

Definition 3.1. Let (F, A) be a soft set over S, then (F, A) is said to be an idealistic soft $\Gamma-$ring over S if $F(x)$ is an ideal of S for all $x \in A$. For the sake of convenience, we simply regard the empty set $\emptyset$ as an ideal of S.

Example 3.2. Let $S = Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ and (F, A) be a soft set over S, where $A = \{\overline{0}, \overline{1}, \overline{2}\}$ and $F : A \to P(S)$ is defined by $F(\overline{x}) = \{\overline{y} \in S \mid \overline{x}\rho\overline{y} \iff \overline{xy} \in \{\overline{0}, \overline{2}, \overline{4}\}\}$ for all $\overline{x} \in A$. Then $F(\overline{0}) = Z_6, F(\overline{1}) = \{\overline{0}, \overline{2}, \overline{4}\}, F(\overline{2}) = Z_6$. And let $\Gamma = Z$, define $\overline{x}\alpha\overline{y} = \overline{x\alpha y}$ for all $\overline{x}, \overline{y} \in S, \alpha \in \Gamma$. It is clear that S is a $\Gamma-$ring and (F, A) is an idealistic soft $\Gamma-$ring over S.

By using the definition of an idealistic soft $\Gamma-$ring, we can list below some important properties of idealistic soft $\Gamma-$rings.

Proposition 3.3. *(F, A) is an idealistic soft $\Gamma-$ring over S and if $B \subset A$, then (F, B) is an idealistic soft $\Gamma-$ring over S.*

Proof. It is straightforward.

Proposition 3.4. *Let (F, A) and (H, B) be two idealistic soft $\Gamma-$rings over S with $A \bigcap B \neq \emptyset$. Then the intersection $(F, A)\widetilde{\bigcap}(H, B)$ is an idealistic soft $\Gamma-$ring over S.*

Proof. We first observe that $(F, A)\widetilde{\bigcap}(H, B) = (U, C)$, where $C = A \bigcap B$. Now, for all $x \in C$, either $U(x) = F(x)$ is an ideal of S or $U(x) = H(x)$ is an ideal of S since (F, A) and (H, B) are idealistic soft $\Gamma-$ring over S. Thus, for all $x \in C$, either $U(x) = F(x)$ or $U(x) = H(x)$ is an ideal of S. Hence, (U, C) is an idealistic soft $\Gamma-$ring over S.

Proposition 3.5. *Let (F, A) and (H, B) be two idealistic soft $\Gamma-$rings over S. If $A \bigcap B = \emptyset$, then $(F, A)\widetilde{\bigcup}(H, B)$ is an idealistic soft $\Gamma-$ring over S.*

Proof. Clearly, $(F, A)\widetilde{\bigcup}(H, B) = (U, C)$. Since $A \bigcap B = \emptyset$, it follows that either $x \in A - B$ or $x \in B - A$ for all $x \in C$. If $x \in A - B$, then $U(x) = F(x)$ is an ideal of S and if $x \in B - A$, then $U(x) = H(x)$ is an ideal of S. Hence we deduce that $U(x)$ is an ideal of S, for all $x \in C$. Thus $(F, A)\widetilde{\bigcup}(H, B) = (U, C)$ is an idealistic soft $\Gamma-$ring over S.

Proposition 3.6. *Let (F, A) and (H, B) be two idealistic soft $\Gamma-$rings over S. Then $(F, A)\widetilde{\bigwedge}(H, B)$ is an idealistic soft $\Gamma-$ring over S.*

Proof. Clearly, $(F, A)\widetilde{\bigwedge}(H, B) = (U, A \times B)$. Since $F(\alpha)$ and $H(\beta)$ are ideals of S, $F(\alpha) \bigcap H(\beta)$ is an ideal of S for all $(\alpha, \beta) \in A \times B$. Hence $(F, A)\widetilde{\bigwedge}(H, B) = (U, A \times B)$ is an idealistic soft $\Gamma-$ring over S.

Proposition 3.7. *Let $f : S \to K$ be a $\Gamma-$epimorphism of $\Gamma-$rings. If (F, A) is an idealistic soft $\Gamma-$ring over S, then $(f(F), A)$ is an idealistic soft $\Gamma-$ring over K.*

Proof. Since (F, A) is an idealistic soft $\Gamma-$ring over S, $F(x)$ is an ideal of S for all $x \in A$. We can deduce that $f(F(x))$ is an ideal of K, for all $x \in A$. Hence, $(f(F), A)$ is an idealistic soft $\Gamma-$ring over K.

Definition 3.8. Let (F, A) be an idealistic soft $\Gamma-$ring over S and (H, B) a soft set over S with $B \subset A$. Then (H, B) is said to be a soft ideal of (F, A) if $H(x)$ is an ideal of $F(x)$ for all $x \in B$.

Proposition 3.9. *Let (F, A) be an idealistic soft $\Gamma-$ring over S and $\{(H_i, K_i) \mid i \in I\}$ a family of soft ideals of (F, A). Then the following properties hold:*

(1) If $\bigcap_{i \in I} K_i \neq \emptyset$, then $\widetilde{\bigcap}_{i \in I}(H_i, K_i)$ is a soft ideal of (F, A).

(2) $\widetilde{\bigwedge}_{i\in I}(H_i, K_i)$ *is a soft ideal of* $\widetilde{\bigwedge}_{i\in I}(F, A)$*.*

(3) If $K_i \bigcap K_j = \emptyset$ *for all* $i, j \in I$*,* $i \neq j$*, then* $\widetilde{\bigcup}_{i\in I}(H_i, K_i)$ *is a soft ideal of* (F, A)*.*

Proof. The proofs follow by Definition 3.8 and Proposition 3.4, 3.5, 3.6.

4. Γ−isomorphism theorems for idealistic soft Γ−rings

Definition 4.1. Let (F, A) and (H, B) be idealistic soft Γ−rings over S and K, respectively. If $f : S \to K$ and $g : A \to B$ are two functions, then (f, g) is called a Γ−homomorphism such that (f, g) is a Γ−homomorphism from (F, A) to (H, B). The latter is written by $(F, A) \sim (H, B)$ if the following conditions are satisfied:
(1) f is a Γ−epimorphism from S to K,
(2) g is a surjective mapping, and
(3) $f(F(x)) = H(g(x))$ for all $x \in A$.

In the above definition, if f is a Γ−isomorphism from S to K and g is a bijective mapping, then (f, g) is called a Γ−isomorphism so that (f, g) is a Γ−isomorphism from (F, A) to (H, B), denoted by $(F, A) \simeq (H, B)$.

Example 4.2. Let $S = Z$, $K = Z_m$ and $\Gamma = Z$, define $\overline{x}\alpha\overline{y} = \overline{x\alpha y}$ for all $\overline{x}, \overline{y} \in K, \alpha \in \Gamma$, let $x\alpha y$ be usual product for all $x, y \in S, \alpha \in \Gamma$. Clearly, S and K are both Γ−rings. Define a Γ−homomorphism f from Z onto Z_m by $f(k) = \overline{k}$ for $k \in Z$, and a mapping g from Z^+ onto Z_m by $g(k) = \overline{k}$, for $k \in Z^+$. Let $F : Z^+ \to P(Z)$ and $F(x) = \{5kx \mid k \in Z\}$, $H : Z_m \to P(Z_m)$ and $H(\overline{u}) = \{\overline{uk} \mid k \in 5Z\}$. It is clear that (F, Z^+) and (H, Z_m) are idealistic soft Γ−rings over Z and Z_m, respectively. Since $f(F(x)) = \{\overline{5xk} \mid k \in Z\}$ and $H(g(x)) = \{\overline{xs} \mid s \in 5Z\}$, we can immediately deduce that $f(F(x)) = H(g(x))$. Hence (f, g) is a Γ−homomorphism from (F, Z^+) to (H, Z_m).

By using the definition of soft Γ−homomorphisms, we can further derive some properties of soft Γ−homomorphisms having similar properties as the classical homomorphisms.

Proposition 4.3. *Let* (F, A) *be an idealistic soft* Γ−*ring over* S *and* (H, B) *a soft set over a* Γ−*ring* K*. If* (f, g) *is a* Γ−*homomorphism from* (F, A) *to* (H, B)*, then* (H, B) *is also an idealistic soft* Γ−*ring over* K*.*

Proof. Since (f, g) is a Γ−homomorphism from (F, A) to (H, B) and (F, A) is an idealistic soft Γ−ring over S, then, $f(S) = K$ is a Γ−ring, and for all $y \in B$, there exists $x \in A$ such that $g(x) = y$. This shows that $H(y) = H(g(x)) = f(F(x))$ is an ideal of K. Consequently, (H, B) is an idealistic soft $\Gamma-$ring over K.

Proposition 4.4. *Let (F, A) and (H, B) be idealistic soft $\Gamma-$rings over S and K, respectively. If (f, g) is a $\Gamma-$homomorphism from (F, A) to (H, B) and (F_1, A_1) is a soft ideal of (F, A), then $(H, g(A_1))$ is a soft ideal of (H, B).*

Proof. Since (f, g) is a $\Gamma-$homomorphism from (F, A) to (H, B) and (F_1, A_1) is a soft ideal of (F, A), we have $g(A_1) \subset B$ and $H(y)$ is a soft ideal of K for all $y \in g(A_1)$. This shows that $(H, g(A_1))$ is a soft ideal of (H, B).

Next, we establish the following three $\Gamma-$isomorphism theorems for idealistic soft $\Gamma-$rings.

Theorem 4.5. *(First $\Gamma-$isomorphism Theorem) Let (F, A) and (H, B) be idealistic soft $\Gamma-$rings over S and K, respectively. If (f, g) is a $\Gamma-$homomorphism from (F, A) to (H, B) and $F(x) \supset kerf$ for all $x \in A$. Then the following conditions hold:*

(1) $(I, A) \simeq (J, A)$, where $I(x) = F(x)/kerf$, $J(x) = f(F(x))$, $x \in A$.

(2) If g is a bijective mapping, then $(I, A) \simeq (H, B)$.

Proof. (1) It is clear that $kerf$ is an ideal of S so that $S/kerf$ is a $\Gamma-$ring, and whence, $kerf$ is an ideal of $F(x)$ and $F(x)/kerf$ is also a $\Gamma-$ring, for all $x \in A$. However, $F(x)/kerf$ is always an ideal of $S/kerf$. This shows that (I, A) is an idealistic soft $\Gamma-$ring over $S/kerf$. For all $x \in A$, we can easily see that $J(x) = f(F(x)) = H(g(x))$ is an ideal of K and hence, (J, A) is an idealistic soft $\Gamma-$ring over K.

Define $\overline{f} : S/kerf \to K$ by $\overline{f}(a + kerf) = f(a)$, for all $a \in S$. Clearly, $\overline{f}$ is a bijective mapping from $S/kerf$ to K. For all $a, b \in S, \alpha \in \Gamma$. $\overline{f}((a + kerf) + (b + kerf)) = \overline{f}((a + b) + kerf) = f(a + b) = \overline{f}(a + kerf) + \overline{f}(b + kerf)$, $\overline{f}((a + kerf)\alpha(b + kerf)) = \overline{f}((a\alpha b) + kerf) = f(a\alpha b) = \overline{f}(a + kerf)\alpha\overline{f}(b + kerf)$. Then $\overline{f}$ is a $\Gamma-$isomorphism.

Let $\overline{g} : A \to A$ be defined by $\overline{g}(x) = x$. Then $\overline{g}$ is a bijective mapping from A onto A. Also, we have $\overline{f}(I(x)) = \overline{f}(F(x)/kerf) = f(F(x)) = J(x) = J(\overline{g}(x))$, and hence, $(\overline{f}, \overline{g})$ is a $\Gamma-$isomorphism so that $(I, A) \simeq (J, A)$.

(2) Define $\overline{f} : S/kerf \to K$ by $\overline{f}(r + kerf) = f(r)$. Then $\overline{f}$ is a $\Gamma-$isomorphism from $S/kerf$ to K. Since g is a bijective mapping and $\overline{f}(I(x)) = \overline{f}(F(x)/kerf) = f(F(x)) = H(g(x))$, we have $(I, A) \simeq (H, B)$.

Theorem 4.6. *Let (F, A) be an idealistic soft $\Gamma-$ring over S. If (H, B) and (I, C) are ideals of (F, A), then*

$(P, B) \sim (Q, B)$ and $(S, C) \sim (T, C)$,

where $P(x) = H(x)/(M \bigcap N)$, $Q(x) = (H(x) + N)/N$, $S(x) = I(x)/(M \bigcap N)$, $T(x) = (I(x) + M)/M$, $M = \bigcap_{x \in B} H(x)$ and $N = \bigcap_{x \in C} I(x)$.

Proof. We first write $K = \langle \bigcup_{x \in B} H(x) \rangle$ and $L = \langle \bigcup_{x \in C} I(x) \rangle$. Then, $M = \bigcap_{x \in B} H(x)$ is an ideal of S. It is clear that M is also an ideal of K so that $M \bigcap N$ is an ideal of K and hence, (P, B) is an idealistic soft $\Gamma-$ring over $K/(M \bigcap N)$. It is trivial that (Q, B) is an idealistic soft $\Gamma-$ring over $(K + N)/N$.

Now, we define $f : K/(M \bigcap N) \to (K + N)/N$ by $f(k + (M \bigcap N)) = k + N$ and define $g : B \to B$ by $g(x) = x$. Then f is a $\Gamma-$homomorphism from $K/(M \bigcap N)$ to $(K + N)/N$, where g is a bijective mapping and $f(P(x)) = f(H(x)/(M \bigcap N)) = (H(x) + N)/N = Q(x) = Q(g(x))$. This shows that $(P, B) \sim (Q, B)$.

$(S, C) \sim (T, C)$ can be proved similarly.

Theorem 4.7. *(Second $\Gamma-$isomorphism Theorem) Let (F, A) be an idealistic soft $\Gamma-$ring over S. If (H, B) and (I, C) are ideals of (F, A) such that $H(x) = M$ for all $x \in B$, then*

$(P, B) \simeq (Q, B)$,

where $P(x) = H(x)/(M \bigcap N)$, $Q(x) = (H(x)+N)/N$, $N = \bigcap_{x \in C} I(x)$.

Proof. From the proof of the above theorem, we can easily prove that $(P, B) \simeq (Q, B)$, and hence omit the proof.

Theorem 4.8. *(Third $\Gamma-$isomorphism Theorem) Let (F, A) be an idealistic soft $\Gamma-$ring over S. If (H, B) and (I, C) are ideals of (F, A) with $B \bigcap C \neq \emptyset$ and $I(x) \subset H(x)$ for all $x \in B \bigcap C$, then*

$(P, B \bigcap C) \simeq (Q, B \bigcap C)$, where $P(x) = (F(x)/N)/(M/N)$, $Q(x) = F(x)/M$ with $M = \bigcap_{x \in B \bigcap C} H(x)$ and $N = \bigcap_{x \in B \bigcap C} I(x)$.

Proof. It can be easily verified that $M = \bigcap_{x \in B \bigcap C} H(x)$ and $N = \bigcap_{x \in B \bigcap C} I(x)$ are ideals of S, and N is an ideal of M. Now, it is easy to see that $(P, B \bigcap C)$ is an idealistic soft $\Gamma-$ring over $(S/N)/(M/N)$ and so $(Q, B \bigcap C)$ is an idealistic soft $\Gamma-$ring over S/M.

Define the mapping $f : (S/N)/(M/N) \to S/M$ by $f((r + N) + (M/N)) = r + M$ and define $g : B \bigcap C \to B \bigcap C$ by $g(x) = x$. Then f is a $\Gamma-$isomorphism from $(S/N)/(M/N)$ to S/M. Obviously, g is a bijective mapping and $f(P(x)) = f((F(x)/N)/(M/N)) = F(x)/M = Q(g(x))$.

Hence, $(P, B \bigcap C) \simeq (Q, B \bigcap C)$. Thus, the third $\Gamma-$isomorphism theorem is proved.

5. Prime idealistic soft $\Gamma-$rings

Definition 5.1. [4] Let S be a $\Gamma-$ring, an ideal P of S is called prime if for all pairs of ideals M and N, $M\Gamma N \subset P$ implies that $M \subset P$ or $N \subset P$.

Definition 5.2. Let (F, A) be a soft set over a $\Gamma-$ring S. Then (F, A) is called a prime idealistic soft $\Gamma-$ring over S if $F(x)$ is a prime ideal of S for all $x \in A$.

Example 5.3. Let $S = Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ and (F, A) be a soft set over S, where $A = \{\overline{1}, \overline{2}\}$ and $F : A \to P(S)$ is defined by $F(\overline{x}) = \{\overline{y} \in S \mid \overline{x}\rho\overline{y} \Longleftrightarrow \overline{xy} = \overline{0}\}$ for all $\overline{x} \in A$. Then $F(\overline{1}) = \{\overline{0}\}, F(\overline{2}) = \{\overline{0}, \overline{3}\}$. And let $\Gamma = Z$, define $\overline{x}\alpha\overline{y} = \overline{x\alpha y}$ for all $\overline{x}, \overline{y} \in S, \alpha \in \Gamma$. We can verify that S is a $\Gamma-$ring and (F, A) is a prime idealistic soft $\Gamma-$ring over S.

Theorem 5.4. *Let (F, A) be an idealistic soft $\Gamma-$ring over S and I an ideal of S with $I \subset F(x)$ for all $x \in A$. Then (F, A) is prime if and only if $(F/I, A)$ is prime.*

Proof. Clearly, $(F/I, A)$ is an idealistic soft $\Gamma-$ring over S/I. If (F, A) is prime, i.e., for all pairs of ideals M and N, $M\Gamma N \subset P$ implies that $M \subset P$ or $N \subset P$. Suppose that $(F/I, A)$ is not prime, then there exist two ideals M_0/I and N_0/I of S/I such that $(M_0/I)\Gamma(N_0/I) \subset F(x)/I$, but $M_0/I \not\subset F(x)/I$ and $N_0/I \not\subset F(x)/I$, then there exists $x+I \in M_0/I$, $x+I \notin F(x)/I$, then $x \in M_0$ but $x \notin F(x)$, then $M_0 \not\subset F(x)$. In the same way, we can get there exists $y \in N_0$ but $y \notin F(x)$. It is a contradiction. So $(F/I, A)$ is prime.

Conversely, assume that $(F/I, A)$ is prime, if (F, A) is not prime, then there exist two ideals M and N of S such that $M\Gamma N \subset F(x)$, but $M, N \not\subset F(x)$. We can deduce that $(M/I)\Gamma(N/I) \subset F(x)/I$, but $M/I, N/I \not\subset F(x)/I$, this is a contradiction.

Theorem 5.5. *Let $f : S \to K$ be a $\Gamma-$epimorphism of $\Gamma-$rings. If (F, A) is a prime idealistic soft $\Gamma-$ring over S, then $(f(F), A)$ is a prime idealistic soft $\Gamma-$ring over K.*

Proof. Since (F, A) is a prime idealistic soft $\Gamma-$ring over S, $F(x)$ is a prime ideal of S for all $x \in A$, i.e., for all pairs of ideals M and N of S, $M\Gamma N \subset F(x)$ implies that $M \subset F(x)$ or $N \subset F(x)$. Now suppose M'

and $N^{'}$ are any two ideals of K, then there exist two ideals M and N of S such that $f(M) = M^{'}$ and $f(N) = N^{'}$. If $M'\Gamma N' \subset f(F(x))$ for all $x \in A$, then $f(M\Gamma N) = f(M)\Gamma f(N) \subset f(F(x))$, $M\Gamma N \subset F(x)$, hence $M \subset F(x)$ or $N \subset F(x)$. Thus $M' \subset f(F(x))$ or $N' \subset f(F(x))$, this shows that $f(F(x))$ is a prime ideal of K, for all $x \in A$ and $(f(F), A)$ is a prime idealistic soft $\Gamma-$ring over K.

Theorem 5.6. *Let (F, A) and (H, B) be idealistic soft $\Gamma-$rings over S and K, respectively. If (F, A) is prime and $\Gamma-$homomorphic to (H, B), then (H, B) is prime.*

Proof. Let (f, g) be a $\Gamma-$homomorphism from (F, A) to (H, B). For any $y \in B$, there exists $x \in A$ such that $y = g(x)$, then $H(y) = H(g(x)) = f(F(x))$. For all pairs of ideals $M^{'}$ and $N^{'}$ of K, there exist two ideals M and N of S such that $f(M) = M^{'}$ and $f(N) = N^{'}$. If $M^{'}\Gamma N^{'} \subset H(y) = f(F(x))$, then $M\Gamma N \subset F(x)$, it follows that $M \subset F(x)$ or $N \subset F(x)$, then $M^{'} \subset f(F(x)) = H(g(x)) = H(y)$ or $N^{'} \subset f(F(x)) = H(g(x)) = H(y)$, so (H, B) is prime.

Theorem 5.7. *Let (F, A) and (H, B) be two prime idealistic soft $\Gamma-$rings over S.*
(1) If $A \bigcap B \neq \emptyset$, then $(F, A)\widetilde{\bigcap}(H, B)$ is a prime idealistic soft $\Gamma-$ring over S.
(2) If $A \bigcap B = \emptyset$, then $(F, A)\widetilde{\bigcup}(H, B)$ is a prime idealistic soft $\Gamma-$ring over S.
(3) $(F, A)\widetilde{\bigwedge}(H, B)$ is a prime idealistic soft $\Gamma-$ring over S.

Proof. The proof is similar to Proposition 3.4, 3.5, 3.6 and hence is omitted.

6. Fuzzy ideals of idealistic soft Γ−rings

Definition 6.1. Let (F, A) be an idealistic soft $\Gamma-$ring over S and μ a fuzzy subset in S. Then μ is said to be a fuzzy ideal of (F, A) if, for any $t \in A$, $x, y \in F(t)$ and $\alpha \in \Gamma$, the following conditions hold :
(1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, and
(2) $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$.

Remark 6.2. (1) Let (F, A) be an idealistic soft $\Gamma-$ring over S and μ a fuzzy subset in S. Then μ is a fuzzy ideal of (F, A) if and only if μ is a fuzzy ideal of $F(t)$ for any $t \in A$.
(2) Let (F, A) be an idealistic soft ring over S. If μ is a fuzzy ideal of S, then μ is a fuzzy ideal of (F, A), but the converse is not true.

Example 6.3. Let $S = Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ and (F, A) be a soft set over S, where $A = \{\overline{1}, \overline{2}\}$, and $F : A \to P(S)$ is defined by $F(\overline{x}) = \{\overline{y} \in S \mid \overline{x}\rho\overline{y} \Longleftrightarrow \overline{xy} = \overline{0}\}$ for all $\overline{x} \in A$. And let $\Gamma = Z$, define $\overline{x}\alpha\overline{y} = \overline{x\alpha y}$ for all $\overline{x}, \overline{y} \in S, \alpha \in \Gamma$. Then $F(\overline{1}) = \{\overline{0}\}$, $F(\overline{2}) = \{\overline{0}, \overline{3}\}$ and (F, A) is an idealistic soft $\Gamma-$ring over S.

Let μ be a fuzzy subset in Z_6, where μ is defined by

$$\mu(\overline{x}) = \begin{cases} 0.8, & \overline{x} \in \{\overline{0}, \overline{5}\}, \\ 0.1, & \overline{x} \in \{\overline{1}, \overline{3}\}, \\ 0.5, & \text{otherwise.} \end{cases}$$

we can verify that for any $t \in A$ and $\overline{x}, \overline{y} \in F(t)$, $\mu(\overline{x}-\overline{y}) \geq \min\{\mu(\overline{x}), \mu(\overline{y})\}$, and $\mu(\overline{x}\alpha\overline{y}) \geq \max\{\mu(\overline{x}), \mu(\overline{y})\}$, hence μ is a fuzzy ideal of (F, A). But μ is not a fuzzy ideal of S, because $\mu(\overline{5} - \overline{2}) = \mu(\overline{3}) < \min\{\mu(\overline{5}), \mu(\overline{2})\}$.

Proposition 6.4. *Let (F, A) be an idealistic soft $\Gamma-$ring over S and μ a fuzzy subset in S. Then μ is a fuzzy ideal of (F, A) if and only if, for any $x \in A$ and $t \in [0, 1]$, $U(t, x) = \{r \in F(x) \mid \mu(r) \geq t\}$ is an ideal of $F(x)$.*

Proof. It is straightforward.

Let μ be a fuzzy ideal of a $\Gamma-$ring S. For any $x, y \in S$, define a binary relation $\sim$ on S by $x \sim y$ if and only if

$$\mu(x - y) = \mu(0),$$

where 0 is the zero element of S.

Theorem 6.5. *$\sim$ is a congruence relation of S.*

Proof. (1) For any $x \in S$, $\mu(x - x) = \mu(0)$, then $x \sim x$.

(2) For any $x, y \in S$, if $x \sim y$, then $\mu(x - y) = \mu(0)$, $\mu(y - x) = \mu(0)$, hence $y \sim x$.

(3) For any $x, y, z \in S$, if $x \sim y$ and $y \sim z$, then $\mu(x - y) = \mu(0)$, $\mu(y - z) = \mu(0)$, $\mu(x - z) = \mu((x - y) - (y - z)) = \mu(0)$, i.e., $x \sim z$.

(4) For any $x, y, z \in S$, if $x \sim y$, then $\mu(x - y) = \mu(0)$, $\mu((x + z) - (y + z)) = \mu(x - y) = \mu(0)$, it means $(x + z) \sim (y + z)$.

(5) For any $x, y, u, v \in S$, if $x \sim y$ and $u \sim v$, it follows from (4) that $(x+u) \sim (y+u)$ and $(y+u) \sim (y+v)$, by (3), we obtain $(x+u) \sim (y+v)$. Similarly, we have $(x + v) \sim (y + u)$. The proof is complete.

Let $\mu_x = \{y \in S \mid y \sim x\}$ be the equivalence class containing x and $S/\mu = \{\mu_x \mid x \in S\}$ the set of all equivalence classes of S. Define two operations by

$$\mu_x + \mu_y = \mu_{x+y},$$

$$\mu_x \alpha \mu_y = \mu_{x\alpha y}$$

for $x, y \in S$, $\alpha \in \Gamma$. We can verify that the two operations are well defined and S/μ is a $\Gamma-$ring, the quotient $\Gamma-$ring of S induced by the fuzzy ideal μ.

Lemma 6.6. *Let I be an ideal of a $\Gamma-$ring S and μ a fuzzy ideal of S. We have*
(1) if μ is restricted to I, written μ_I , then μ_I is fuzzy ideal of I;
(2) I/μ_I is an ideal of S/μ.

Let μ be a fuzzy ideal of a $\Gamma-$ring S and (F, A) an idealistic soft $\Gamma-$ring over S. Now, we restrict μ to $F(x)$ for all $x \in A$, then for all $x \in A$, μ is fuzzy ideal of $F(x)$ and $F(x)/\mu$ is an ideal of S/μ. Thus, we can define a set-valued function $F/\mu : A \to P(S/\mu)$ by $(F/\mu)(x) = F(x)/\mu$.

By Lemma 6.6, we can very easily deduce the following.

Theorem 6.7. *If μ is a fuzzy ideal of a $\Gamma-$ring S and (F, A) is an idealistic soft $\Gamma-$ring over S, then*
(1) μ is a fuzzy ideal of (F, A).
(2) $(F/\mu, A)$ is an idealistic soft $\Gamma-$ring over S/μ.
The $(F/\mu, A)$ is called the quotient soft $\Gamma-$ring of (F, A) induced by the fuzzy ideal μ.

Example 6.8. Let $S = Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$, and (F, A) be a soft set over S, where $A = \{\overline{0}, \overline{1}, \overline{2}\}$ and $F : A \to P(S)$ is defined by $F(\overline{x}) = \{\overline{y} \in S \mid \overline{x}\rho\overline{y} \Longleftrightarrow \overline{xy} = \overline{0}\}$ for all $\overline{x} \in A$. And let $\Gamma = Z$, define $\overline{x}\alpha\overline{y} = \overline{x\alpha y}$ for all $\overline{x}, \overline{y} \in S, \alpha \in \Gamma$. Then $F(\overline{0}) = Z_6$, $F(\overline{1}) = \{\overline{0}\}$, $F(\overline{2}) = \{\overline{0}, \overline{3}\}$ are ideals of S and (F, A) is an idealistic soft $\Gamma-$ring over S.

A fuzzy subset μ in Z_6 is defined by

$$\mu(\overline{x}) = \begin{cases} 0.8, & \overline{x} \in \{\overline{0}, \overline{3}\}, \\ 0.2, & \text{otherwise.} \end{cases}$$

It is easy to verify that μ is a fuzzy ideal of S, then μ is a fuzzy ideal of (F, A). We can obtain that $(F/\mu)(\overline{0}) = F(\overline{0})/\mu = \{\{\overline{0}, \overline{3}\}, \{\overline{2}, \overline{5}\}, \{\overline{1}, \overline{4}\}\}$, $(F/\mu)(\overline{1}) = F(\overline{1})/\mu = \{\{\overline{0}\}\}$, $(F/\mu)(\overline{2}) = F(\overline{2})/\mu = \{\{\overline{0}, \overline{3}\}\}$. Then $(F/\mu, A)$ is the quotient soft $\Gamma-$ring of (F, A) induced by the fuzzy ideal μ.

Lemma 6.9. *If μ, ν are two fuzzy ideals of a $\Gamma-$ring S, then $\mu \cap \nu$ is also a fuzzy ideal of S.*

By Lemma 6.9, we can very easily deduce the following proposition.

Theorem 6.10. *If (F, A) is an idealistic soft $\Gamma-$ring over S and μ, ν are two fuzzy ideals of (F, A), then $\mu \cap \nu$ is also a fuzzy ideal of (F, A).*

Let μ be a fuzzy ideals of a $\Gamma-$ring S. We denote $S_\mu = \{x \in S \mid \mu(x) = \mu(0)\}$. Clearly, S_μ is an ideal of S.

Theorem 6.11. *If μ is a fuzzy ideal of a $\Gamma-$ring S and (F, A) is an idealistic soft $\Gamma-$ring over S, let $F_\mu(t) = \{x \in F(t) \mid \mu(x) = \mu(0)\}$ for all $t \in A$. Then (F_μ, A) is an idealistic soft $\Gamma-$ring over S_μ.*

Proof. The proof is trivial and is omitted.

Lemma 6.12. *Let μ be a fuzzy ideal of a $\Gamma-$ring S. Then the fuzzy subset μ^* of S/S_μ defined by $\mu^*(x + S_\mu) = \mu(x)$ for all $x \in S$, is a fuzzy ideal of S/S_μ.*

Theorem 6.13. *If μ is a fuzzy ideal of a $\Gamma-$ring S and (F, A) is an idealistic soft $\Gamma-$ring over S with $F(t) \supset S_\mu$ for all $t \in A$, then*

(1) $(F/S_\mu, A)$ is an idealistic soft $\Gamma-$ring over S/S_μ, where $(F/S_\mu)(t) = F(t)/S_\mu$ for all $t \in A$.

(2) the fuzzy subset μ^* of S/S_μ defined in Lemma 6.12, is a fuzzy ideal of the idealistic soft $\Gamma-$ring $(F/S_\mu, A)$.

Proof. (1) Since μ is a fuzzy ideal of a $\Gamma-$ring S and (F, A) is an idealistic soft $\Gamma-$ring over S with $F(t) \supset S_\mu$ for all $t \in A$, $F(t)$ is an ideal of S and S_μ is an ideal of $F(t)$, for all $t \in A$. Let $x + S_\mu, y + S_\mu \in F(t)/\mu$, where $x, y \in F(t)$, then $(x + S_\mu) - (y + S_\mu) = ((x - y) + S_\mu) \in F(t)/S_\mu$ and $(x + S_\mu)\alpha(y + S_\mu) = ((x\alpha y) + S_\mu) \in F(t)/\mu$. Then we can easily deduce that $F(t)/S_\mu$ is an ideal of S/S_μ for all $t \in A$, hence $(F/S_\mu, A)$ is an idealistic soft $\Gamma-$ring over S/S_μ.

(2) By Lemma 6.12, μ^* is a fuzzy ideal of $S/S\mu$, then μ^* is a fuzzy ideal of the idealistic soft $\Gamma-$ring $(F/S_\mu, A)$.

7. Conclusion

In this paper, we introduced idealistic soft $\Gamma-$rings based soft sets, and defined soft $\Gamma-$homomorphisms and soft $\Gamma-$isomorphisms, then focused on establishing the First, Second and Third $\Gamma-$isomorphism Theorems of idealistic soft $\Gamma-$rings respectively. Moreover, the notions of prime idealistic soft $\Gamma-$rings and their properties are proposed. Finally, fuzzy ideals of idealistic soft $\Gamma-$rings are defined and some classes of quotient $\Gamma-$rings are characterized by their fuzzy ideals.

To extend this work, one could study other algebraic structures such as modules and fields, and do some further work on the properties of them.

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