Journal of Hyperstructures 11 *(2) (2022), 214-224.*
ISSN: 2322-1666 print/2251-8436 online

ON VAGUE IDEALS IN NEAR-RINGS

PRITAM VIJAYSIGH PATIL, JANARDHAN D. YADAV*

ABSTRACT. Using Vague ideals of near-ring R, authors introduce the concepts of normal vague ideals, complete vague ideals and maximal vague ideals of near-ring R with few properties.

Key Words: Vague Ideals, Normal Vague Ideals, Maximal Vague Ideals, Complete Vague Ideal.
2010 Mathematics Subject Classification: Primary: 03E72; Secondary: 06F35, 03G25.

1. Introduction

The concept of fuzzy sets is introduced by L. A. Zadeh [3], W. Liu [13] wrote on ideals of Fuzzy sets and some authors have extended that work further. D. L. Prince Williams [1] introduced fuzzy ideals in Near-subtraction Semigroups in 2008. Concepts of fuzzy ideals are extended to vague sets by different authors [4,6,7,11]. W. L. Gau and D. J. Buehrer [12] introduced vague sets with truth membership and false membership function. R. Biswas [7] introduced Vague groups and then S. Zaid [8] studied fuzzy sub near-ring and fuzzy ideals of near-ring. In 2017 L. Bhasker [4] extended that part of fuzzy ideals in the near-ring to the Vague ideal of a near-ring. There are some required definitions are given below:

Definition 1.1. [2] A non-empty set R with two binary operations "+" and "." satisfying the following axioms:
(1) (R,+) is a group,

Received: 11 August 2021, Accepted: 20 December 2022. Communicated by Mirela Stefanescu;
∗Address correspondence to Pritam Vijaysigh Patil; E-mail: mailme.prit@rediffmail.com.

(2) (R,.) is a semigroup,
(3) $x.(y+z) = x.y + x.z$ for all $x, y, z \in R$.
Precisely speaking, it is a left near-ring because it satisfies the left distributive law. We will use "near-ring", instead of "left near-ring". We denote xy instead of x.y. Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for some $x, y \in R$. Let R and S be near-rings. A map $f : R \to S$ is called a (near-ring) homomorphism if $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$ for any $x, y \in R$. An ideal I of a near-ring R is a subset of R such that
(4)(I,+) is a normal subgroup of (R,+),
(5) $RI \subseteq I$,
(6) $(r+i)s - rs \in I$ for any $r, s \in R$
Note that I is a left ideal of R if I satisfies (4) and (5), and I is a right ideal of R if I satisfies (4) and (6).
Throughout this paper let $I = (I, +, -, \vee, \wedge, 0, 1)$ be a dually residuated lattice ordered semigroup satisfying $1 - (1 - a) = a$ for all $a \in I$.

Definition 1.2. [12] A vague set A on a non-empty set X is characterized by two membership function given by:
1. A true membership function

$$t_A : X \to [0, 1]$$

2. A false membership function

$$f_A : X \to [0, 1]$$

where $t_A(x)$ is a lower bound on the grade of membership of x derived from the "evidence for x", $f_A(x)$ is a lower bound on the negation of x derived from the "evidence against x" and $t_A(x) \leq 1 - f_A(x)$ for all $x \in X$.

Definition 1.3. [12] The interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of $x \in X$ and is denoted by $V_A(x)$.

Definition 1.4. [4] Let A be a vague set of a near-ring R. Then A is called vague sub near-ring of R if for all $x, y \in R$, it satisfies
(i) $V_A(x+y) \geq min\{V_A(x), V_A(y)\}$
(ii) $V_A(-x) = V_A(x)$
(iii) $V_A(xy) \geq min\{V_A(x), V_A(y)\}$.

Definition 1.5. [4] Let A be a vague set of near-ring R, then A is said to be a Vague ideal of R if for all $x, y, z, i \in R$, it satisfies
(i) $V_A(x+y) \geq min\{V_A(x), V_A(y)\}$

(ii) $V_A(-x) = V_A(x)$
(iii) $V_A(z + x - z) \geq V_A(x)$
(iv) $V_A(xy) \geq V_A(x)$
(v) $V_A[(x + y)z - xz] \geq V_A(y)$ or $V_A(xz - xy) \geq V_A(z - y)$.
A is said to be vague right ideal if it satisfies (i), (ii), (iii) and (iv).
And A is said to be vague left ideal if it satisfies (i), (ii), (iii) and (v).

2. Vague ideals of near-rings

Definition 2.1. Let $\{A_i/i \in I\}$ (here I is index set) be a family of vague ideals in near-ring R, then the intersection $\underset{i\in\Lambda}{\cap} A_i$ of $\{A_i/i \in I\}$ is defined by $V_{(\underset{i\in\Lambda}{\cap} A_i)}(x) = min\{V_{A_i}(x)/i \in I\}$

Theorem 2.2. *Intersection of a family of left (resp. right) vague ideals of near-ring R is left (resp. right) vague ideal of near-ring R.*
Proof. *Let Let $\{A_i/i \in I\}$ be a family of left (resp. right) vague ideals in near-ring R.*
Let $A = \underset{i\in\Lambda}{\cap} A_i$ and $\forall x, y, z, \in R$,

$$\begin{aligned}
V_A(x-y) &= min\{V_{A_i}(x-y)/i \in I\} \\
&\geq min\{min[V_{A_i}(x), V_{A_i}(y)]/i \in I\} \\
&= min\{min[V_{A_i}(x)/i \in I], min[V_{A_i}(y)/i \in I]\} \\
&= min\{V_A(x), V_A(y)\}. \\
V_A(xy) &= min\{V_{A_i}(xy)/i \in I\} \\
&\geq min\{min[V_{A_i}(x), V_{A_i}(y)]/i \in I\} \\
&= min\{min[V_{A_i}(x)/i \in I], min[V_{A_i}(y)/i \in I]\} \\
&= min\{V_A(x), V_A(y)\}. \\
V_A(y+x-y) &= min\{V_{A_i}(y+x-y)/i \in I\} \\
&\geq min\{V_{A_i}(x)/i \in I\} \\
&= V_A(x). \\
V_A(xy) &= min\{V_{A_i}(xy)/i \in I\} \\
&\geq min\{V_{A_i}(y)/i \in I\} \\
&= V_A(y).
\end{aligned}$$

$$
\begin{aligned}
V_A[(x+y)z - xz] &= min\{V_{A_i}[(x+y)z - xz]/i \in I\} \\
&\geq min\{V_{A_i}(y)/i \in I\} \\
&= V_A(y).
\end{aligned}
$$

Hence the proof.

Definition 2.3. Any vague ideal of near-ring R is said to be normal if $\exists\ a \in R$ such that $V_A(a) = 1$.

Remark 2.4. Any vague ideal is normal if and only if $V_A(0) = 1$.

Remark 2.5. Here $F_N(R)$ denotes the set of normal vague ideals of near-ring R.

Theorem 2.6. *If A^+ be a vague set in near-ring R defined by $V_{A^+}(x) = V_A(x) + 1 - V_A(0), \forall x \in R$ for any vague left (resp. right) ideal A of near-ring R, then A^+ is a normal vague ideal containing A.*

Proof. Proof. we have,$V_{A^+}(x) = V_A(x) + 1 - V_A(0), \forall x \in R$
Now, $\forall x, y, z \in R$

$$
\begin{aligned}
V_{A^+}(x-y) &= V_A(x-y) + 1 - V_A(0) \\
&\geq min\{V_A(x), V_A(y)\} + 1 - V_A(0) \\
&= min\{V_A(x) + 1 - V_A(0), V_A(y) + 1 - V_A(0)\} \\
&= min\{V_{A^+}(x), V_{A^+}(y)\} \\
V_{A^+}(xy) &= V_A(xy) + 1 - V_A(0) \\
&\geq min\{V_A(x), V_A(y)\} + 1 - V_A(0) \\
&= min\{V_A(x) + 1 - V_A(0), V_A(y) + 1 - V_A(0)\} \\
&= min\{V_{A^+}(x), V_{A^+}(y)\} \\
V_{A^+}(y+x-y) &= V_A(y+x-y) + 1 - V_A(0) \\
&\geq V_A(x) + 1 - V_A(0) \\
&= V_{A^+}(x)
\end{aligned}
$$

$$
\begin{aligned}
V_{A^+}(xy) &= V_A(xy) + 1 - V_A(0) \\
&\geq V_A(y) + 1 - V_A(0) \\
&= V_{A^+}(y). \qquad (Similarly \quad V_{A^+}(xy) \geq V_{A^+}(x)) \\
V_{A^+}[(x+y)z - xz] &= V_A[(x+y)z - xz] + 1 - V_A(0) \\
&\geq V_A(y) + 1 - V_A(0) \\
&= V_{A^+}(y)
\end{aligned}
$$

It shows A^+ is a vague left (resp. right) ideal of near-ring R.
Now, we have $V_{A^+(x)} = V_A(x) + 1 - V_A(0), \forall x \in R$
Put $x = 0$ we get, $V_{A^+}(0) = V_A(0) + 1 - V_A(0) = 1$
It shows A^+ is normal that is, $A^+ \in F_N(R)$
Also as, $V_{A^+}(x) = V_A(x) + 1 - V_A(0), \forall x \in R$ shows, $V_{A^+}(x) \geq V_A(x), \forall x \in R$
$\implies A \subseteq A^+$. Hence the proof. □

Corollary 2.7. *Let us consider a vague left (resp. right) ideal A of near-ring R satisfying $V_{A^+}(a) = 0$ for some $a \in R$ then $V_A(a) = 0$.*

Proof. Let A be vague left (resp. right) ideal of R. On contrary let us consider that $\exists\ a \in R$ such that $V_{A^+}(a) = 0 \quad \& \quad V_A(a) \neq 0$
Now, we have $V_{A^+(x)} = V_A(x) + 1 - V_A(0), \forall x \in R$
Put $x = a$ we get,
$V_{A^+(a)} = V_A(a) + 1 - V_A(0) \implies 0 = V_A(a) + 1 - V_A(0)$
$\implies V_A(0) = V_A(a) + 1 \geq 1$ which is contradiction, $\implies V_A(a) = 0$. □

Theorem 2.8. *If ϕ be an increasing function defined on $[0, V_A(0)]$ to $[0,1]$ where A is an vague left (resp. right) ideal of near-ring R. Also A_ϕ be a vague set in near-ring R such that $V_{A_\phi}(x) = \phi[V_A(x)], \forall x \in R$, then A_ϕ is a vague left (resp. right)ideal of near-ring R. Moreover if $\phi[V_A(0)] = 1$ then, A_ϕ is normal vague left (resp. right) ideal of near-ring R and if $\phi(x) \geq t, \forall\ t \in [0, V_A(0)]$ then $A \subseteq A_\phi$*

Proof. Let $x, y, z \in R$ then,

$$\begin{aligned}
V_{A_\phi}(x-y) &= \phi[V_A(x-y)] \\
&\geq \phi[min\{V_A(x), V_A(y)\}] \\
&= min[\phi\{V_A(x)\}, \phi\{V_A(y)\}] \\
&= min\{V_{A_\phi}(x), V_{A_\phi}(y)\} \\
V_{A_\phi}(xy) &= \phi[V_A(xy)] \\
&\geq \phi[min\{V_A(x), V_A(y)\}] \\
&= min[\phi\{V_A(x)\}, \phi\{V_A(y)\}] \\
&= min\{V_{A_\phi}(x), V_{A_\phi}(y)\} \\
V_{A_\phi}(y+x-y) &= \phi[V_A(y+x-y)] \\
&\geq \phi[V_A(x)] \\
&= V_{A_\phi}(x). \\
V_{A_\phi}(xy) &= \phi[V_A(xy)] \\
&\geq \phi[V_A(y)] \\
&= V_{A_\phi}(y). \\
V_{A_\phi}[(x+y)z-xz] &= \phi[V_A\{(x+y)z-xz\}] \\
&\geq \phi[V_A(y)] \\
&= V_{A_\phi}(y)
\end{aligned}$$

So A_ϕ is vague left (resp. right) ideal of R.
If $\phi[V_A(0)] = 1$ then Obivously $A_\phi \in F_N(R)$.
Let $\phi(x) \geq t, \forall t \in [0, V_A(0)]$, then $V_{A_\phi}(x)) = \phi[V_A(x)] \geq V_A(x)$
$\implies A \subseteq A_\phi, \forall a \in R$. Hence the proof. □

Theorem 2.9. *Let A be non- constant and maximal element of poset $(F_N(R) \subseteq)$then either $V_A(x) = 0$ or $V_A(x) = 1 \quad \forall x \in R$.*

Proof. As $A \in F_N(R)$ implies A is normal vague left (resp. right) ideal of near-ring R. $\implies V_A(0) = 1$.
Let $V_A(x) \neq 1$ for some $x \in R$.
Claim- $V_A(x) = 0$
On contrary if not, $\exists\ x_0 \in R$ such that $1 > V_A(x_0) > 0$.
Let us define vague set B on R such that $V_B(x) = \frac{V_A(x)+V_A(x_0)}{2}, \forall x \in R$.
Clearly it is well defined.

Now $\forall x, y, z \in R$.

$$\begin{aligned} V_B(x-y) &= \frac{V_A(x-y)+V_A(x_0)}{2} \\ &\geq \frac{min\{V_A(x),V_A(y)\}+V_A(x_0)}{2} \\ &= \frac{min\{V_A(x)+V_A(x_0),V_A(y)+V_A(x_0)\}}{2} \\ &= min\{\frac{V_A(x)+V_A(x_0)}{2},\frac{V_A(y)+V_A(x_0)}{2}\} \\ &= min\{V_B(x),V_B(y)\} \end{aligned}$$

$$\begin{aligned} V_B(xy) &= \frac{V_A(xy)+V_A(x_0)}{2} \\ &\geq \frac{min\{V_A(x),V_A(y)\}+V_A(x_0)}{2} \\ &= \frac{min\{V_A(x)+V_A(x_0),V_A(y)+V_A(x_0)\}}{2} \\ &= min\{\frac{V_A(x)+V_A(x_0)}{2},\frac{V_A(y)+V_A(x_0)}{2}\} \\ &= min\{V_B(x),V_B(y)\}. \end{aligned}$$

$$\begin{aligned} V_B(y+x-y) &= \frac{V_A(y+x-y)+V_A(x_0)}{2} \\ &\geq \frac{V_A(x)+V_A(x_0)}{2} \\ &= V_B(x). \end{aligned}$$

$$\begin{aligned} V_B(xy) &= \frac{V_A(xy)+V_A(x_0)}{2} \\ &\geq \frac{V_A(y)+V_A(x_0)}{2} \\ &= V_B(y). \qquad (Similarly \quad V_B(xy) \geq V_B(x)). \end{aligned}$$

$$\begin{aligned} V_B[(x+y)z-xz] &= \frac{V_A[(x+y)z-xz]+V_A(x_0)}{2} \\ &\geq \frac{V_A(y)+V_A(x_0)}{2} \\ &= V_B(y). \end{aligned}$$

So B is vague left (resp. right) ideal of R.
Now by theorem 3.8 B^+ is maximal element of R. $\implies B^+ \in F_N(R)$.
Now,

$$\begin{aligned}
V_{B^+}(x_0) &= V_B(x_0) + 1 - V_B(0) \\
&= \frac{V_A(x_0) + V_A(x_0)}{2} + 1 - \frac{V_A(0) + V_A(x_0)}{2} \\
&= V_A(x_0) + 1 - \frac{1}{2} - \frac{V_A(x_0)}{2} \\
&= \frac{V_A(x_0) + 1}{2} \\
&< 1 \\
&= V_{B^+}(0)
\end{aligned}$$

$\implies B^+$ is non-constant maximal element of $F_N(R)$. Which is contradiction. So A is maximal element of poset such that either $V_A(x) = 0$ or $V_A(x) = 1 \quad \forall x \in R$. Hence the proof. □

Definition 2.10. Any vague left (resp. right) ideal A of near-ring R is said to be maximal vague left (resp. right) ideal of near-ring R if it satisfies,
i) A is non-constant.
ii) A^+ is a maximal element of $(F_N(R), \subseteq)$.

Theorem 2.11. *If R has a vague ideal which is maximal, then*
i) A is normal vague ideal of near-ring R.
ii) $V_A(x) = 0$ or $V_A(x) = 1 \quad \forall x \in R$
iii) A^0 is a maximal vague left (resp. right) ideal of near-ring R, where $A^0 = \{x \in R / V_A(0) = 1\}$.

Proof. Let A is maximal vague ideal of R. $\implies A^+$ is a non-constant maximal element of poset $(F_N(R), \subseteq)$.
By theorem 2.9, A^+ takes only two values 0 and 1, $\forall x \in R$.
Also note that $V_{A^+}(x) = 1$ if and only if $V_A(x) = V_A(0)$ and $V_{A^+}(x) = 0$ if and only if $V_A(x) = V_A(0) - 1$.
By corollary 2.7 $V_A(x) = 0 \implies V_A(0) - 1$
It implies that A is normal vague left (resp. right) ideal of near-ring R and $A^+ = A$.
It proves i) and ii).
iii) Clearly A^0 is proper ideal of R as it takes only two values 0 and 1.
Let $A^0 \supseteq B$ be an ideal of near-ring R.

$\implies A = A^0 \supseteq B$ as A is normal.
B is also normal and it takes only two values 0 and 1, but by assumption A is maximal $\implies A = B$ or $A = \phi$, where $V_A(x) = 1 \quad \forall x \in R$.
In case if $A^0 = R$ which is not possible.
So $A = B$, that is $\chi_A(x) = V_A(x), \quad x \in R$.
$\implies A^0 = B$. □

Definition 2.12. Let A be a vague left (resp. right) ideal of near-ring R, then it is said to be complete if it satisfies
i) A is normal vague left (resp. right) ideal of near-ring R.
ii) $\exists\ y \in R$ such that $V_A(y) = 0$.

Theorem 2.13. *Let A any vague left (resp. right) ideal of near-ring R and a be a fixed element of R. Let us define a vague set of R such that $V_{A^*}(x) = \frac{V_A(x)-V_A(a)}{V_A(1)-V_A(a)}, \quad \forall x \in R$. Then A^* is a complete vague left (resp. right) ideal of R.*

Proof. . $\forall x, y, z \in R$,

$$\begin{aligned}
V_{A^*}(x-y) &= \frac{V_A(x-y) - V_A(a)}{V_A(1) - V_A(a)} \\
&\geq \frac{min\{V_A(x), V_A(y)\} - V_A(a)}{V_A(1) - V_A(a)} \\
&= min\{\frac{V_A(x) - V_A(a)}{V_A(1) - V_A(a)}, \frac{V_A(y) - V_A(a)}{V_A(1) - V_A(a)}\} \\
&= min\{V_{A^*}(x), V_{A^*}(y)\}. \\
V_{A^*}(xy) &= \frac{V_A(xy) - V_A(a)}{V_A(1) - V_A(a)} \\
&\geq \frac{min\{V_A(x), V_A(y)\} - V_A(a)}{V_A(1) - V_A(a)} \\
&= min\{\frac{V_A(x) - V_A(a)}{V_A(1) - V_A(a)}, \frac{V_A(y) - V_A(a)}{V_A(1) - V_A(a)}\} \\
&= min\{V_{A^*}(x), V_{A^*}(y)\}.
\end{aligned}$$

$$
\begin{aligned}
V_{A^*}(y+x-y) &= \frac{V_A(y+x-y)-V_A(a)}{V_A(1)-V_A(a)} \\
&\geq \frac{V_A(x)-V_A(a)}{V_A(1)-V_A(a)} \\
&= V_{A^*}(x) \\
V_{A^*}(xy) &= \frac{V_A(xy)-V_A(a)}{V_A(1)-V_A(a)} \\
&\geq \frac{V_A(y)-V_A(a)}{V_A(1)-V_A(a)} \\
&= V_{A^*}(y). \qquad (Similarly \quad V_{A^*}(xy) \geq V_{A^*}(x)). \\
V_{A^*}[(x+y)z-xz] &= \frac{V_A[(x+y)z-xz]-V_A(a)}{V_A(1)-V_A(a)} \\
&\geq \frac{V_A(y)-V_A(a)}{V_A(1)-V_A(a)} \\
&= V_{A^*}(y).
\end{aligned}
$$

$\Longrightarrow$ A^* is vague left (resp. right) ideal of near-ring R. Now at $x = 1$, $V_{A^*}(1) = \frac{V_A(1)-V_A(a)}{V_A(1)-V_A(a)} = 1$ and at $x = a, V_{A^*}(a) = 0$. So $A^* \in F_N(R)$ and A^* is complete vague left (resp. right) ideal of near-ring R. □

3. Conclusions

In this paper, authors have introduced the concepts of normal vague ideals, complete vague ideals and maximal vague ideals of near-ring R with few properties.

Acknowledgments

The authors wish to thank reviewer of this paper, for the valuable inputs and suggestions.

References

[1] D. R Prince Williams, *Fuzzy Ideals in Near-subtraction semigroups*, International Journal of Computational and Mathematical Sciences, **2** (2008), 39–46.

[2] J. D. P. Meldrum, *Near-rings and their links with groups*, Pitman, Boston (1985).
[3] L. A. Zadeh, *Fuzzy set, information and control*, **8** (1965), 338–353.
[4] L. Bhaskar, *Vague ideals of a near-rings*, International Journal of Pure and Applied Mathematics, **117 (20)** (2017), 219–227.
[5] P. N. Swamy, *A note on fuzzy ideal of near-ring*, International Journal of Mathematical Sciences and Engineering Applications, **4(4)** (2010), 423–435.
[6] Pritam Vijaysigh Patil & Janardhan D. Yadav, *I- Vague ideals in near-rings*, Journal of Hyperstructures, vol. **10(1)** (2020) 13–21.
[7] R. Biswas, *Vague groups*, International Journal of Computational Cognition, **4(2)** (2006), 20–23.
[8] S. Abou-Zaid, *On fuzzy sub near-rings and ideals*, Fuzzy Sets and Systems, **44** (1991) 139–146.
[9] S. Abou-Zaid, *On fuzzy ideals and fuzzy quotient rings of a ring*, Fuzzy Sets and Systems, **59** (1993), 205–210.
[10] S. M. Hong., Y. B. Jun & H. S. kim, *Fuzzy ideals in near-rings*, Bulletin of the Korean Mathematical Society, **35(3)** (1998), 455–464.
[11] T. Eswarlal and N. Ramakrishna, *Vague fields and Vague vector spaces*, International Journal of pure and applied Mathematics, 94(3)(2014), 295–305.
[12] W. L. Gau and D. J. Buehrer, *Vague sets*, IEEE Transactions on Systems, Man and Cybernetics, **23** (1993), 610–614.
[13] W. Liu, *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Sets and Systems, **8** (1982), 133–139.

Pritam Vijaysigh Patil
1. Department of Mathematics, Shivaji University, Kolhapur, P.O.Box 416004, Kolhapur, India,
2. Sanjay Ghodawat University, Kolhapur, P.O.Box 416118, Atigre, India
Email: mailme.prit@rediffmail.com

Janardhan D. Yadav
Department of Mathematics, Sadguru Ghadage Maharaj College, Karad, P.O.Box 415124, Karad, India,
Email: jdy1560@yahoo.co.in