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GRAPH BASED ON RESIDUATED LATTICES

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Abstract. In this paper, the residuated graph of residuated lattices will be studied. To do so, the notion of zero divisors of a nonempty subset of a residuated lattice is first introduced and some related properties are investigated. By means of the set of all zero-divisors of an element of a residuated lattice L, the residuated graph $\Gamma(L)$ is defined and several examples are given. This graph is connected and also some necessary conditions for the residuated graph to be a star graph are found. Finally, the relation between a residuated lattice L and the residuated graph $\Gamma(L)$ are studied.

Key Words: Residuated lattice, Graph, zero divisors, Residuated graph.
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1. Introduction

Non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered in a frame of residuated lattices [1]. For example, Hajek's BL (basic logic [2]), Lukasiewiczs MV (many-valued logic[3]) and MTL (monoidal t-norm based logic [4]) are determined by the class of BL-algebras, MV-algebras and MTL-algebras, respectively. All of these algebras have lattices with residuation as a common support set. Thus it is very important to investigate properties of algebras with residuation. Residuated lattices, introduced by Ward and Dilworth [5], are on one hand a generalization of lattice-ordered groups (L-groups),

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and on the other hand provide algebraic semantics to the non-classical logical calculi known as substructural logics [6,7]. It is well known that residuated lattices are also one of the important algebraic structures associated with fuzzy logic and have been extensively studied for their importance in fuzzy logic and in some related areas [8].
Many authors studied the graph theory in connection with semigroups and rings. Beck in [9] associated to any commutative ring R its zero divisors graph $G(R)$, whose vertices are the zero divisors of R, with two vertices a, b jointed by an edge in case $ab = 0$. In [10], Jun and Lee introduced the notion of associated graph of BCK/BCI-algebras by zero divisors in BCK/BCI-algebras and verified some properties of this graph.
The main object of this paper is to study the interplay of residuated-theoretic properties of L with graph-theoretic properties of residuated graph $\Gamma(L)$.
In the following, some preliminary theorems and definitions are stated from [5, 9, 11]. In section 3, we define the notion of zero divisors of a nonempty subset A of a residuated lattice L and we obtain some related results . After that we introduce the set of all zero divisors D_x of an element x of L and we show that $D_0 = D_s(L)$(the set of dense elements of L). We associate a graph to a residuated lattice L, denoted by $\Gamma(L)$, and prove some theorems. We show $\Gamma(L)$ is always a connected graph and its diameter is at most two.

2. **Preliminaries**

At first we recall the definition of a residuated lattice. By a residuated lattice, we mean an algebraic structure L=$(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$, where
(LR_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(LR_2) $(L, \otimes, 1)$ is a commutative monoid with the unit element 1,
(LR_3) For all $a, b, c \in L$, $c \leq a \rightarrow b$ if and only if $a \otimes c \leq b$.

In [8] , residuated lattices are called commutative, integral, residuated l-monoids.

Let L be a residuated lattice. We have the following results.

Theorem 2.1. *The following properties hold for all $x, y, z \in L$:*
(lr_1) $x \rightarrow x = 1, 1 \rightarrow x = x$,
(lr_2) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
(lr_3) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
(lr_4) $x \leq y \Leftrightarrow x \rightarrow y = 1$,

(lr_5) $x \to (y \to z) = y \to (x \to z) = (x \otimes y) \to z$,
(lr_6) $x \otimes (x \to y) \leq y$, $x \leq y \to x$,
(lr_7) If $x \leq y$, *then* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$,
(lr_8) If $(x \vee y) \otimes a = (x \otimes a) \vee (y \otimes a)$.

For each $a \in L$ and $\emptyset \neq A \subseteq L$, we define $\bar{a} := a \to 0$ and $\bar{A} = \{\bar{x} : x \in A\}$.
$B(L)$ denotes the boolean algebra of all complemented elements in the lattice L. We have $a \in B(L)$ if and only if $a \vee \bar{a} = 1$.[7]
A nonempty subset I of L is called an ideal of the lattice L, if it satisfies the following properties:
(LI_1) $\forall a, b \in I,\ a \vee b \in I$,
(LI_2) If $a \in I,\ b \in L$ and $b \leq a$, then $b \in I$.
A nonempty subset F of L is called a filter of L if
(F_1) $x \in F$ and $x \leq y$ imply $y \in F$,
(F_2) for all $x, y \in F$, $x \otimes y \in F$.
The set of all filters of L is denoted by $F(L)$.
Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. The edge which connects two distinct vertices x and y is denoted by $x - y$. Note that, $x - y$ and $y - x$ are the same. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph $G = (V, E)$ is connected , if any two distinct vertices x and y of G linked by a path in G, otherwise the graph is disconnected. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y. If there is no $x - y$ path, then $d(x, y) = \infty$. The diameter of G is $diam(G) = sup\{d(x, y) \mid x$ and y are distinct vertices of $V(G)\}$. A tree is a connected graph with no cycles. A graph G is called complete graph if $x - y \in E(G)$, for any distinct elements $x, y \in V(G)$. A graph G is called a star graph in case there is a vertex x in G such that every other vertex in G is an end, connected to x and no other vertex by an edge.[11]

3. Residuated graph $\Gamma(L)$

From now on, in this paper, L=$(L, \wedge, \vee, \otimes, \to, 0, 1)$ or simply L is a residuated lattice.

Definition 3.1. Let A be a nonempty subset of L. The set of all zero divisors of A is denoted by Z_A and is defined as follows:

$$Z_A = \{x \in L : \ a \otimes x = 0, \ \forall a \in A\}.$$

Proposition 3.2. *Let A and B be nonempty subsets of L. Then the following statements hold:*

(1) $0 \in Z_A$,
(2) $A \subseteq B$ *implies* $Z_B \subseteq Z_A$,
(3) *If* $\overline{Z_A} - \{1\} \neq \emptyset$, *then* $Z_{\overline{Z_A}-\{1\}} \subseteq Z_A$,
(4) *If* $1 \in A$, *then* $Z_A = \{0\}$,
(5) $Z_F = \{0\}$, *for all* $F \in F(L)$,
(6) $1 \in Z_A$ *if and only if* $A = \{0\}$ *if and only if* $Z_A = L$,
(7) *If* $0 \in A$, *then* $Z_A = Z_{A-\{0\}}$,
(8) $Z_A \cap Z_B = Z_{A\vee B} = Z_{A\cup B}$, *where* $A \vee B = \{a \vee b \mid a \in A$ *and* $b \in B\}$,
(9) *If* $a \in B(L)$, *then* $Z_{\{a,\bar{a}\}} = \{0\}$.

Proof. (1) The proof is easy by (LR_3).
(2) Let $x \in Z_B$. Then $x \otimes b = 0$, for all $b \in B$ and so $x \otimes a = 0$, for all $a \in A$. Thus $x \in Z_A$, that is $Z_B \subseteq Z_A$.
(3) Assume that $x \in Z_{\overline{Z_A}-\{1\}}$, then $x \otimes y = 0$, for all $y \in \overline{Z_A} - \{1\}$. Put $y = \bar{t}$, for $t \in Z_A - \{0\}$. Hence $x \otimes \bar{t} = 0$, for all $t \in Z_A - \{0\}$, and we get that $\bar{t} \leq \bar{x}$, for all $t \in Z_A - \{0\}$. Since $t \in Z_A - \{0\}$, then $t \otimes a = 0$, for all $a \in A$, i.e $a \leq \bar{t}$, for all $a \in A$. Thus we can obtain $a \leq \bar{x}$, that is $x \otimes a = 0$, for all $a \in A$. Therefore $x \in Z_A$.
(4) Let $1 \in A$ and $x \in Z_A$. Then $x = x \otimes 1 = 0$, and so $Z_A = \{0\}$.
(5) The proof follows from (4).
(6) Let $1 \in Z_A$. Then $1 \otimes a = 0$, for all $a \in A$ and so $A = \{0\}$. Conversely, let $A = \{0\}$. Then $Z_A = L$. We get that $1 \in Z_A$. It is easy to prove that $Z_A = L$ if and only if $1 \in Z_A$.
(7) Straightforward.
(8) By definition 3.1 we have:

$$\begin{aligned} x \in Z_A \cap Z_B &\Leftrightarrow x \otimes a = x \otimes b = 0,\ \forall a \in A,\ b \in B \\ &\Leftrightarrow x \otimes (a \vee b) = 0,\ \forall a \in A,\ b \in B \\ &\Leftrightarrow x \in Z_{A\vee B} \end{aligned}$$

Similarly, we can prove $Z_A \cap Z_B = Z_{A\cup B}$.
(9) Let $a \in B(L)$. Then $a \vee \bar{a} = 1$. Hence by parts (4) and (8) we have $Z_{\{a,\bar{a}\}} = Z_{\{a\}\cup\{\bar{a}\}} = Z_{\{a\vee\bar{a}\}} = Z_{\{1\}} = \{0\}$. □

By the following example we show that the inverse inclusion of Proposition 3.2, part (3) and the converse of Proposition 3.2, part (9) may not be true.

Example 3.3. (1) Let $L = \{0, a, b, c, d, e, f, 1\}$, with $0 < d < c < b < a < 1$ and $0 < d < e < f < a < 1$ and elements b, f and c, e are pairwise incomparable. The binary operations " $\rightarrow$ " and " $\otimes$ " are given by the tables below:

$\otimes$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	f	f	f	1
c	f	1	1	1	f	f	f	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	0	a	b	c	d	e	f	1

Then L is a residuated lattice (see [1]). Consider $A = \{c, d\}$, we have: $Z_A = \{0, d, e, f\}$, $\overline{Z_A} = \{a, b, c, 1\}$ and $Z_{\overline{Z_A}-\{1\}} = \{0, d\}$. Thus $Z_A \nsubseteq Z_{\overline{Z_A}-\{1\}}$. Therefore the inverse inclusion of Proposition 3.2, part (3) dose not hold.

(2) Let $L = \{0, a, b, c, 1\}$, with $0 < a, b < c < 1$ but a, b are incomparable. The binary operations " $\to$ " and " $\otimes$ " are given by the tables below:

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then L is a residuated lattice (see [12]). Also we have $B(L) = \{0, 1\}$ and $Z_{\{a,\bar{a}\}} = Z_{\{a,b\}} = \{0\}$ while $a \notin B(L)$. So the converse of Proposition 3.2, part (8) dose not hold.

Lemma 3.4. *Let A be a nonempty subset of L. Then Z_A is an ideal of the lattice L.*

Proof. Since $0 \in Z_A$, Z_A is nonempty. Let $x \leq y$, $y \in Z_A$ and $x \in L$. Then $y \otimes a = 0$, for all $a \in A$ and so $x \otimes a = 0$, for all $a \in A$. Thus $x \in Z_A$, hence (LI_1) holds.
Now let $x, y \in Z_A$. Then $x \otimes a = y \otimes a = 0$, for all $a \in A$. Hence $(x \vee y) \otimes a = (x \otimes a) \vee (y \otimes a) = 0$, for all $a \in A$, that is $x \vee y \in Z_A$. Therefore Z_A is an ideal of lattice L. □

For $x \in L$, the set $D_x = \{y \in L : Z_{\{x,y\}} = \{0\}\}$ is called the set of all zero divisors of x. By Poroposition 3.2 part (2) we have, $Z_{\{x\}} = \{0\}$ if and only if $D_x = L$.
By Proposition 3.2, we can get that $D_1 = L$ and $1 \in D_x$, for all $x \in L$.

Lemma 3.5. *D_x is a filter of L, for $x \in L$.*

Proof. By Proposition 3.2 part (4), $Z_{\{1,x\}} = \{0\}$, then $1 \in D_x$ i.e. $D_x \neq \emptyset$. We show that (F_1) and (F_2) hold.
(F_1) Let $y \leq t$, $y \in D_x$ and $t \in L$. We show that $t \in D_x$. Let $h \in Z_{\{x,t\}}$. We get that $h \otimes x = h \otimes t = 0$. Since $y \leq t$, we have $h \otimes y = 0$. Thus $h \in Z_{\{x,y\}} = \{0\}$, that is $h = 0$. Therefore $Z_{\{x,t\}} = \{0\}$, i.e. $t \in D_x$.
(F_2) Let $y_1, y_2 \in D_x$ and $h \in Z_{\{x,y_1 \otimes y_2\}}$. Then we have

$$h \otimes (y_1 \otimes y_2) = h \otimes x = 0,$$

Thus

$$(h \otimes y_1) \otimes x = (h \otimes y_1) \otimes y_2 = 0.$$

Hence $h \otimes y_1 \in Z_{\{x,y_2\}} = \{0\}$, we get that $h \otimes y_1 = 0$. Thus $h \otimes x = h \otimes y_1 = 0$, and we conclude $h \in Z_{\{x,y_1\}} = \{0\}$. Therefore $h = 0$, that is $Z_{\{x,y_1 \otimes y_2\}} = \{0\}$ and so $y_1 \otimes y_2 \in D_x$. □

The set of dense elements of a residuated lattice L is denoted by $D_s(L) = \{x \in L : \bar{x} = 0\}$.

Theorem 3.6. *$D_0 = D_s(L)$.*

Proof. Let $y \in D_0$. Then

$$\{0\} = Z_{\{0,y\}} = \{t : t \otimes y = 0\} = \{t : t \leq \bar{y}\}.$$

Since $\bar{y} \in Z_{\{0,y\}}$, hence $\bar{y} = 0$. Thus $y \in D_s(L)$, that is $D_0 \subseteq D_s(L)$. Conversely, let $y \in D_s(L)$. Then $\bar{y} = 0$. Since $t \leq \bar{y}$, for all $t \in Z_{\{0,y\}}$, so $t = 0$. Thus $Z_{\{0,y\}} = \{0\}$ i.e. $y \in D_0$. Therefore $D_0 = D_s(L)$. □

Definition 3.7. $\Gamma(L)$ is called a residuated graph if vertices are just the elements of L, and for distinct $x, y \in L$, there is an edge connecting x and y if and only if $Z_{\{x,y\}} = \{0\}$.
The edge which connects two vertices x and y is denoted by $x - y$.

Example 3.8. Let $L = \{0, a, b, c, d, 1\}$, with $0 < a, b < c < 1$ and $0 < b < d < 1$. The binary operations " $\rightarrow$ " and " $\otimes$ " are given by the tables below:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Then L is a residuated lattice (see [13]). We have:

$$\begin{aligned} Z_{\{0,a\}} &= \{0, b, d\}, \\ Z_{\{a,1\}} = Z_{\{b,1\}} = Z_{\{c,1\}} &= Z_{\{d,1\}} = Z_{\{c,d\}} = Z_{\{b,c\}} \\ = Z_{\{0,1\}} = Z_{\{0,c\}} &= Z_{\{a,b\}} = Z_{\{a,c\}} = Z_{\{a,d\}} = \{0\} \\ Z_{\{0,b\}} = Z_{\{0,d\}} &= Z_{\{b,d\}} = \{0, a\}. \end{aligned}$$

Therefore, $E(\Gamma(L)) = \{0 - 1, 0 - c, a - 1, b - 1, c - 1, d - 1, a - b, a - c, a - d, b - c, c - d\}$ and $\Gamma(L)$ is given by Fig. 1.

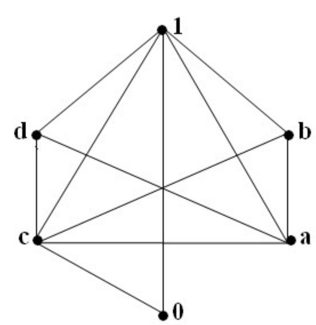

FIGURE 1. Associated graph $\Gamma(L)$ of L

Theorem 3.9. $\Gamma(L)$ *is a connected graph with diameter at most two. .*

Proof. Since $Z_{\{1,x\}} = \{0\}$ for all $x \in L$, then 1 is connected to all points of L. Hence both vertices are connected by a path, and so $\Gamma(L)$ is connected. Let x, y be two vertices in $\Gamma(L)$. If $Z_{\{x,y\}} = \{0\}$, then $d(x,y) = 1$. If $Z_{\{x,y\}} \neq \{0\}$, then we have path $(x - 1 - y)$ and so $d(x,y) = 2$. Therefore $diam(\Gamma(L)) = sup\{d(x,y) \mid x, y \in L\} \leq 2$. □

Theorem 3.10. $D_s(L) = L - \{0\}$ *if and only if* $\Gamma(L)$ *is complete.*

Proof. Let $D_s(L) = L - \{0\}$. For $a, b \in L - \{0\}$ such that $a \neq b$, we have:

$$\begin{aligned} Z_{\{a,b\}} &= \{t : t \otimes a = 0,\ t \otimes b = 0\} \\ &= \{t : t \leq \bar{a},\ t \leq \bar{b}\} \\ &= \{t : t \leq 0\} \\ &= \{0\}. \end{aligned}$$

Also we can obtain $Z_{\{0,a\}} = \{0\}$ for $a \in L - \{0\}$. Therefore, $\Gamma(L)$ is complete.
Conversely, suppose that $\Gamma(L)$ is complete. Then $Z_{\{0,x\}} = \{0\}$, for all $x \in L - \{0\}$ and so $D_0 = L - \{0\}$. Therefore by Theorem 3.6, $D_s(L) = L - \{0\}$. □

Recall that nontrivial residuated lattice L is directly indecomposable if and only if $B(L) = \{0, 1\}$(see [1]).
From Theorem 3.10 we have the following corollaries:

Corollary 3.11. *If* $\Gamma(L)$ *is complete, then* $B(L) = \{0, 1\}$ *and moreover* L *is subdirectly irreducible.*

Corollary 3.12. *If* $|L| > 2$ *and* $D_s(L) = L - \{0\}$, *then* $\Gamma(L)$ *is not tree.*

Theorem 3.13. *If* $\Gamma(L)$ *is tree, then* $|D_s(L)| = 1$.

Proof. Let $\Gamma(L)$ is tree. We know $1 \in D_s(L)$. Now suppose that $|D_s(L)| > 1$. Then, there is $1 \neq x \in L$ such that $\bar{x} = 0$. Thus by Theorem 3.6 we have $Z_{\{0,x\}} = \{0\}$ and so $(0 - x - 1 - 0)$ is cycle, which is a contradiction, since $\Gamma(L)$ is tree . Therefore $|D_s(L)| = 1$. □

The converse of the above theorem does not hold in general.

Example 3.14. Let $L = \{0, a, b, c, d, 1\}$, with $0 < a, b < c < 1$ and $0 < b < d < 1$, but a, b are incomparable. The binary operations " $\rightarrow$ "

and " $\otimes$ " are given by the tables below:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Then L is a residuated lattice (see [1]). We have:

$$Z_{\{0,1\}} = Z_{\{a,1\}} = Z_{\{b,1\}} = Z_{\{c,1\}} = Z_{\{d,1\}} = Z_{\{a,d\}} = Z_{\{c,d\}} = \{0\}$$
$$Z_{\{0,a\}} = \{0, b, d\},\ Z_{\{0,b\}} = \{0, a, b, c\},\ Z_{\{0,d\}} = Z_{\{b,d\}} = \{0, a\}$$
$$Z_{\{a,b\}} = Z_{\{a,c\}} = Z_{\{b,c\}} = Z_{\{0,c\}} = \{0, b\}.$$

Therefore, $\Gamma(L)$ is given by Fig. 2. Also it is easy to see that $|D_s(L)| = 1$ and $\Gamma(L)$ is not tree.

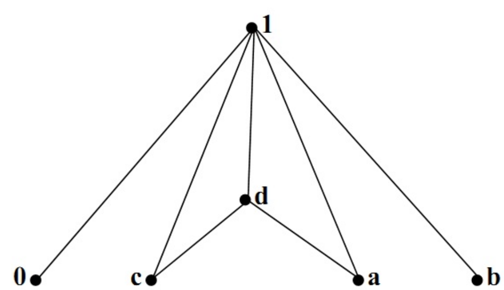

FIGURE 2. Associated graph $\Gamma(L)$ of L is not a tree

Example 3.15. On $L = [0, 1]$, the real unit interval, for $x, y \in L$ we define $x \otimes y = max\{0, x + y - 1\}$ and $x \rightarrow y = min\{1, y - x + 1\}$, Then $(L, max, min, \otimes, \rightarrow, 0, 1)$ is a residuated lattice (called Lukasiewicz structure). Thus, $|D_s(L)| = 1$ and for all $x, y \in L$ we have :

$$Z_{\{x,y\}} = [0, 1 - x] \cap [0, 1 - y].$$

We can see that $\Gamma(L)$ is tree and also it is not complete.

Theorem 3.16. $\Gamma(L)$ *is a star graph if it satisfies the following conditions:*

(i) $|D_s(L)| = 1$,
(ii) There is $a \in L - \{0\}$ *such that* $a \leq x$*, for all* $x \in L - \{0\}$.

Proof. By $Z_{\{1,x\}} = \{0\}$ for all $x \in L$, we have 1 is connected to all points. Now suppose that $x, y \in L - \{1\}$ and $x \neq y$. By (i) we have $\bar{x},\ \bar{y} \neq 0$ and so by (ii) we get $a \leq \bar{x},\ \bar{y}$. Thus $a \in Z_{\{x,y\}}$, i.e. $Z_{\{x,y\}} \neq \{0\}$. Hence $\Gamma(L)$ is a star graph. □

Corollary 3.17. *Under two conditions of the above theorem,* $\Gamma(L)$ *is a tree.*

By the following examples we show that any two conditions listed in Theorem 3.16, are necessary.

Example 3.18. (i) Consider the residuated lattice $L = \{0, a, b, c, d, 1\}$ in Example 3.14. Then there is not $a \in L - \{0\}$ such that $a \leq x$, for all $x \in L - \{0\}$. Also $Z_{\{a,d\}} = \{0\}$, thus $\Gamma(L)$ is not a star graph.
(ii) Let $L = \{0, a, b, c, 1\}$, with $0 < a, b < c < 1$. The binary operations "$\rightarrow$" and "$\otimes$" are given by the tables below:

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Then L is a residuated lattice (see [13]). Also $|D_s(L)| \neq 1$ and $a \leq x$, for all $x \in L - \{0\}$. We have:

$$Z_{\{a,b\}} = Z_{\{a,c\}} = Z_{\{b,c\}} = \{0\}.$$

Clearly, $\Gamma(L)$ is not a star graph.

If A_1 and A_2 are two residuated lattices we have $A_1 \times A_2$ is a residuated lattice. Consider $\emptyset \neq B \subseteq A_1 \times A_2$, then there exist $\emptyset \neq B_1 \subseteq A_1$, $\emptyset \neq B_2 \subseteq A_2$ such that $B = B_1 \times B_2$, so $Z_B^{A_1 \times A_2} = Z_{B_1}^{A_1} \times Z_{B_2}^{A_2}$. Since $Z^{A_1 \times A_2}_{\{(0,0),(0,x)\}} \neq \{(0,0)\}$ for all $x \in A_2$, $(0,0)$ does not connect to $(0,x)$ in the graph $\Gamma(A_1 \times A_2)$. Thus this graph is not complete. Also $Z^{A_1 \times A_2}_{\{(x,1),(1,0)\}} = Z^{A_1 \times A_2}_{\{(1,0),(1,1)\}} = Z^{A_1 \times A_2}_{\{(x,1),(1,1)\}} = \{(0,0)\}$ for all $x \in A_1$, we obtain that $\Gamma(A_1 \times A_2)$ is not tree and star.

Let A_1 and A_2 be two residuated lattices such that $A_1 \cap A_2 = \{1\}$. We define $\otimes$ on $A = A_1 \cap A_2$ (see[8]).

$$x \otimes y = \begin{cases} x \otimes_i y & \text{if } x, y \in A_i,\ i = 1,2, \\ x & \text{if } x \in A_1 - \{1\},\ y \in A_2, \\ y & \text{if } x \in A_2,\ y \in A_1 - \{1\}. \end{cases}$$

$$x \to y = \begin{cases} x \to_i y & \text{if } x, y \in A_i,\ i = 1,2, \\ y & \text{if } x \in A_2,\ y \in A_1 - \{1\}, \\ 1 & \text{if } x \in A_1 - \{1\},\ y \in A_2. \end{cases}$$

Then A is a residuated lattice and we denote the ordinal sum $A = A_1 \oplus A_2$.

Proposition 3.19. *Let A_1 and A_2 be two residuated lattices, $A = A_1 \oplus A_2$ and $\emptyset \neq B \subseteq A$. Then*

$$Z_B^A = \begin{cases} A & \textit{if } B = \{0_1\}, \\ Z_B^{A_1} & \textit{if } B \neq \{0_1\}, B \subseteq A_1, \\ \{0_1\} & o.w\ , \end{cases}$$

Proof. Let $B = \{0_1\}$. Then by Proposition 3.2 part (6) $Z_B^A = A$. Now let $B \neq \{0_1\}$ and $B \subseteq A_1$. Assume that $t \in Z_B^A$, we obtain that $t \otimes b = \{0_1\}$ for all $b \in B$, by definition $\otimes$ on A we get that $t \in A_1$ and so $t \otimes_1 b = \{0_1\}$ for all $b \in B$. Therefore $t \in Z_B^{A_1}$. It is easy to see that $Z_B^{A_1} \subseteq Z_B^A$. Thus $Z_B^{A_1} = Z_B^A$ for $\emptyset \neq B \subseteq A_1$. If $B \neq \{0_1\}$, $B \cap A_2 \neq \emptyset$ and $t \in Z_B^A$, then $t \otimes b = \{0_1\}$ for some $b \in B \cap A_2$ and so we get that $t = 0_1$. Thus $Z_B^A = \{0_1\}$. □

By the above proposition we can conclude that in $\Gamma(A)$, every vertex of A_2 connects to all vertices of A_1 and A_2.

Corollary 3.20. *$\Gamma(A)$ is complete if and only if $\Gamma(A_1)$ is complete.*

Let F be a filter of L. Define a binary relation θ on L as follows: $(a, b) \in \theta$ if and only if $a \to b, b \to a \in F$ if and only if $(a \to b) \otimes (b \to a) \in F$. Then, θ is a congruence relation and it is called the equivalence relation induced by F. If $L/F = \{[x] \mid x \in L\}$, then L/F is a residuated lattice. Moreover, let Π be a partition of L. The graph whose vertexes are the elements of Π and for distinct elements $u, v \in \Pi$, there is an edge connecting u and v if and only if $x - y \in E(\Gamma(L))$, for some $x \in u$ and $y \in v$, is denoted by $\Gamma(L)/\Pi$. Now, we want to verify the relation between the $\Gamma(L/F)$ and $\Gamma(L)/\Pi$, where F is a filter of L, θ is a congruence relation induced by F and Π is the partition induced by θ.

Theorem 3.21. *Let F be a filter of L and Π be the partition of L induced by F.Then $\Gamma(L)/\Pi$ is a subgraph of $\Gamma(L/F)$.*

Proof. Clearly, $V(\Gamma(L/F)) = \{[x] \mid x \in L\} = V(\Gamma(L)/\Pi)$. Let $[x]-[y] \in E(\Gamma(L)/\Pi)$. Then there are $u \in [x]$ and $v \in [y]$, such that $u - v \in E(\Gamma(L))$. Hence, $Z^L_{\{u,v\}} = \{0\}$. Let $[t] \in Z^{L/F}_{\{[x],[y]\}}$. Since $[u] = [x]$ and $[v] = [y]$, then $[t \otimes u] = [t] \otimes [u] = [0] = [t] \otimes [v] = [t \otimes v]$ and so $\overline{(t \otimes u)}$ and $\overline{(t \otimes v)} \in F$. Put $\overline{(t \otimes u)} = a$ and $\overline{(t \otimes v)} = b$, for some $a, b \in F$, then $(a \otimes b) \otimes t \in Z^L_{\{u,v\}} = \{0\}$ and we get that $a \otimes b \leq \bar{t}$. Since $a, b \in F$ and F be a filter of L, we have $\bar{t} \in F$, that is $[t] = [0]$. Hence $Z^{L/F}_{\{[x],[y]\}} = \{[0]\}$. Therefore, $[x] - [y] \in E(\Gamma(L/F))$ and so $\Gamma(L)/\Pi$ is a subgraph of $\Gamma(L/F)$. □

4. Conclusion and Future Research

We have introduced the set of all zero divisors of an element of a residuated lattice L and investigated some properties. We have associated to any residuated lattice L the graph $\Gamma(L)$, whose vertices are just the elements of L, with two distinct vertices a and b joined by an edge in case $Z_{\{a,b\}} = \{0\}$. we have shown that $\Gamma(L)$ is connected and found some conditions that $\Gamma(L)$ be complete. We have investigated some relationships between a residuated lattice L and the residuated graph $\Gamma(L)$. Our future work is how to find some kinds of filters in a residuated lattice by the residuated graph $\Gamma(L)$.

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