Journal of Hyperstructures 3 *(1) (2014), 40-52.*
ISSN: 2251-8436 print/2322-1666 online

NUMERICAL SOLUTION OF VARIATIONAL PROBLEMS VIA PARAMETRIC QUINTIC SPLINE METHOD

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Abstract. In this paper, the parametric quintic spline method is used for finding the solution of variational problems associated in engineering and physics. The present approximation reduce the problems to an explicit system of algebraic equations. Some numerical examples are also given to illustrate the accuracy and applicability of the presented method.

Key Words: Calculus of variation, Parametric quintic spline, Boundary value problem.
2010 Mathematics Subject Classification: Primary: 34L99; Secondary: 65D05, 65D07.

1. Introduction

The need for an optimum function, rather than an optimal point, arises in numerous problems from a wide range of fields in engineering and physics, which include optimal control, transport phenomena, optics, elasticity, vibrations, statics and dynamics of solid bodies and navigation [1]. The calculus of variations and its extensions are devoted to finding the optimum function that gives the best value of the economic model and satisfies the constraints of a system. . In computer vision the calculus of variations has been applied to such problems as estimating optical flow [2] and shape from shading [3]. Several numerical methods for approximating the solution of problems in the calculus of variations are known. Galerkin method is used for solving variational problems in [4]. The Ritz method [5], usually based on the subspaces of kinematically admissible

Received: 6 January 2013, Accepted: 18 April 2014. Communicated by A. Borhanifar;
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complete functions, is the most commonly used approach in direct methods of solving variational problems. Chen and Hsiao [6] introduced the Walsh series method to variational problems. Due to the nature of the Walsh functions, the solution obtained was piecewise constant. Some orthogonal polynomials are applied on variational problems to find the continuous solutions for these problems [7]-[9]. A simple algorithm for solving variational problems via Bernstein orthonormal polynomials of degree six is proposed by Dixit et al. [10]. Razzaghi et al. [11] applied a direct method for solving variational problems using Legendre wavelets. Chebyshev finite difference method has been employed for solving some problems in calculus of variations in [12].

Spline functions are special functions in the space of which approximate solutions of ordinary differential equations. In other words spline function is a piecewise polynomial satisfying certain conditions of continuity of the function and its derivatives. The applications of spline as approximating, interpolating and curve fitting functions have been very successful[13]-[16]. In [17], a non-polynomial spline technique has been developed for the numerical solutions of a system of fourth order boundary value problems associated with obstacle, unilateral and contact problems. Polynomial and non-polynomial spline functions based methods have been presented to find approximate solutions to second order boundary value problems [18]. Khan [19] used parametric cubic spline function to develop a numerical method, which is fourth order for a specific choice of the parameter. Parametric spline approach to the solution of a system of second-order boundary-value problems has been proposed by Khan et al. [20]. Rashidinia et al. [21]-[22] used non-polynomial quintic spline method for the solution of a system of obstacle problems. Also Sinc-Galerkin method has been used for the solution of problems in calculus of variations in [23]. The main purpose of the present paper is to use parametric quintic spline method for numerical solution of boundary value problems which arise from problems of calculus of variations. The method consists of reducing the problem to a set of algebraic equations.

The outline of the paper is as follows. First, in Section 2 we introduce the problems in calculus of variations and explain their relations with boundary value problems. Section 3 outlines parametric quintic spline and basic equations that are necessary for the formulation of the discrete system. Also in this section, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering two numerical examples.

2. Statement of the problem

The genaral form of a variational problem is finding extremum of the functional

$$J[u_1(t), u_2(t), \ldots, u_n(t)] = \int_a^b G\big(t, u_1(t), u_2(t), \ldots, u_n(t), u_1'(t), u_2'(t), \ldots, u_n'(t)\big) dt. \tag{2.1}$$

To find the extreme value of J, the boundary conditions of the admissible curves are known in the following form:

$$u_i(a) = \gamma_i, \quad i = 1, 2, \ldots, n, \tag{2.2}$$

$$u_i(b) = \delta_i, \quad i = 1, 2, \ldots, n. \tag{2.3}$$

The necessary condition for $u_i(t), \quad i = 1, 2, \ldots, n$ to extremize $J[u_1(t), u_2(t), \ldots, u_n(t)]$ is to satisfy the Euler-Lagrange equations that is obtained by applying the well known procedure in the calculus of variation [5],

$$\frac{\partial G}{\partial u_i} - \frac{d}{dt}\Big(\frac{\partial G}{\partial u_i'}\Big) = 0, \quad i = 1, 2, \ldots, n, \tag{2.4}$$

subject to the boundary conditions given by Eqs. (2.2)-(2.3).

In this paper, we consider the spacial forms of the variational problem (2.1) as

$$J[u(t)] = \int_a^b G\big(t, u(t), u'(t)\big) dt, \tag{2.5}$$

with boundary conditions

$$u(a) = \gamma, \quad u(b) = \delta, \tag{2.6}$$

and

$$J[u_1(t), u_2(t)] = \int_a^b G\big(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\big)dt, \tag{2.7}$$

subject to boundary conditions

$$u_1(a) = \gamma_1, \quad u_1(b) = \delta_1, \tag{2.8}$$

$$u_2(a) = \gamma_2, \quad u_1(b) = \delta_2. \tag{2.9}$$

Thus, for solving the variational problems (2.5), by using Euler Lagrange Eq. (2.4) we consider the second-order differential equation

$$\frac{\partial G}{\partial u} - \frac{d}{dt}\Big(\frac{\partial G}{\partial u'}\Big) = 0, \tag{2.10}$$

with the boundary condition (2.6). And also, for solving the variational problems (2.7), we find the solution of the system of second-order differential equations

$$\frac{\partial G}{\partial u_i} - \frac{d}{dt}\Big(\frac{\partial G}{\partial u_i'}\Big) = 0, \quad i = 1, 2, \tag{2.11}$$

with the boundary conditions (2.8)-(2.9). Therefore, by applying parametric quintic spline method for the Euler-Lagrange equations (2.10) and (2.11) we can obtain an approximate solution to the variational problems (2.5) and (2.7).

3. Parametric quintic spline method

Consider the partition Δ of $[a, b] \subset R$. Let $S_k(\Delta)$ denote the set of piecewise polynomials of degree k on subinterval $I_i = [t_{i-1}, t_i]$ of partition Δ. In this work, we consider parametric quintic spline method for finding approximate solution of variational problems.

Consider the grid points t_i on the interval $[a, b]$ as follows:

$$a = t_0 < t_1 < t_2 < \ldots, t_{n-1} < t_n = b, \tag{3.1}$$

$$t_i = t_0 + ih, \quad i = 0, 1, 2, \ldots, n, \tag{3.2}$$

$$h = \frac{b-a}{n}, \tag{3.3}$$

where n is a positive integer. Let $S_\Delta(t, \tau)(t)$ be quintic spline function of class $C^4[a, b]$ that interpolates $u(t)$ at the grid points $\{t_i\}_{i=0}^{n}$. Also,

$S_\Delta(t,\tau)$ depends on a parameter $\tau > 0$ that is called a parametric spline function also, $S_\Delta(t,\tau)$ reduces to a ordinary quintic spline as $\tau \to 0$. By considering parametric quintic spline $S_\Delta(t,\tau) = S_\Delta(t)$, the spline function $S_\Delta(t)$ satisfies in the following equation:

$$S_\Delta^{(4)}(t)+\tau^2 S_\Delta^{(2)}(t) = \big(S_\Delta^{(4)}(t_i)+\tau^2 S_\Delta^{(2)}(t_i)\big)\big[\frac{t-t_{i-1}}{h}\big]+\big(S_\Delta^{(4)}(t_{i-1})+\tau^2 S_\Delta^{(2)}(t_{i-1})\big)\big[\frac{t_i-t}{h}\big], \tag{3.4}$$

where $t \in [t_{i-1}, t_i]$, $S_\Delta(t_i) = u(t_i)$, and $h = t_i - t_{i-1}$. The Eq.(3.4) is a inhomogeneous ordinary differential equation. We solve the Eq.(3.4) and obtain the constants of integration by using interpolation conditions at the endpoints of the interval $[t_{i-1}, t_i]$, then we get:

$$\begin{aligned} S_\Delta(t) = {} & (\frac{t-t_{i-1}}{h})u_i + (\frac{t_i-t}{h})u_{i-1} + (\frac{h^2}{3!})\Big[M_i\Big((\frac{t-t_{i-1}}{h})^3 - (\frac{t-t_{i-1}}{h})\Big) \\ & +M_{i-1}\Big((\frac{t_i-t}{h})^3 - (\frac{t_i-t}{h})\Big)\Big] + (\frac{h}{w})^4\Big[\frac{w^2}{3!}\Big((\frac{t-t_{i-1}}{h})^3 - (\frac{t-t_{i-1}}{h})\Big) \\ & -\Big((\frac{t-t_{i-1}}{h}) - \frac{1}{\sin w}\big(\sin w(\frac{t-t_{i-1}}{h})\big)\Big)\Big]F_i + (\frac{h}{w})^4\Big[\frac{w^2}{3!}\Big((\frac{t_i-t}{h})^3 - \\ & (\frac{t_i-t}{h})\Big) - \Big((\frac{t_i-t}{h}) - \frac{1}{\sin w}\big(\sin w(\frac{t_i-t}{h})\big)\Big)\Big]F_{i-1}, \end{aligned} \tag{3.5}$$

where

$$\begin{gathered} S_\Delta(t_i) = u(t_i) = u_i, \quad S''_\Delta(t_i) = M_i, \\ S_\Delta^{(4)}(t_i) = F_i, \;\; w = \tau h, \;\; \tau > 0. \end{gathered} \tag{3.6}$$

We use the continuity of first and third derivatives of spline function (3.5) at t_i, and obtain the following result:

$$M_{i+1} + 4M_i + M_{i-1} = \frac{6}{h^2}\big(u_{i+1} - 2u_i + u_{i-1}\big) - 6h^2\big(\alpha_1 F_{i+1} + 2\beta_1 F_i + \alpha_1 F_{i-1}\big), \tag{3.7}$$

$$M_{i+1} - 2M_i + M_{i-1} = h^2\big(\alpha F_{i+1} + 2\beta F_i + \alpha F_{i-1}\big), \tag{3.8}$$

where

$$\alpha = \frac{1}{w^2}\big(w \csc w - 1\big), \qquad \beta = \frac{1}{w^2}\big(1 - w\cot w\big), \tag{3.9}$$

$$\alpha_1 = \frac{1}{w^2}\big(\frac{1}{6} - \alpha\big), \qquad \beta_1 = \frac{1}{w^2}\big(\frac{1}{3} - \beta\big). \tag{3.10}$$

Considering Eqs. (3.7)-(3.8) and also some simple calculations, we can obtain the value of F_i as follows:

$$F_i = \frac{1}{12h^2(\alpha_1\beta - \alpha\beta_1)}\Big[(\alpha + 6\alpha_1)\big(M_{i+1} + M_{i-1}\big) + (4\alpha - 12\alpha_1)M_i$$

$$- \frac{6\alpha}{h^2}\big(u_{i+1} - 2u_i + u_{i-1}\big)\Big]. \tag{3.11}$$

Having used Eq. (3.11) and replaced F_{i-1}, F_i and F_{i+1} in Eq. (3.8), the following result is obtained:

$$ph^2\big(M_{i+2} + M_{i-2}\big) + h^2 sM_i + h^2 q\big(M_{i+1} + M_{i-1}\big) = \alpha\big(u_{i+2} + u_{i-2}\big)$$

$$+2(\beta - \alpha)\big(u_{i+1} + u_{i-1}\big) + \big(2\alpha - 4\beta\big)u_i, \quad i = 2, 3, \ldots, n-2, \tag{3.12}$$

where

(3.13)

$$s = 2\Big[\frac{1}{6}\big(\alpha+4\beta\big)+\big(\alpha_1-2\beta_1\big)\Big], \quad q = 2\Big[\frac{1}{6}\big(2\alpha+\beta\big)-\big(\alpha_1-\beta_1\big)\Big], \quad p = \frac{\alpha}{6}+\alpha_1.$$

Remark. To study the convergence analysis,you can see [21]-[22].
In order to illustrate the performance of the parametric quintic spline method, we present two examples.The observed maximum absolute errors are given in tables 1 and 2, also we have compared our computed results with the results obtained by others in [23].

Example 3.1. We first consider the following variational problem with the exact solution $u(t) = e^{3t}$ in [12] and [23]:

$$\min J = \int_0^1 \big(u(t) + u'(t) - 4e^{3t}\big)^2 dt, \tag{3.14}$$

subject to boundary conditions

$$u(0) = 1, \quad u(1) = e^3. \tag{3.15}$$

Considering the Eq. (3.8), the Euler-Lagrange equation of this problem can be written in the following form:

$$u''(t) - u(t) - 8e^{3t} = 0. \tag{3.16}$$

The solution of the second-order differential equation (3.16) with boundary conditions (3.15) is approximated by the presented parametric spline method. For our purpose, we consider the boundary value problem (3.16) in general form as follows:

$$u''(t) = g(t)u(t) + f(t), \tag{3.17}$$

where $g(t) = 1$ and $f(t) = 8e^{3t}$. The exact solution of this problem is $u(t) = e^{3t}$. For a numerical solution of the boundary-value problem (3.17), the interval $[0, 1]$ is divided into a set of grid points with step size h. Setting $t = t_i = t_0 + ih$, in Eq. (3.17), we obtain

$$u''(t_i) = g(t_i)u(t_i) + f(t_i), \tag{3.18}$$

by using the assumption $S''_\Delta(t_i) = M_i$ in (3.18) we have

$$M_i = g(t_i)u(t_i) + f(t_i). \tag{3.19}$$

Substituting M_i as Eq. (3.19) into Eq.(3.12), we get

$$\begin{aligned}&\big(ph^2 g_{i-2} - \alpha\big)u_{i-2} + \big(qh^2 g_{i-1} - 2(\beta - \alpha)\big)u_{i-1} + \big(h^2 s g_i - (2\alpha - 4\beta)\big)u_i\\ &+ \big(qh^2 g_{i+1} - 2(\beta - \alpha)\big)u_{i+1} + \big(ph^2 g_{i+2} - \alpha\big)u_{i+2} =\\ &- ph^2\big(f_{i+2} + f_{i-2} - h^2 s f_i - h^2 q\big(f_{i+1} + f_{i-1}\big)\big), \quad i = 2, 3, \ldots, n-2.\end{aligned} \tag{3.20}$$

where $u_0 = 1,\ u_n = e^3$. Using Taylor's series for Eq. (3.20), we can obtain local truncation error as follows:

$$\begin{aligned}t_i = {}& h^4\Big[\frac{1}{6}\big(7\alpha + \beta\big) - \big(4p + q\big)\Big]u_i^{(4)} + h^6\Big[\frac{1}{180}\big(31\alpha + \beta\big) - \frac{1}{12}\big(16p + q\big)\Big]u_i^{(6)}\\ &+ h^8\Big[\frac{1}{10080}\big(127\alpha + \beta\big) - \frac{1}{360}\big(64p + q\big)\Big]u_i^{(8)} + O(h^9).\end{aligned} \tag{3.21}$$

In Eq. (3.21), if $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$, the presented method is a six-order convergence method[21]. The linear system (3.20) consists of $(n-3)$ equations with $(n-1)$ unknowns $u_i,\ i =, 1, \ldots, n-1$. To obtain unique solution, we need two equations. For this purpose, we can use the following equations that arise from boundary conditions:

$$\begin{aligned}4u_{i-1} - 7u_i + 2u_{i+1} + u_{i+2} = h^2\Big[&\frac{71}{240}u''_{i-1} + \frac{43}{12}u''_i + \frac{7}{8}u''_{i+1} + \frac{1}{3}u''_{i+2}\\ &- \frac{5}{48}u''_{i+3} + \frac{1}{60}u''_{i+4}\Big], \qquad i = 1,\end{aligned} \tag{3.22}$$

$$\begin{aligned}4u_{i+1} - 7u_i + 2u_{i-1} + u_{i-2} = h^2\Big[&\frac{71}{240}u''_{i+1} + \frac{43}{12}u''_i + \frac{7}{8}u''_{i-1} + \frac{1}{3}u''_{i-2}\\ &- \frac{5}{48}u''_{i-3} + \frac{1}{60}u''_{i-4}\Big], \qquad i = n-1,\end{aligned} \tag{3.23}$$

The local truncation errors $t_i,\ i = 1, n-1$ corresponding $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$ are given by $t_i = \frac{7677}{544320} h^8 u_i^{(8)}$.

Now, Eqs. (3.20), (3.22) and (3.23) yield $(n-1)$ equations with $(n-1)$ unknowns $u_i,\ i = 1, 2, \ldots, n-1$. Solving this linear system, we obtain the approximations $u_1, u_2, \ldots, u_{n-1}$ of the solution $u(t)$ at the grid points $t_1, t_2, \ldots, t_{n-1}$.

The errors are reported on the set of uniform grid points

$$S = \{a = t_0, \ldots, t_1, \ldots, t_n = b\},$$

$$t_i = t_0 + ih, \quad i = 0, 1, 2, \ldots, n, \ \ h = \frac{b-a}{n}. \tag{3.24}$$

The maximum error on the uniform grid points S is

$$\|E_u(h)\|_\infty = \max_{0 \le j \le n} |u(t_j) - u_n(t_j)|, \tag{3.25}$$

where $u(t_j)$ is the exact solution of the given example, and u_j is the computed solution by the parametric quintic spline method. The maximum absolute errors in numerical solution of the Example 3.1 are tabulated in table 1. From compared results with SGM [23] in Table 2 we conclude that our results show the efficiency and applicability of the presented method.

TABLE 1. $\|E_u(h)\|_\infty$for Example 3.1.

n	Our method	Method in [23]
10	1.17518×10^{-6}	6.47961×10^{-3}
20	4.58948×10^{-9}	1.39879×10^{-4}
30	6.73661×10^{-10}	6.10976×10^{-6}
40	1.40647×10^{-10}	4.30248×10^{-7}
50	3.96359×10^{-11}	6.92302×10^{-8}

Example 3.2. For the sake of comparison, we consider the following problem to find the extremals of the functional, discussed in [11] and [23]:

$$J[u_1(t), u_2(t)] = \int_0^{\frac{\pi}{2}} \left(u_1'^2(t) + u_2'^2(t) + 2u_1(t)u_2(t)\right) dt, \tag{3.26}$$

with boundary conditions

$$u_1(0) = 0, \quad u_1(\frac{\pi}{2}) = 1, \tag{3.27}$$

$$u_2(0) = 0, \quad u_2(\frac{\pi}{2}) = -1, \tag{3.28}$$

which has the exact solution given by $\big(u_1(t), u_2(t)\big) = \big(\sin(t), -\sin(t)\big)$. For this problem, the corresponding Euler-Lagrange equations are

$$\begin{cases} u_1''(t) - u_2(t) = 0, \\ u_2''(t) - u_1(t) = 0, \end{cases} \tag{3.29}$$

with boundary conditions(3.27) and (3.28).

In a similar manner and applying (3.4) and (3.5), we assume that functions $u_1(t)$ and $u_2(t)$ defined over the interval $[0, \frac{\pi}{2}]$ are approximated by

$$\begin{aligned} u_1(t) \simeq S_{1\Delta}(t) = (\frac{t-t_{i-1}}{h})u_{1,i} + (\frac{t_i - t}{h})u_{1,i-1} + (\frac{h^2}{3!})\Big[M_{1,i}\Big((\frac{t-t_{i-1}}{h})^3 - (\frac{t-t_{i-1}}{h})\Big) \\ +M_{1,i-1}\Big((\frac{t_i-t}{h})^3 - (\frac{t_i-t}{h})\Big)\Big] + (\frac{h}{w})^4\Big[\frac{w^2}{3!}\Big((\frac{t-t_{i-1}}{h})^3 - (\frac{t-t_{i-1}}{h})\Big) \\ -\Big((\frac{t-t_{i-1}}{h}) - \frac{1}{\sin w}\big(\sin w(\frac{t-t_{i-1}}{h})\big)\Big)\Big]F_{1,i} + (\frac{h}{w})^4\Big[\frac{w^2}{3!}\Big((\frac{t_i-t}{h})^3 - \\ (\frac{t_i-t}{h})\Big) - \Big((\frac{t_i-t}{h}) - \frac{1}{\sin w}\big(\sin w(\frac{t_i-t}{h})\big)\Big)\Big]F_{1,i-1}, \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} u_2(t) \simeq S_{2\Delta}(t) = (\frac{t-t_{i-1}}{h})u_{2,i} + (\frac{t_i - t}{h})u_{2,i-1} + (\frac{h^2}{3!})\Big[M_{2,i}\Big((\frac{t-t_{i-1}}{h})^3 - (\frac{t-t_{i-1}}{h})\Big) \\ +M_{2,i-1}\Big((\frac{t_i-t}{h})^3 - (\frac{t_i-t}{h})\Big)\Big] + (\frac{h}{w})^4\Big[\frac{w^2}{3!}\Big((\frac{t-t_{i-1}}{h})^3 - (\frac{t-t_{i-1}}{h})\Big) \\ -\Big((\frac{t-t_{i-1}}{h}) - \frac{1}{\sin w}\big(\sin w(\frac{t-t_{i-1}}{h})\big)\Big)\Big]F_{2,i} + (\frac{h}{w})^4\Big[\frac{w^2}{3!}\Big((\frac{t_i-t}{h})^3 - \\ (\frac{t_i-t}{h})\Big) - \Big((\frac{t_i-t}{h}) - \frac{1}{\sin w}\big(\sin w(\frac{t_i-t}{h})\big)\Big)\Big]F_{2,i-1}, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} S_{j\Delta}(t_i) = u_j(t_i) = u_{j,i}, \ \ j = 1,2, \quad S''_{j\Delta}(t_i) = M_{j,i}, \ \ j = 1,2, \\ S^{(4)}_{j\Delta}(t_i) = F_{j,i}, \ \ j = 1,2, \quad w = h\sqrt{\tau}. \end{aligned} \tag{3.32}$$

Having used the continuity of first and third derivatives of the spline functions $S_{1\Delta}(t)$ and $S_{2\Delta}(t)$, and substituted $t = t_i$ for $i = 1, 2, \ldots, n-1$, where t_i are uniform grid points, we obtain:

$$
\begin{aligned}
F_{1,i} = {} & \frac{1}{12h^2(\alpha_1\beta - \alpha\beta_1)}\Big[(\alpha + 6\alpha_1)\big(M_{1,i+1} + M_{1,i-1}\big) + (4\alpha - 12\alpha_1)M_{1,i} \\
& - \frac{6\alpha}{h^2}\big(u_{1,i+1} - 2u_{1,i} + u_{1,i-1}\big)\Big],
\end{aligned} \tag{3.33}
$$

$$
\begin{aligned}
F_{2,i} = {} & \frac{1}{12h^2(\alpha_1\beta - \alpha\beta_1)}\Big[(\alpha + 6\alpha_1)\big(M_{2,i+1} + M_{2,i-1}\big) + (4\alpha - 12\alpha_1)M_{2,i} \\
& - \frac{6\alpha}{h^2}\big(u_{2,i+1} - 2u_{2,i} + u_{2,i-1}\big)\Big],
\end{aligned} \tag{3.34}
$$

and consequently, we can obtain the following results:

$$
\begin{aligned}
& ph^2\big(M_{1,i+2} + M_{1,i-2}\big) + h^2 s M_{1,i} + h^2 q\big(M_{1,i+1} + M_{1,i-1}\big) = \alpha\big(u_{1,i+2} + u_{1,i-2}\big) \\
& \qquad +2(\beta - \alpha)\big(u_{1,i+1} + u_{1,i-1}\big) + \big(2\alpha - 4\beta\big)u_{1,i}, \quad i = 2, 3, \ldots, n-2,
\end{aligned} \tag{3.35}
$$

$$
\begin{aligned}
& ph^2\big(M_{2,i+2} + M_{2,i-2}\big) + h^2 s M_{2,i} + h^2 q\big(M_{2,i+1} + M_{2,i-1}\big) = \alpha\big(u_{2,i+2} + u_{2,i-2}\big) \\
& \qquad +2(\beta - \alpha)\big(u_{2,i+1} + u_{2,i-1}\big) + \big(2\alpha - 4\beta\big)u_{2,i}, \quad i = 2, 3, \ldots, n-2,
\end{aligned} \tag{3.36}
$$

where α, β, α_1 and β_1 are defined in (3.7)-(3.9). Now, consider the system (3.29) and substitute $t = t_i$, then we get:

$$
u''_{1,i} = u_{2,i}, \qquad u''_{2,i} = u_{1,i} \tag{3.37}
$$

Considering Eq. (3.37) and assumption (3.32), we have:

$$
M_{1,i} = u_{2,i}, \qquad M_{2,i} = u_{1,i}. \tag{3.38}
$$

Now, by using relations (3.35) – (3.38), we can write

(3.39)
$$\begin{cases} ph^2\big(u_{2,i+2}+u_{2,i-2}\big)+h^2su_{2,i}+h^2q\big(u_{2,i+1}+u_{2,i-1}\big)=\alpha\big(u_{1,i+2}+u_{1,i-2}\big) \\ +2(\beta-\alpha)\big(u_{1,i+1}+u_{1,i-1}\big)+\big(2\alpha-4\beta\big)u_{1,i}, \quad i=2,3,\ldots,n-2, \\ \\ ph^2\big(u_{1,i+2}+u_{1,i-2}\big)+h^2su_{1,i}+h^2q\big(u_{1,i+1}+u_{1,i-1}\big)=\alpha\big(u_{2,i+2}+u_{2,i-2}\big) \\ +2(\beta-\alpha)\big(u_{2,i+1}+u_{2,i-1}\big)+\big(2\alpha-4\beta\big)u_{2,i}, \quad i=2,3,\ldots,n-2. \end{cases}$$

The system (3.39) contains $2(n-3)$ equations with $2(n-1)$ unknown coefficients $u_{j,i},\ j=1,2,\ i=1,2\ldots,n-1$. To obtain unique solution, we consider four equations from boundary conditions as:

$$(3.40)\quad \begin{cases} 4u_{1,i-1}-7u_{1,i}+2u_{1,i+1}+u_{1,i+2} & =h^2\Big[\frac{71}{240}u_{2,i-1}+\frac{43}{12}u_{2,i}+\frac{7}{8}u_{2,i+1}+\frac{1}{3}u_{2,i+2} \\ & -\frac{5}{48}u_{2,i+3}+\frac{1}{60}u_{2,i+4}\Big], \qquad i=1, \\ \\ 4u_{1,i+1}-7u_{1,i}+2u_{1,i-1}+u_{1,i-2} & =h^2\Big[\frac{71}{240}u_{2,i+1}+\frac{43}{12}u_{2,i}+\frac{7}{8}u_{2,i-1}+\frac{1}{3}u_{2,i-2} \\ & -\frac{5}{48}u_{2,i-3}+\frac{1}{60}u_{2,i-4}\Big], \qquad i=n-1, \end{cases}$$

and

$$(3.41)\quad \begin{cases} 4u_{2,i-1}-7u_{2,i}+2u_{2,i+1}+u_{2,i+2} & =h^2\Big[\frac{71}{240}u_{1,i-1}+\frac{43}{12}u_{1,i}+\frac{7}{8}u_{1,i+1}+\frac{1}{3}u_{1,i+2} \\ & -\frac{5}{48}u_{1,i+3}+\frac{1}{60}u_{1,i+4}\Big], \qquad i=1, \\ \\ 4u_{2,i+1}-7u_{2,i}+2u_{2,i-1}+u_{2,i-2} & =h^2\Big[\frac{71}{240}u_{1,i+1}+\frac{43}{12}u_{1,i}+\frac{7}{8}u_{1,i-1}+\frac{1}{3}u_{1,i-2} \\ & -\frac{5}{48}u_{1,i-3}+\frac{1}{60}u_{1,i-4}\Big], \qquad i=n-1. \end{cases}$$

The Eqs. (3.39)-(3.41) produce a linear system that contains $2\times(n-1)$ equations with $2\times(n-1)$ unknown coefficients $u_{j,i}, j=1,2,\ i=1,2,\ldots,n-1$. Solving this linear system, we can obtain the approximate solution of the system of second-order boundary value problems (3.29).

Suppose $\|E_{u_1}(h)\|_\infty$ and $\|E_{u_2}(h)\|_\infty$ be the maximum absolute errors. We solved Example 3.2 for different values of n. The maximum of absolute errors on the uniform grid points (3.24) are tabulated in Table 2. We compare the results with the SGM [23] applied to same equation. Table 2 exhibits the compared results.

Table 2. Our method for Example 3.2.

n	$\|E_{u_1}(h)\|_\infty$	$\|E_{u_2}(h)\|_\infty$	$\|E_{u_1}(h)\|_\infty - SGM[23]$	$\|E_{u_2}(h)\|_\infty - SGM[23]$
10	6.70171×10^{-10}	6.70171×10^{-10}	2.72950×10^{-4}	2.72950×10^{-4}
20	7.07034×10^{-12}	7.07034×10^{-12}	8.69675×10^{-6}	8.69675×10^{-6}
30	8.10463×10^{-13}	8.10463×10^{-13}	5.65263×10^{-7}	5.65263×10^{-7}
40	1.55653×10^{-13}	1.55653×10^{-13}	5.47391×10^{-8}	5.47391×10^{-8}
50	4.21885×10^{-14}	4.21885×10^{-14}	6.93422×10^{-9}	6.93422×10^{-9}

4. Conclusion

In this paper parametric quintic spline method employed for finding the extremum of a functional over the specified domain. The main purpose is to find the solution of boundary value problems which arise from the variational problems. The parametric quintic spline method reduce the computation of boundary value problems to some algebraic equations. The proposed scheme is simple and computationally attractive. Applications are demonstrated through illustrative examples.

References

[1] R. Weinstock, *Calculus of Variations: With Applications to Physics and Engineering*, Dover, 1974.

[2] B. Horn and B. Schunck, *Determining optical flow*, Artificial Intelligence, **17** (1981), 185–203.

[3] K. Ikeuchi and B. Horn, *Numerical shape from shading and occluding boundaries*, Artificial Intelligence, **17** (1981), 141–184.

[4] L. Elsgolts, *Differential Equations and Calculus of Variations*, Mir: Moscow, 1977 (translated from the Russian by G. Yankovsky).

[5] I.M. Gelfand and S.V. Fomin, *Calculus of Variations*, Prentice-Hall: Englewood Cliffs, NJ, 1963.

[6] C.F. Chen and C.H. Hsiao, it A walsh series direct method for solving variational problems, J. Franklin Inst., **300** (1975), 265–280.

[7] R.Y. Chang and M.L. Wang, *Shifted Legendre direct method for variational problems*, J. Optim. Theory Appl., **39** (1983), 299–306.

[8] . I.R. Horng and J.H. Chou, *Shifted Chebyshev direct method for solving variational problems*, Internat. J. Systems Sci., **16** (1985), 855–861.

[9] C. Hwang and Y.P. Shih, *Laguerre series direct method for variational problems*, J. Optim. Theory Appl., **39** (1983), 143–149.

[10] S. Dixit, V.K. Singh, A.K. Singh and O.P. Singh, *Bernstein Direct Method for Solving Variational Problems*, International Mathematical Forum, **5** (2010), 2351–2370.

[11] M. Razzaghi and S. Yousefi,*Legendre wavelets direct method for variational problems*, Mathematics and Computers in Simulation, **53** (2000), 185–192.

[12] M. Tatari and M. Dehghan, *The numerical solution of problems in calculus of variation using Chebyshev finite difference method*, Physics Letters A **372** (2008), 4037–4040.

[13] J.H. Ahlberg, E.N. Nilson and J.L. Walsh, *The Theory of Splines and Their Applications*, Academic Press: New York 1967.

[14] T.N.E. Greville, *Introduction to spline functions, in: Theory and Application of Spline Functions*, Academic Press: New York 1969.

[15] P.M. Prenter, *Splines and Variational Methods*, John Wiley & Sons INC. 1975.

[16] G. Micula and Sanda Micula, *Hand Book of Splines*, Kluwer Academic Publisher's 1999.

[17] S.S. Siddiqi and G. Akram, *Numerical solution of a system of fourth order boundary value problems using cubic non-polynomial spline method*, Applied Mathematics and Computation, **190** (2007), 652–661.

[18] M.A. Ramadan, I.F. Lashien and W.K. Zahra, *Polynomial and nonpolynomial spline approaches to the numerical solution of second order boundary value problems*, Applied Mathematics and Computation, **184** (2007), 476-484.

[19] A. Khan, *Parametric cubic spline solution of two point boundary value problems*, Applied Mathematics and Computation, **154** (2004), 175–182.

[20] A. Khan and T. Aziz, *Parametric cubic spline approach to the solution of a system of second-order boundary-value problems*, Journal of Optimization Theory and Applications, **118** (2003), 45–54.

[21] J. Rashidinia, R. Jalilian and R. Mohammadi, *Non-polynomial spline methods for the solution of a system of obstacle problems*, Applied Mathematics and Computation, **188** (2007) 1984–1990.

[22] J. Rashidinia, R. Jalilian and R. Mohammadi, *Convergence analysis of spline solution of certain two-point boundary value problems*, Computer Science and Engineering and Electrical Engineering, **16** (2009), 128–136.

[23] M. Zarebnia and N. Aliniya , *Sinc-Galerkin method for the solution of problems in calculus of variations*, International Journal of Engineering and Natural Sciences, **5** (2011), 140–145.

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