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SOME PROPERTIES OF AFFINE INTUITIONISTIC FUZZY SETS

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Abstract. Intuitionistic fuzzy sets (IFSs) introduced by Atanassov are generalisations of fuzzy sets which are powerful tools in dealing with vagueness. In this paper, concept of convex (concave) IFSs and its characteristics using cut sets of IFSs were studied. In particular, we introduced affine intuitionistic fuzzy sets and investigate some of its characteristics.

Key Words: Fuzzy set, Intuitionistic fuzzy set, Convex intuitionistic fuzzy set, Affine intuitionistic fuzzy set.

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1. Introduction

The classical mathematics methods which have been successful in solving problems that are exact in nature are only necessary to model many fields that deal with uncertain data. To deal with this short coming of classical mathematics methods, many well-known theories such as theory of fuzzy sets [16], of rough sets [14], of vague sets [8], of soft sets [13] have been introduced. In addition, the theory of *intuitionistic fuzzy sets* (IFSs) was introduced by Atanassov [2] as an alternative theory to handle uncertainties accurately. This theory is characterised by membership and non-membership functions. IFSs have been widely studied and applied in such areas as convexity, decision making (see [2, 3, 4, 6, 9, 10, 13] for details).

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In the field of convex analysis, convex sets and their properties were extended to fuzzy sets by Zadeh [16], Lowen [10] and Feiyue [7]. The notion of affine fuzzy sets and their properties have been extensively studied by Lowen [10], Feiyue [7], AL-Mayahi and Selah-Mahdi Ali [1], Sangodapo and Ajayi [15]. Recently, convex intuitionistic fuzzy sets were initiated by Huang [9] as an intuitionistic version of Zadeh's definition of convex sets [16]. Huang [9] defined convex intuitionistic fuzzy sets and studied some of their properties using the cut sets of intuitionistic fuzzy sets.

In this paper, we contribute to the presented work of Huang [9] on convex (concave) intuitionistic fuzzy sets and their properties. We also introduce affine intuitionistic fuzzy sets and provide some of their characteristics.

The organisation of this paper is as follows. In Section 2, we present basic definitions and preliminaries used throughout this paper. In Section 3, we define affine intuitionistic fuzzy sets and investigate some of their characteristics.

2. Preliminaries

In this section, we recall some basic notions of intuitionistic fuzzy sets.

Throughout this paper, $\mathbb{I}$ denotes the unit interval $[0,1]$, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{E}$ denotes the Euclidean space.

Definition 2.1 ([16])**.** Let X be a nonempty set. A fuzzy set A on X is an object of the form

$$A = \{\langle x, \sigma_A(x)\rangle | x \in X\},$$

with a membership function

$$\sigma_A : X \longrightarrow \mathbb{I},$$

where the function $\sigma_A(x)$ denotes the degree of membership of $x \in X$.

Definition 2.2 ([2])**.** An intuitionistic fuzzy set A of $\mathbb{E}$ is defined as

$$A = \{\langle x, \sigma_A(x), \tau_A(x)\rangle | x \in \mathbb{E}\},$$

where the functions

$$\sigma_A(x) : \mathbb{E} \to \mathbb{I} \text{ and } \tau_A(x) : \mathbb{E} \to \mathbb{I},$$

are the membership degree and non-membership degree of x, respectively, and for every,

$$x \in \mathbb{E} \quad 0 \le \sigma_A(x) + \tau_A(x) \le 1.$$

For each IFS A of $\mathbb{E}$,

$$\pi_A(x) = 1 - \sigma_A(x) - \tau_A(x)$$

is the IFS index of $x \in \mathbb{E}$. The IFS index $\pi_A(x)$ is the degree of non-determinacy of $x \in \mathbb{E}$ to A and $\pi_A(x) \in \mathbb{I}$. It is the function that expresses lack of knowledge of whether $x \in \mathbb{E}$ or $x \notin \mathbb{E}$. Thus,

$$\sigma_A(x) + \tau_A(x) + \pi_A(x) = 1.$$

We denote the set of all IFSs over $\mathbb{E}$ by $I^{\mathbb{E}}$.

Definition 2.3 ([2, 3, 4, 5])**.** Let $A,\ B \in I^{\mathbb{E}}$, $A = \{\langle x, \sigma_A(x), \tau_A(x)\rangle | x \in \mathbb{E}\}$ and $B = \{\langle x, \sigma_B(x), \tau_B(x)\rangle | x \in \mathbb{E}\}$ be two IFSs of $\mathbb{E}$. Then,
(i) $A \subseteq B$ if for all $x \in \mathbb{E}$, $\sigma_A(x) \leq \sigma_B(x)$ and $\tau_A(x) \geq \tau_B(x)$;
(ii) $A = B$ if for all $x \in \mathbb{E}$, $\sigma_A(x) = \sigma_B(x)$ and $\tau_A(x) = \tau_B(x)$;
(iii) $A' = \{\langle x, \tau_A(x), \sigma_A(x)\rangle | x \in \mathbb{E}\}$;
(iv) $A \cap B = \{\langle x, \sigma_A(x) \wedge \sigma_B(x), \tau_A(x) \vee \tau_B(x)\rangle | x \in \mathbb{E}\}$;
(v) $A \cup B = \{\langle x, \sigma_A(x) \vee \sigma_B(x), \tau_A(x) \wedge \tau_B(x)\rangle | x \in \mathbb{E}\}$.

Proposition 2.4 ([4])**.** *For IFSs* $A = \{\langle x, \sigma_A(x), \tau_A(x)\rangle | x \in \mathbb{E}\}$, $B = \{\langle x, \sigma_B(x), \tau_B(x)\rangle | x \in \mathbb{E}\}$ *and* $C = \{\langle x, \sigma_C(x), \tau_C(x)\rangle | x \in \mathbb{E}\}$ *of* $I^{\mathbb{E}}$*, the following properties hold:*
(i) $(A')' = A$;
(ii) $(A' \cap B')' = A \cup B$;
(iii) $(A' \cup B')' = A \cap B$;
(iv) $A \cap A = A$;
(v) $A \cup A = A$;
(vi) $A \cap B = B \cap A$;
(vii) $A \cup B = B \cup A$;
(viii) $(A \cap B) \cap C = A \cap (B \cap C)$;
(ix) $(A \cup B) \cup C = A \cup (B \cup C)$;
(x) $(A \cap B)' = A' \cup B'$;
(xi) $(A \cup B)' = A' \cap B'$.

Definition 2.5 ([10])**.** Let A be an IFS of $I^{\mathbb{E}}$. Then, (r, s)-level of A, denoted by $A_{(r,s)}$ is defined as

$$A_{(r,s)} = \{x \in \mathbb{E} | \sigma_A(x) \geq r,\ \tau_A(x) \leq s\}.$$

Definition 2.6 ([9])**.** Given an IFS $A \in I^{\mathbb{E}}$. A is called *convex intuitionistic fuzzy set* (CIFS) if for any $x, y \in \mathbb{E}$ and for any $\lambda \in \mathbb{I}$,

$$\sigma_A[\lambda x + (1 - \lambda) y] \geq \sigma_A(x) \wedge \sigma_A(y)$$

and

$$\tau_A[\lambda x + (1-\lambda)y] \leq \tau_A(x) \vee \tau_A(y).$$

Definition 2.7 ([12])**.** For a CIFS $A \in I^{\mathbb{E}}$, $a_1, a_2, \cdots, a_p \in A$, $p \in \mathbb{N}$, where $\mathbb{N}$ is the natural numbers, convex combination of A is defined as $x = \sum_{i=1}^{p} \lambda_i a_i = \lambda_1 a_1 + \lambda_1 a_2 + \cdots + \lambda_p a_p,\ \forall\ a_i \in A,\ \lambda_i \in \mathbb{I}$, where

$$x = \langle \sigma_x, \tau_x \rangle,\ \lambda_i = \langle \lambda_{i\sigma}, \lambda_{i\tau} \rangle,\ a_i = \langle \sigma_A(a_i), \tau_A(a_i) \rangle,$$

$$\sigma_x = \sum_{i=1}^{p} \lambda_{i\sigma} \sigma_A(a_i),\ \tau_x = \sum_{i=1}^{p} \lambda_{i\tau} \tau_A(a_i),$$

$\sum_{i=1}^{p} (\lambda_{i\sigma} + \lambda_{i\tau}) = 1$ and $0 \leq \lambda_{i\sigma} + \lambda_{i\tau} \leq 1$ for each $i \in \mathbb{N}$.

Definition 2.8 ([12])**.** Let $A = \{\langle x, \sigma_A(x), \tau_A(x) \rangle | x \in \mathbb{E}\} \in I^{\mathbb{E}}$ be a CIFS. Then, the convex hull of A is denoted as $\mathcal{E}_A$ whose membership degree and non-membership degree are defined as:

$$\sigma_{\mathcal{E}_A}(a) = inf\{\sigma_B(a) | \sigma_B \geq \sigma_A(a)\}$$

and

$$\tau_{\mathcal{E}_A}(a) = sup\{\tau_B(a) | \tau_B \leq \tau_A(a)\},$$

respectively, where B is a CIFS of the form $B = \{\langle x, \sigma_B(x), \tau_B(x) \rangle | x \in \mathbb{E}\}$.

Proposition 2.9 ([9])**.** *A CIFS $A = \{\langle x, \sigma_A(x), \tau_A(x) \rangle | x \in \mathbb{E}\}$ is convex if and only if $A_{(r,s)}$ is convex.*

Proposition 2.10 ([12])**.** *The convex hull of CIFS A is the intersection of all convex sets that contains A.*

Proposition 2.11 ([12])**.** *The convex hull of A, $\mathcal{E}_A$ is the set of all convex combinations of points in A.*

Definition 2.12 ([12])**.** Let $A \in I^{\mathbb{E}}$ be an IFS. Then, A is called intuitionistic concave fuzzy set if for any $x, y \in \mathbb{E}$ and for any $\lambda \in \mathbb{I}$,

$$\sigma_A[\lambda x + (1-\lambda)y] \leq \sigma_A(x) \vee \sigma_A(y)$$

and

$$\tau_A[\lambda x + (1-\lambda)y] \geq \tau_A(x) \wedge \tau_A(y).$$

Definition 2.13 ([12]). Let $A = \{\langle x, \sigma_A(x), \tau_A(x)\rangle | x \in \mathbb{E}\}$ be a concave intuitionistic fuzzy set of $I^{\mathbb{E}}$. Then, the concave hull of A is denoted as E_A, whose membership degree and non-membership degree are defined as:

$$\sigma_{E_A}(a) = sup\{\sigma_B(a) | \sigma_B \leq \sigma_A(a)\}$$

and

$$\tau_{E_A}(a) = inf\{\tau_B(a) | \tau_B \geq \tau_A(a)\},$$

respectively, where B is a concave intuitionistic fuzzy set of the form

$$B = \{\langle x, \sigma_B(x), \tau_B(x)\rangle | x \in \mathbb{E}\}.$$

3. Affine Intuitionistic Fuzzy Sets

In this section, we introduce *affine intuitionistic fuzzy sets* (AIFSs) and investigate some of their characteristics.

Definition 3.1. Given an IFS $X \in I^{\mathbb{E}}$. Then, X is called *affine intuitionistic fuzzy set* (AIFS) if for any $x, y \in \mathbb{E}$ and for any $\lambda \in \mathbb{R}$,

$$\sigma_X[\lambda x + (1-\lambda)y] \geq \sigma_X(x) \wedge \sigma_X(y)$$

and

$$\tau_X[\lambda x + (1-\lambda)y] \leq \tau_X(x) \vee \tau_X(y).$$

Definition 3.2. Let $X \in I^{\mathbb{E}}$ be an AIFS, $a_1, a_2, \cdots, a_d \in A$, where $d \in \mathbb{N}$. Then, the *affine combination* of X is defined as $a = \sum_{i=1}^{d} \lambda_i a_i$, where $a = \langle \sigma_a, \tau_a\rangle$, $\lambda_i = \langle \lambda_{i\sigma}, \lambda_{i\tau}\rangle$, $a_i = \langle \sigma_X(a_i), \tau_X(a_i)\rangle$ with $\sigma_a = \sum_{i=1}^{d} \lambda_{i\sigma}\sigma_X(a_i)$, $\tau_a = \sum_{i=1}^{d} \lambda_{i\tau}\tau_X(a_i)$, $\sum_{i=1}^{d}(\lambda_{i\sigma} + \lambda_{i\tau}) = 1$ and $\lambda_{i\sigma} + \lambda_{i\tau} \in \mathbb{R}$ for all $i \in \mathbb{N}$.

The set of all affine combination of affine intuitionistic fuzzy set X is denoted by $\mathcal{C}(a_i, p)$.

AIFS can also be defined in terms of affine combination as follows.

Definition 3.3. Let $X \in I^{\mathbb{E}}$ be an IFS and $a_1, a_2, \cdots, a_d \in A$, where $d \in \mathbb{N}$. Then, X is called an AIFS if for all $a_i \in X$, $\lambda \in \mathbb{R}$, $i = 1, 2, \cdots, d$,

$$\sigma_X\{\sum_{i=1}^{d} \lambda_i a_i\} \geq \sigma_X(a_1) \wedge \sigma_X(a_2) \wedge \cdots \wedge \sigma_X(a_d),$$

$$\tau_X\{\sum_{i=1}^{d} \lambda_i a_i\} \leq \tau_X(a_1) \vee \tau_X(a_2) \vee \cdots \vee \tau_A(a_d)$$

and $\sum_{i=1}^{d} \lambda_i = 1$, where λ_i and a_i are as defined above.

The next proposition demonstrates the relationship between Definitions 3.1 and 3.3.

Proposition 3.4. *An IFS A is an AIFS if and only if for all $a_i \in \mathbb{E}$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \cdots p$, $p \in \mathbb{N}$ such that $\sum_{i=1}^{p} \lambda_i = 1$, we have*

$$\sigma_A\left(\sum_{i=1}^{p} \lambda_i a_i\right) \geq \bigwedge_{i=1}^{p} \sigma_A(a_i), \quad \tau_A\left(\sum_{i=1}^{p} \lambda_i a_i\right) \leq \bigvee_{i=1}^{p} \tau_A(a_i), \tag{1}$$

where $\lambda_i = \langle \lambda_{i\sigma}, \lambda_{i\tau} \rangle$ and $a_i = \langle \sigma_A(a_i), \tau_A(a_i) \rangle$.

Proof. Suppose that A contains all intuitionistic affine combination of its points, that is, for all $a_i \in \mathbb{E}$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \cdots p$, $p \in \mathbb{N}$ such that $\sum_{i=1}^{p} \lambda_i = 1$. In particular, let b_1, $b_2 \in \mathbb{E}$, where $b_1 = \sum_{i=1}^{p} \lambda_i a_i$, with $b_1 = \langle \sigma_{b_1}, \tau_{b_1} \rangle$, $\lambda_i = \langle \lambda_{i\sigma_1}, \lambda_{i\tau_1} \rangle$, $a_i = \langle \sigma_A(a_i), \tau_A(a_i) \rangle$, $\sigma_{b_1} = \sum_{i=1}^{p} \lambda_{i\sigma_1} \sigma_A(a_i)$, $\tau_{b_1} = \sum_{i=1}^{p} \lambda_{i\tau_1} \tau_A(a_i)$, $\sum_{i=1}^{p} (\lambda_{i\sigma_1} + \lambda_{i\tau_1}) = 1$, $(\lambda_{i\sigma_1} + \lambda_{i\tau_1}) \in \mathbb{R}$ for all $i \in \mathbb{N}$ and $b_2 = \sum_{i=1}^{p} \phi_i a_i$, with $b_2 = \langle \sigma_{b_2}, \tau_{b_2} \rangle$, $\phi_i = \langle \phi_{i\sigma_2}, \phi_{i\tau_2} \rangle$, $a_i = \langle \sigma_A(a_i), \tau_A(a_i) \rangle$, $\sigma_{b_2} = \sum_{i=1}^{p} \phi_{i\sigma_2} \sigma_A(a_i)$, $\tau_{b_2} = \sum_{i=1}^{p} \phi_{i\tau_2} \tau_A(a_i)$, $\sum_{i=1}^{p} (\phi_{i\sigma_2} + \phi_{i\tau_2}) = 1$, $(\phi_{i\sigma_2} + \phi_{i\tau_2}) \in \mathbb{R}$ for all $i \in \mathbb{N}$.

Thus,

$$\sigma_A\left(\gamma b_1 + (1-\gamma) b_2\right) \geq \sigma_A(b_1) \wedge \sigma_A(b_2)$$

and

$$\tau_A\left(\gamma b_1 + (1-\gamma) b_2\right) \leq \tau_A(b_1) \vee \tau_A(b_2).$$

The degrees of membership and degrees of non-membership of b_1 and b_2 are given by

$$\begin{aligned} \sigma_a &= \gamma \sum_{i=1}^{p} \lambda_{i\sigma_1} \sigma_A(a_i) + (1-\gamma) \sum_{i=1}^{p} \phi_{i\sigma_2} \sigma_A(a_i) \\ &= \sum_{i=1}^{p} \left(\gamma \lambda_{i\sigma_1} + (1-\gamma) \phi_{i\sigma_2}\right) \sigma_A(a_i) \end{aligned}$$

and

$$\begin{aligned} \tau_a &= \gamma \sum_{i=1}^{p} \lambda_{i\tau_1} \tau_A(a_i) + (1-\gamma)\phi_{i\tau_2}\tau_A(a_i), \\ &= \sum_{i=1}^{p} \left(\gamma\lambda_{i\tau_1} + (1-\gamma)\phi_{i\tau_2}\right) \tau_A(a_i), \end{aligned}$$

respectively with

$$\sum_{i=1}^{p} [\gamma\lambda_{i\sigma_1} + (1-\gamma)\phi_{i\sigma_2}] + \sum_{i=1}^{p} [\gamma\lambda_{i\tau_1} + (1-\gamma)\phi_{i\tau_2}]$$

$$\begin{aligned} &= \left(\sum_{i=1}^{p} \gamma\lambda_{i\sigma_1} + \sum_{i=1}^{p} \gamma\lambda_{i\tau_1}\right) + \left(\sum_{i=1}^{p} (1-\gamma)\phi_{i\sigma_2} + \sum_{i=1}^{p} (1-\gamma)\phi_{i\tau_2}\right) \\ &= \gamma \left(\sum_{i=1}^{p} \lambda_{i\sigma_1} + \sum_{i=1}^{p} \lambda_{i\tau_1}\right) + (1-\gamma) \left(\sum_{i=1}^{p} \phi_{i\sigma_2} + \sum_{i=1}^{p} \phi_{i\tau_2}\right) \\ &= \gamma + 1 - \gamma \\ &= 1. \end{aligned}$$

and since

$$(\lambda_{i\sigma} + \lambda_{i\tau}) \in \mathbb{R}, \ (\phi_{i\sigma} + \phi_{i\tau}) \in \mathbb{R},$$

then

$$((\gamma\lambda_{i\sigma_1} + (1-\gamma)\phi_{i\sigma_2}) + (\gamma\lambda_{i\tau_1} + (1-\gamma)\phi_{i\tau_2})) \in \mathbb{R}.$$

Hence, A is an AIFS.

To prove the converse, we make use of Definition 3.2. Using induction, we show equation (1) for all $p \in \mathbb{N}, \ p > 1$. Let $p = 2$. Then, there exist $\{b_1, \ b_2\} \in \mathcal{C}(a, 2)$ and $\gamma \in \mathbb{R}$ such that $a = \gamma b_1 + (1-\gamma)b_2$. Since A is an AIFS,

$$\begin{aligned} \sigma_A(a) &= \sigma_A(\gamma b_1 + (1-\gamma)b_2) \\ &\geq \sigma_A(b_1) \wedge \sigma_A(b_2) \end{aligned}$$

and

$$\begin{aligned} \tau_A(a) &= \tau_A(\gamma b_1 + (1-\gamma)b_2) \\ &\leq \tau_A(b_1) \vee \tau_A(b_2). \end{aligned}$$

Now suppose that equation (1) holds for $p = k$ that is

$$\sigma_A(a) \geq \bigwedge_{i=1}^{k} \sigma_A(a_i), \quad \tau_A(a) \leq \bigvee_{i=1}^{k} \tau_A(a_i) \tag{2}$$

for $\{a_1, a_2, \cdots, a_k\} \in \mathcal{C}(a, k)$ such that $\sum_{i=1}^{k} \lambda_i a_i$ and $\sum_{i=1}^{k} \lambda_i = 1$. We must prove that equation (1) is true for $p = k+1$. Given $\{a_1, a_2, \cdots, a_{k+1}\} \in \mathcal{C}(a, k+1)$ and $\lambda_1, \lambda_2, \cdots, \lambda_{k+1} \in \mathbb{R}$ such that $\sum_{i=1}^{k+1} \lambda_i = 1$. Suppose that at least one $\lambda_i \in \mathbb{R}$ say $\lambda_1 \neq 1$, let b be an affine combination of k points of $\mathbb{E}$.

Thus, $b = \sum_{i=2}^{k+1} \lambda'_i a_i$ where $\lambda'_i = \dfrac{\lambda_i}{1-\lambda_1} \geq 0$, for $i = 1, 2, \cdots, k+1$, $\sigma_b = \sum_{i=2}^{k+1} (\dfrac{\lambda_i}{1-\lambda_1}) \sigma_A(a_i)$ and $\tau_b = \sum_{i=2}^{k+1} (\dfrac{\lambda_i}{1-\lambda_1}) \tau_A(a_i)$. Thus,

$$\sum_{i=2}^{k+1} \lambda'_i = \sum_{i=2}^{k+1} (\frac{\lambda_i}{1-\lambda_1}) = \frac{1-\lambda_1}{1-\lambda_1} = 1.$$

Hence, $\{a_2, a_3, \cdots, a_{k+1}\} \in \mathcal{C}(b, k)$. From equation (2),

$$\sigma_A(b) \geq \bigwedge_{i=2}^{k+1} \sigma_A(a_i) \text{ and } \tau_A(b) \leq \bigvee_{i=2}^{k+1} \tau_A(a_i).$$

Now, $a_1,\ b \in \mathbb{E}$ and A is an AIFS. Then, from induction,

$$\begin{aligned}
\sigma_A(\lambda a_1 + (1-\lambda)b) &\geq \sigma_A(a_1) \wedge \sigma_A(b) \\
&\geq \sigma_A(a_1) \wedge \sigma_A(\sum_{i=2}^{k+1} (\frac{\lambda_i}{1-\lambda_1})(a_i)) \\
&\geq \bigwedge_{i=1}^{k+1} \sigma_A(a_i)
\end{aligned}$$

and

$$\begin{aligned}\tau_A(\lambda a_1 + (1-\lambda)b) &\le \tau_A(a_1) \vee \tau_A(b)\\ &\le \tau_A(a_1) \vee \tau_A(\sum_{i=2}^{k+1}(\frac{\lambda_i}{1-\lambda_1})(a_i))\\ &\le \bigvee_{i=1}^{k+1} \tau_A(a_i).\end{aligned}$$

Therefore, equation (1) is established. □

Definition 3.5. Let $X = \{\langle a, \sigma_X(a), \tau_X(a)\rangle | a \in \mathbb{E}\} \in I^{\mathbb{E}}$ be an AIFS. An *affine hull* of X, F_X is an AIFS, whose membership degree and non-membership degree are defined as

$$\sigma_{F_X}(a) = \wedge\{\sigma_Y(a) | \sigma_Y \ge \sigma_X(a)\}$$

and

$$\tau_{F_X}(a) = \vee\{\tau_Y(a) | \tau_Y \le \tau_X(a)\},$$

respectively, where Y is an AIFS of the form $Y = \{\langle a, \sigma_Y(a), \tau_Y(a)\rangle | a \in \mathbb{E}\}$.

Proposition 3.6. *An AIFS* $X = \{\langle a, \sigma_X(a), \tau_X(a)\rangle | a \in \mathbb{E}\}$ *is affine if and only if* $X_{(r,s)}$ *is affine.*

Proof. Suppose that X is an AIFS. Let $a, b \in X_{(r,s)}$, then $\sigma_X(a) \ge r$, $\sigma_X(b) \ge r$, $\tau_X(a) \le s$, and $\tau_X(b) \le s$, thus $\sigma_X(a) \wedge \sigma_X(b) \ge r$ and $\tau_X(a) \vee \tau_X(b) \le s$. Since X is an AIFS, then

$$\sigma_X[\lambda a + (1-\lambda)b] \ge \sigma_X(a) \wedge \sigma_X(b) \ge r$$

and

$$\tau_X[\lambda a + (1-\lambda)b] \le \tau_A(a) \vee \tau_A(b) \le s,$$

for all $\lambda \in \mathbb{R}$. Thus, $\lambda a + (1-\lambda)b \in X_{(r,s)}$. Hence, $X_{(r,s)}$ is affine.

Conversely, suppose that $X_{(r,s)}$ is an affine set for all $r, s \in \mathbb{I}$, let $a, b \in \mathbb{E}$, $\lambda \in \mathbb{R}$. Let $r = \sigma_X(a) \wedge \sigma_X(b)$ and $s = \tau_X(a) \vee \tau_X(b)$. Now

$$\sigma_X(a) \ge \sigma_X(a) \wedge \sigma_X(b) = r,\ \sigma_X(b) \ge \sigma_X(a) \wedge \sigma_X(b) = r$$

and

$$\tau_X(a) \le \tau_X(a) \vee \tau_X(b) = s,\ \tau_X(b) \le \tau_X(a) \vee \tau_X(b) = s,$$

which imply that $a, b \in X_{(r,s)}$. By affinity of $X_{(r,s)}$, we have $\lambda a + (1-\lambda)b \in X_{(r,s)}$. Thus,

$$\sigma_X[\lambda a + (1-\lambda)b] \ge r = \sigma_X(a) \wedge \sigma_X(b)$$

and

$$\tau_X[\lambda a + (1-\lambda)b] \leq s = \tau_X(a) \vee \tau_X(b).$$

Therefore, X is an AIFS. □

Proposition 3.7. *Let A and B be AIFSs. Then, the intersection of A and B, $A \cap B$ is an AIFS.*

Proof. Let $C = A \cap B$ and $a, b \in \mathbb{E}$, take $\lambda \in \mathbb{I}$. Then,

$$\sigma_C[\lambda a + (1-\lambda)b] = \sigma_A[\lambda a + (1-\lambda)b] \cap \sigma_B[\lambda a + (1-\lambda)b]$$

and

$$\tau_C[\lambda a + (1-\lambda)b] = \tau_A[\lambda a + (1-\lambda)b] \cup \tau_B[\lambda a + (1-\lambda)b].$$

Since A and B are AIFSs, then

$$\sigma_A[\lambda a + (1-\lambda)b] \geq \sigma_A(a) \wedge \sigma_A(b),\ \sigma_B[\lambda a + (1-\lambda)b] \geq \sigma_B(a) \wedge \sigma_B(b)$$

and

$$\tau_A[\lambda a + (1-\lambda)b] \leq \tau_A(a) \vee \tau_A(b),\ \tau_B[\lambda a + (1-\lambda)b] \leq \tau_B(a) \vee \tau_B(b).$$

Thus,

$$\begin{aligned}
\sigma_C[\lambda a + (1-\lambda)b] &\geq [\sigma_A(a) \wedge \sigma_A(b)] \cap [\sigma_B(a) \wedge \sigma_B(b)] \\
&\geq [\sigma_A(a) \cap \sigma_B(a)] \wedge [\sigma_A(b) \cap \sigma_B(b)] \\
&\geq [\sigma_A \cap \sigma_B](a) \wedge [\sigma_A \cap \sigma_B](b) \\
&\geq [\sigma_{A\cap B}(a) \wedge [\sigma_{A\cap B}(b) \\
&\geq \sigma_C(a) \wedge \sigma_C(b),
\end{aligned}$$

and

$$\begin{aligned}
\tau_C[\lambda a + (1-\lambda)b] &\leq [\tau_A(a) \vee \tau_A(b)] \cup [\tau_B(a) \vee \tau_B(b)] \\
&\leq [\tau_A(a) \cup \tau_B(a)] \vee [\tau_A(b) \cup \tau_B(b)] \\
&\leq [\tau_A \cup \tau_B](a) \vee [\tau_A \cup \tau_B](b) \\
&\leq [\tau_{A\cup B}(a) \vee [\tau_{A\cup B}(b) \\
&\leq \tau_C(a) \vee \tau_C(b).
\end{aligned}$$

□

Corollary 3.8. *The intersection of any family $\{A_i, i = 1, 2, \cdots\}$ of AIFSs is an AIFS.*

Proposition 3.9. *Affine hull of A, F_A consists of all affine combinations of points of A.*

Proof. The points of A belong to F_A, thus all their affine combinations belong to F_A by Proposition 3.3. On the other hand, let b_1, λ_i and a_i, σ_{b_1}, τ_{b_1}, be defined as in the proof of Proposition 3.3 and $b_2 = \sum_{j=p+1}^{q} \phi_j a_j$, $b_2 = \langle \sigma_{b_2}, \tau_{b_2}\rangle$, $\phi_j = \langle \lambda_{j\sigma_2}, \phi_{j\tau_2}\rangle$ with

$$a_j = \langle \sigma_A(a_j), \tau_A(a_j)\rangle,$$

$$\sigma_{b_2} = \sum_{j=p+1}^{q} \phi_{j\sigma_2}\sigma_A(a_j),$$

$$\tau_{b_2} = \sum_{j=p+1}^{q} \phi_{j\tau_2}\tau_A(a_j),$$

$$\sum_{j=p+1}^{q} (\phi_{j\sigma_2} + \phi_{j\tau_2}) = 1,$$

and

$$(\phi_{j\sigma_2} + \phi_{j\tau_2}) \in \mathbb{R},$$

for all $j \in \mathbb{N}$. So, we have

$$(\gamma\lambda_{i\sigma_1} + (1-\gamma)\phi_j\sigma_2) + (\gamma\lambda_{i\tau_1} + (1-\gamma)\phi_{j\tau_2})) \in \mathbb{R},$$

since $(\lambda_{i\sigma_1} + \lambda_{i\tau_1}) \in \mathbb{R}$, and $(\phi_{j\sigma_2} + \phi_{j\tau_2}) \in \mathbb{R}$ by the above proof. This imply

$$\begin{aligned}
&\sigma_A(b_1) \wedge \sigma_A(b_2)\\
&\quad \geq \sigma_A(\sum_{i=1}^{p} \lambda_i a_i) \wedge \sigma_A(\sum_{j=p+1}^{q} \phi_j a_j)\\
&\quad \geq \sigma_A(a_1) \wedge \sigma_A(a_2) \wedge \cdots \wedge \sigma_A(a_p) \wedge \sigma_A(a_{p+1}) \wedge \cdots \wedge \sigma_A(a_q),
\end{aligned}$$

and

$$\begin{aligned}
&\tau_A(b_1) \vee \tau_A(b_2)\\
&\quad \leq \tau_A(\sum_{i=1}^{p} \lambda_i a_i) \vee \tau_A(\sum_{j=p+1}^{q} \phi_j a_j)\\
&\quad \leq \tau_A(a_1) \vee \tau_A(a_2) \vee \cdots \vee \tau_A(a_p) \vee \tau_A(a_{p+1}) \vee \cdots \vee \tau_A(a_q).
\end{aligned}$$

Thus, $\lambda b_1 + \phi b_2$ is another affine combination of points of A, which itself is an AIFS. Hence, it contains A and it must coincide with F_A. □

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