Journal of Hyperstructures 3 (2) (2014), 155-164.
ISSN: 2322-1666 print/2251-8436 online

PRICING FORMULA FOR EXCHANGE OPTION IN FRACTIONAL BLACK-SCHOLES MODEL WITH JUMPS

KYONG-HUI KIM, MYONG-GUK SIN AND UN-HUA CHONG

Abstract. In this paper pricing formula for exchange option in a fractional Black-Scholes model with jumps is derived. We found out some errors in proof of pricing formula for European call option [7]. At first we revise these errors and then extend this result to pricing formula for exchange option in fractional Black-Scholes model with jumps.

Key Words: Pricing formula, Exchange option, Fractional Black-Scholes model, Jump noise.

2010 Mathematics Subject Classification: Primary: 60G22; Secondary: 91B25.

1. Introduction

Fractional Black-Scholes model with jumps is as follows [7].

$$\begin{aligned} &dB(t) = (r_d - r_f)B(t)dt, \; B(0) = 1, \\ (1.1) \qquad &dS(t) = S(t)\left((\mu - \lambda\mu_\xi)dt + \sigma dB_H(t) + (e^\xi - 1)dN(t)\right), \\ &S(0) = S, \end{aligned}$$

where r_d, r_f are the short-term domestic interest rate and foreign interest rate respectively, and these are known. $S(t)$ denotes the spot exchange rate at time t and μ, σ are assumed to be constants. $B_H(t)$ is a fractional Brownian motion and $N(t)$ is a Poisson process with rate λ. $\xi(t)$ is jump size percent at time t which is sequence of independent

Received: 25 April 2014, Accepted: 1 October 2014. Communicated by Hossein Jafari;
∗Address correspondence to Kyong-Hui Kim; E-mail: kim.kyonghui@yahoo.com

identically distributed, and $\left(e^{\xi(t)}-1\right) \sim N\left(\mu_{\xi(t)}, \sigma^2_{\xi(t)}\right)$. In addition, all three sources of randomness, the fractional Brownian motion $B_H(t)$, the Poisson process $N(t)$ and jump size $e^{\xi(t)}-1$ are assumed to be independent.

Currencies are different with stocks; moreover since geometric Brownian motion cannot represent movement currency returns precisely, some papers have provided evidence of mispricing for currency options by standard option price model [1]. Merton proposed a jump-diffusion process with Poisson jump to match the abnormal fluctuation of stock price [3, 5].

Non-normality, non-independence and nonlinearity were discovered in empirical researches of currency return processes. To capture these non-normal behaviors, scholars have considered other distributions with fat tails such as Pareto-stable distribution and tried to interpret long memory and self-similarity using fractional Brownian motion. Research interest for interpreting these abnormal phenomena was re-encouraged by new insights in stochastic analysis based on the Wick integration [2]. Neucula and Meng et al. derived fractional Black-Scholes formula for option pricing using geometric fractional Brownian motion [6, 4]. Model (1.1), the combination of Poisson jumps and fractional Brownian motion was introduced and pricing formula for European call option was derived in [7], but we found out some errors in evaluation of quasi-expectation. In this paper we revise pricing formula for European call option and derive pricing formula for exchange option in fractional Black-Scholes model with jumps and so generalize previous pricing formula for European call option.

2. Preliminaries

We describe some necessary lemmas.

Lemma 2.1. ([6]) (Geometric fractional Brownian motion) *Consider the fractional differential equation*

$$dX(t) = X(t)\left(\mu dt + \sigma dB_H(t)\right),\ X(0) = x.$$

We have that

$$X(t) = x\ \exp\left(\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right).$$

Lemma 2.2. ([6]) *f be a function such that $E[f(B_H(T))] < \infty$. Then for every $t < T$ we have*

$$\tilde{E}_t[f(B_H(T))] = \int_R \frac{1}{\sqrt{2\pi(T^{2H} - t^{2H})}} \exp\left(-\frac{(x - B_H(t))^2}{2(T^{2H} - t^{2H})}\right) f(x)dx,$$

where $\tilde{E}[\cdot]$ denotes quasi-conditional expectation with respect to $\mathbf{F}_t^H = \mathbf{B}(B_H(s), s < t)$. That is for $G = \sum_{n=0}^{\infty} \int_{R^n} g_n dB_H^{\otimes n} \in G^*$ we define as

$$\tilde{E}_t[G] := \tilde{E}\left[G|F_t^H\right] = \sum_{n=0}^{\infty} \int_{R^n} g_n(s)\chi_{0\le s\le t}(s) dB_H^{\otimes n}(s).$$

Let $\theta \in \mathbf{R}$. Consider the process

$$B_H^*(t) = B_H(t) + \theta t^{2H} = B_H(t) + \int_0^t 2H\theta\tau^{2H-1}d\tau,\ 0 \le t \le T$$

This process is a fractional Brownian motion under new measure μ^* by fractional Girsanov theorem, where measure μ^* is defined as $\frac{d\mu^*}{d\mu} = Z(t) = \exp\left(-\theta B_H(t) - \frac{\theta^2}{2} t^{2H}\right)$. We will denote by $\tilde{E}_t^*[\cdot]$ the quasi-conditional expectation with respect to μ^*.

Lemma 2.3. ([6]) *Let f be a function such that $E[f(B_H(T))] < \infty$. Then for every $t < T$*

$$\tilde{E}_t^*[f(B_H(T))] = \frac{1}{Z(t)} \tilde{E}_t[f(B_H(T))Z(T)].$$

Lemma 2.4. ([6]) (fractional risk-neutral evaluation) *The price at every $t \in [0, T]$ of a bounded F_T^H-measurable claim $F \in L^2(\mu)$ is given by $F(t) = e^{-r(T-t)}\tilde{E}_t[F]$.*

3. Main results

Theorem 3.1. *In fractional Black-Scholes model (1.1) with jumps, pricing formula for European call option is as follows.*

$$V(S(t), t) = \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n}{n!} e^{-\lambda(T-t)} \varepsilon_n$$
$$\times \left\{ S(t)\exp\Big(-\lambda\mu_\xi(T-t) + \sum_{j=1}^{n} \xi_j\Big)\Phi(d_+) - Ke^{-(r_d - r_f)(T-t)}\Phi(d_-)\right\},$$

where ε_n *denotes the expectation operator over the distribution of* $\exp\left(\sum_{j=1}^{n}\xi_j\right)$ *and*

$$d_{\pm}=\frac{\ln(S(t)/K)+\sum_{j=1}^{n}\xi_j+(r_d-r_f-\lambda\mu_\xi)(T-t)}{\sigma\sqrt{T^{2H}-t^{2H}}}\pm\frac{1}{2}\sigma\sqrt{T^{2H}-t^{2H}}.$$

Proof. It was proved in [2] that model (1.1) is complete and does not have an arbitrage opportunity. Thus under risk-neutral measure $\hat{P}_H$ model (1.1) can be expressed as

$$dS(t)=S(t)\left\{(r_d-r_f)dt+\sigma d\hat{B}_H(t)+(e^{\xi}-1)dN(t)\right\}, \tag{3.1}$$

where risk-neutral measure $\hat{P}_H$ is defined as

$$\frac{d\hat{P}_H}{dP_H}=\exp\left\{-\theta B_H(t)-\frac{\theta^2}{2}t^{2H}\right\},$$

under this measure process $\hat{B}_H(t)=B_H(t)+\theta t^{2H}$ is a fractional Brownian motion and $\theta=(\mu-\lambda\mu_\xi+r_f-r_d)/\sigma$. By [7] the solution of Eq.(3.1) is expressed as

$$S(T)=S(t)\exp\left\{(r_d-r_f-\lambda\mu_\xi)(T-t)-\frac{1}{2}\sigma^2(T^{2H}-t^{2H})\right.$$
$$\left.+\sigma(\hat{B}_H(T)-\hat{B}_H(t))+\sum_{j=1}^{N(T-t)}\xi_j\right\},$$

By Lemma 2.4 the price at t for European call option $F=(S(T)-K)^+$ is

$$V(S(t),t)=e^{-(r_d-r_f)(T-t)}\tilde{E}_t[F]=e^{-(r_d-r_f)(T-t)}\tilde{E}_{\hat{P}_H}[F|\mathbf{F}_t^H].$$

If we define as

$$S_n(T)=S(t)\exp\left\{(r_d-r_f-\lambda\mu_\xi)(T-t)-\frac{1}{2}\sigma^2(T^{2H}-t^{2H})\right.$$
$$\left.+\sigma(\hat{B}_H(T)-\hat{B}_H(t))+\sum_{j=1}^{n}\xi_j\right\},$$

then

$$
\begin{aligned}
V(S(t),t) &= e^{-(r_d-r_f)(T-t)}\tilde{E}_{\hat{P}_H}[(S(T)-K)^+|\mathbf{F}_t^H] \\
&= e^{-(r_d-r_f)(T-t)}\sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n}{n!}e^{-\lambda(T-t)}\tilde{E}_{\hat{P}_H} \\
&\quad [(S_n(T)-K)^+|\mathbf{F}_t^H].
\end{aligned} \tag{3.2}
$$

Since

$$
\begin{aligned}
\tilde{E}_{\hat{P}_H}[(S_n(T)-K)^+|\mathbf{F}_t^H] &= \tilde{E}_{\hat{P}_H}[S_n(T)\chi_{\{S_n(T)>K\}}|\mathbf{F}_t^H] \\
&\quad - K\tilde{E}_{\hat{P}_H}[\chi_{\{S_n(T)>K\}}|\mathbf{F}_t^H],
\end{aligned} \tag{3.3}
$$

we firstly estimate $\tilde{E}_{\hat{P}_H}[\chi_{\{S_n(T)>K\}}|\mathbf{F}_t^H]$. From Lemma 2.2, we have

$$
\begin{aligned}
\tilde{E}_{\hat{P}_H}&[\chi_{\{S_n(T)>K\}}|\mathbf{F}_t^H] = \tilde{E}_{\hat{P}_H}[\chi_{\{\hat{B}_H(T)>d_-^*\}}|\mathbf{F}_t^H] \\
&= \frac{1}{\sqrt{2\pi(T^{2H}-t^{2H})}}\int_{d_-^*}^{\infty}\exp\left(-\frac{(x-\hat{B}_H(t))^2}{2(T^{2H}-t^{2H})}\right)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{\frac{d_-^*-\hat{B}_H(t)}{\sqrt{T^{2H}-t^{2H}}}}^{\infty}\exp\left(-\frac{y^2}{2}\right)dy \\
&= \Phi\left(\frac{\hat{B}_H(t)-d_-^*}{\sqrt{T^{2H}-t^{2H}}}\right) \\
&= \Phi(d_-),
\end{aligned} \tag{3.4}
$$

where

$$
\begin{aligned}
d_-^* = \Bigg(&\ln(K/S(t)) - (r_d - r_f - \lambda\mu_\xi)(T-t) + \frac{1}{2}\sigma^2\left(T^{2H}-t^{2H}\right) \\
&-\sum_{j=1}^{n}\xi_j + \sigma\hat{B}_H(t)\Bigg)/\sigma.
\end{aligned}
$$

Next we estimate $\tilde{E}_{\hat{P}_H}[S_n(T)\chi_{\{S_n(T)>K\}}|\mathbf{F}_t^H]$. Let $B_H^*(t) = \hat{B}_H(t) - \sigma t^{2H}$, then from fractional Girsanov formula, there exists a probability measure P_H^* such that $B_H^*(t)$ is a fractional Brownian motion. In fact, the probability measure P_H^* is defined as follows:

$$
\frac{dP_H^*}{d\hat{P}_H} = \exp\left\{\sigma d\hat{B}_H(t) - \frac{1}{2}\sigma^2 t^{2H}\right\} = Z(t).
$$

From Lemma 2.3 we have

$$
\begin{aligned}
&\tilde{E}_{\hat{P}_H}[S_n(T)\chi_{\{S_n(T)>K\}}|\mathbf{F}_t^H] \\
&\qquad = S\exp\Big((r_d - r_f - \lambda\mu_\xi)T + \sum_{j=1}^{n}\xi_j\Big) \\
&\qquad\qquad \times \tilde{E}_{\hat{P}_H}[Z(T)\chi_{\{S_n(T)>K\}}\Big|\,\mathbf{F}_t^H] \\
&\qquad = S\exp\Big((r_d - r_f - \lambda\mu_\xi)T + \sum_{j=1}^{n}\xi_j\Big) \\
&\qquad\qquad \times Z(t)\tilde{E}_{P_H^*}[\chi_{\{\hat{B}_H(T)>d_-^*\}}\Big|\,\mathbf{F}_t^H] \\
&\qquad = S_n(t)\exp\big((r_d - r_f - \lambda\mu_\xi)(T-t)\big) \qquad (3.5)\\
&\qquad\qquad \times \tilde{E}_{P_H^*}[\chi_{\{\hat{B}_H(T)>d_-^*\}}\Big|\,\mathbf{F}_t^H] \\
&\qquad = S(t)\exp\Big((r_d - r_f - \lambda\mu_\xi)(T-t) + \sum_{j=1}^{n}\xi_j\Big) \\
&\qquad\qquad \times \tilde{E}_{P_H^*}[\chi_{\{B_H^*(T)>d_+^*\}}\Big|\,\mathbf{F}_t^H] \\
&\qquad = S(t)\exp\Big((r_d - r_f - \lambda\mu_\xi)(T-t) + \sum_{j=1}^{n}\xi_j\Big)\Phi(d_+),
\end{aligned}
$$

where

$$
d_+^* = d_-^* - \sigma T^{2H},\ d_+ = \frac{B_H^*(t) - d_+^*}{\sqrt{T^{2H} - t^{2H}}} = d_- + \sigma(T^{2H} - t^{2H}).
$$

Now substituting Eq.(3.4) and Eq.(3.5) into Eq.(3.2) and Eq.(3.3) implies the statement of the theorem. □

Theorem 3.2. *In fractional Black-Scholes model (1.1) with jump noise, pricing formula for exchange option of two foreign currencies is as follows.*

$$
\begin{aligned}
V(S(t),t) = \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n}{n!}e^{-\lambda(T-t)}\Bigg\{&S_1(t)\exp\Big(-\lambda\mu_\xi(T-t) \\
&+\sum_{j=1}^{n}\xi_j^{(1)}\Big)\Phi(\tilde{d}_+) - S_2(t)\exp\Big(-\lambda\mu_\xi(T-t) + \sum_{j=1}^{n}\xi_j^{(2)}\Big)\Phi(\tilde{d}_-)\Bigg\},
\end{aligned}
$$

where ε_n *denotes the expectation operator over the distribution of* $\exp\left(\sum_{j=1}^{n}\xi_j\right)$ *and*

$$\tilde{d}_{\pm} = \frac{\ln\left(\frac{S_1(t)}{S_2(t)}\right) + \frac{1}{2}(\sigma_1-\sigma_2)^2(T^{2H}-t^{2H}) + \sum_{j=1}^{n}\left(\xi_j^{(1)}-\xi_j^{(2)}\right)}{(\sigma_1-\sigma_2)\sqrt{T^{2H}-t^{2H}}}.$$

Proof. Under the risk-neutral measure $\hat{P}_H$, Exchange rates for two foreign currencies $S_1(t), S_2(t)$ satisfy the following equations:

$$dS_1(t) = S_1(t)\left\{(r_d-r_f)dt + \sigma_1 d\hat{B}_H(t) + (e^{\xi^{(1)}}-1)dN(t)\right\},$$
$$dS_2(t) = S_2(t)\left\{(r_d-r_f)dt + \sigma_2 d\hat{B}_H(t) + (e^{\xi^{(2)}}-1)dN(t)\right\}.$$

Using Lemma 2.4, we have the price of exchange option at t

$$\begin{aligned} V(S_1(t),S_2(t),t) &= e^{-(r_d-r_f)(T-t)}\tilde{E}_{\hat{P}_H}\left[(S_1(T)-S_2(T))^+ \middle| \mathbf{F}_t^H\right] \\ &= e^{-(r_d-r_f)(T-t)}\sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n}{n!}e^{-\lambda(T-t)}\tilde{E}_{\hat{P}_H} \\ &\quad\times\left[(S_{1n}(T)-S_{2n}(T))^+ \middle| \mathbf{F}_t^H\right], \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} S_{in}(T) = S_i(t)\exp\Big\{&(r_d-r_f-\lambda\mu_\xi)(T-t) - \frac{1}{2}\sigma_i^2(T^{2H}-t^{2H}) \\ &+ \sigma_i(\hat{B}_H(T)-\hat{B}_H(t)) + \sum_{j=1}^{n}\xi_j^{(i)}\Big\}. \end{aligned}$$

Also we see that the following facts hold:

$$\tilde{E}_{\hat{P}_H}\left[(S_{1n}(T)-S_{2n}(T))^+ \middle| \mathbf{F}_t^H\right] = \tilde{E}_{\hat{P}_H}\left[S_{2n}(T)\left(\frac{S_{1n}(T)}{S_{2n}(T)}-1\right)^+ \middle| \mathbf{F}_t^H\right].$$

Now let

$$\frac{dQ_H}{d\hat{P}_H} = \exp\left\{\sigma_2\hat{B}_H(t) - \frac{1}{2}\sigma_2^2 t^{2H}\right\} = \tilde{Z}(t).$$

Then under this measure Q_H, $\tilde{B}_H(t) = \hat{B}_H(t) - \sigma_2^2 t^{2H}$ is a fractional Brownian motion and from Lemma 2.3 we have

$$\begin{aligned}
&\tilde{E}_{\hat{P}_H}\left[S_{2n}(T)\left(\frac{S_{1n}(T)}{S_{2n}(T)}-1\right)^+ \middle| \mathbf{F}_t^H \right] \\
&\quad = \tilde{E}_{\hat{P}_H}\left[S_2 \exp\left\{ (r_d - r_f - \lambda\mu_\xi)T + \sum_{j=1}^n \xi_j^{(2)} \right\} \tilde{Z}(T) \right. \\
&\qquad \left. \times \left(\frac{S_{1n}(T)}{S_{2n}(T)}-1\right)^+ \middle| \mathbf{F}_t^H \right] \\
&\quad = S_2 \exp\left\{ (r_d - r_f - \lambda\mu_\xi)T + \sum_{j=1}^n \xi_j^{(2)} \right\} \tilde{Z}(t) \tilde{E}_{Q_H} \\
&\qquad \times \left[\left(\frac{S_{1n}(T)}{S_{2n}(T)}-1\right)^+ \middle| \mathbf{F}_t^H \right] \\
&\quad = S_{2n}(t) \exp\{ (r_d - r_f - \lambda\mu_\xi)(T-t) \} \tilde{E}_{Q_H} \\
&\qquad \times \left[\left(\frac{S_{1n}(T)}{S_{2n}(T)}-1\right)^+ \middle| \mathbf{F}_t^H \right] \\
&\quad = S_2(t) \exp\left\{ (r_d - r_f - \lambda\mu_\xi)(T-t) + \sum_{j=1}^n \xi_j^{(2)} \right\} \tilde{E}_{Q_H} \\
&\qquad \times \left[\left(\frac{S_{1n}(T)}{S_{2n}(T)}-1\right)^+ \middle| \mathbf{F}_t^H \right].
\end{aligned} \tag{3.7}$$

Setting $t = 0, T = t$ and considering the expression of $S_{in}(T)$, we have

$$\begin{aligned}
\frac{S_{1n}(t)}{S_{2n}(t)} &= \frac{S_1}{S_2} \exp\left\{ (\sigma_1 - \sigma_2) d\hat{B}_H(t) - \frac{1}{2}(\sigma_1^2 - \sigma_2^2)t^{2H} + \sum_{j=1}^n \left(\xi_j^{(1)} - \xi_j^{(2)}\right) \right\} \\
&= \frac{S_1}{S_2} \exp\left\{ (\sigma_1 - \sigma_2) d\hat{B}_H(t) - \frac{1}{2}(\sigma_1 - \sigma_2)^2 t^{2H} + \sum_{j=1}^n \left(\xi_j^{(1)} - \xi_j^{(2)}\right) \right\}.
\end{aligned}$$

Thus stochastic process $\frac{S_{1n}(t)}{S_{2n}(t)}$ satisfies the following stochastic differential equation

$$d\left(\frac{S_{1n}(t)}{S_{2n}(t)}\right) = \frac{S_{1n}(t)}{S_{2n}(t)}\left((\sigma_1-\sigma_2)d\tilde{B}_H(t) + \left(e^{\xi_j^{(1)}-\xi_j^{(2)}}\right)dN(t)\right),$$

so quasi-conditional expectation in Eq.(3.7) can be considered as a price for European call option with exercise price K=1. Since this is the special case of Theorem 3.1 with the parameters

$$S_n(T) = \frac{S_{1n}(T)}{S_{2n}(T)},\ r_d - r_f = 0, \sigma = \sigma_1 - \sigma_2,\ \xi = \xi^{(1)} - \xi^{(2)},\ \mu_\xi = 0,\ K = 1,$$

we have

$$\begin{aligned}
&\tilde{E}_{Q_H}\left[\left(\frac{S_{1n}(T)}{S_{2n}(T)} - 1\right)^+ \middle| \mathbf{F}_t^H\right] \\
&\qquad = \frac{S_1(t)}{S_2(t)}\exp\left\{\sum_{j=1}^{n}\left(\xi_j^{(1)} - \xi_j^{(2)}\right)\right\}\Phi(\tilde{d}_+) - \Phi(\tilde{d}_-).
\end{aligned}$$

Thus substituting above equation into Eq.(3.7) and again into Eq.(3.6), we obtain the result of theorem. □

Acknowledgments

The authors wish to thank anonymous referees for their comments and suggestions.

References

[1] R. Cookson, *Models of imperfection*, Risk, **29** (5) (1992), 55–60.

[2] Y.Z. Hu, B. Øksendal, *Fractional white noise calculus and applications to finance*, Infin. Dimens. Anal. Quantum. Probab. Relat. Top., **6** (1) (2003), 1–32.

[3] C.H. Ma, *Intertemporal recursive utility and an equilibrium asset pricing model in the presence of Lvy jumps*, Journal of Mathematical Economics, **42** (2) (2006), 131–160.

[4] L. Meng, M. Wang, *Comparison of Black-Scholes formula with fractional Black-Scholes formula in the foreign exchange option market with changing volatility*, Asia-Pacific Finan Markets, **17** (2) (2010), 99–111.

[5] R.C. Merton, *Option pricing when underlying stock returns are discontinuous*, Journal of Financial Economics, **3** (1-2) (1976), 125–144.

[6] C. Necula, *Option pricing in fractional Brownian motion environment*, Academy of Economics Studies Bucharest, Romania, Preprint, 2002.
[7] W.L. Xiao, W.G. Zhang, X.L. Zhang, Y.L. Wang, *Pricing currency options in a fractional Brownian motion with jumps*, Econ. Modell., **27** (5) (2010), 935–942.

Kyong-Hui Kim
Faculty of Mathematics, University of **Kim Il Sung** University, Pyongyang, D.P.R. Korea
Email: kim.kyonghui@yahoo.com

Myong-Guk Sin
Faculty of Mathematics, University of **Kim Il Sung** University, Pyongyang, D.P.R. Korea
Email: sinmyongguk@yahoo.com

Un-Hua Chong
Faculty of Mathematics, University of **Kim Il Sung** University, Pyongyang, D.P.R. Korea