

## REGULAR EQUIVALENCE AND STRONGLY REGULAR EQUIVALENCE ON MULTIPLICATIVE TERNARY HYPERRING

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**ABSTRACT.** We introduce the notion of a multiplicative ternary hyperring, consider regular equivalences and strongly regular equivalences of a multiplicative ternary hyperring and investigate their properties. As a consequence, three isomorphism theorems on multiplicative ternary hyperrings are obtained.

**Key Words:** Binary relation, equivalence relation, regular and strongly regular equivalence relation, ternary hyperrings.

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### 1. INTRODUCTION

The theory of hyperstructure was introduced by F. Marty[7] in 1934. He first studied the hypergroups and analyzed their properties and then applied them to groups and rational algebraic functions. Nowadays, there has been a remarkable growth of hyperstructure theory. Many mathematicians have taken interest to explore the theory of hyperstructure which has many applications in both Pure and Applied sciences.

The notion of a multiplicative hyperring has been introduced by Rota [12] in which the addition is a binary operation and the multiplication is a multiplicative hyperoperation. Krasner also introduced the notion of

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hyperring, called krasner hyperring [11]. In krasner hyperring  $(R, +, \cdot)$ , '+' is a binary hyperoperation and ' $\cdot$ ' is a binary operation. D. Salvo [10] and Assokumar and Velrajan [1] also studied hyperrings in which both addition and multiplication are binary hyperoperations. In 2014, Davvaz[2] also studied krasner hyperring and obtained three isomorphism theorems in krasner hyperring where the hyperideals are normal.

The introduction of mathematical literature of ternary algebraic system dated back to 1924. The notion of ternary algebraic system was first introduced by H.Prüfer[8] by the name 'Schar'. After that W.Dörnte further studied this type of algebraic system. In 1932, D.H.Lehmer[5] investigated certain ternary algebraic systems called triplexes which turn out to be a commutative ternary groups. Ternary groups are the special case of polyadic groups(in terminologies which are known as n-groups) introduced by E.L.Post[6]. In 1971, W.G.Lister[14] introduced the notion of ternary ring and study some important properties of it. According to Lister[14], a ternary ring is an algebraic system consisting of a non-empty set  $R$  together with a binary operation, called addition and a ternary multiplication, which forms a commutative group relative to addition, a ternary semigroup relative to multiplication and left, right, lateral distributive laws hold.

In 2010, Davvaz and Mirvakili [3] introduced a new class of n-ary multivalued hyper algebra called an  $(m, n)$ -hyperring in which both the m-ary operation and the n-ary operation are hyperoperations and studied it. Recently Anvariyeheh and Mirvakili [13] studied  $(m, n)$ -hypermodule over  $(m, n)$ -hyperring. J. R. Castillo and Jocelyn S. Paradero-Vilela in the year 2014 [9] introduced ternary hyperrings, called Krasner ternary hyperring. In Krasner ternary hyperring  $(R, +, \cdot)$ , '+' is a binary hyperoperation and ' $\cdot$ ' is a ternary multiplication.

In this paper we introduce the notion of a multiplicative ternary hyperring. Our notion of a multiplicative ternary hyperring differs from

the notion of Krasner multiplicative ternary hyperring. In our multiplicative ternary hyperring addition is a binary operation and multiplication is a ternary hyperoperation, whereas in Krasner ternary hyperring addition is a binary hyperoperation and multiplication is a ternary operation. We consider regular equivalence and strongly regular equivalence on a multiplicative ternary hyperring and study some properties of them. The regular equivalence plays the same role as the congruence does in algebra. As a consequence of regular equivalence relation and strongly regular equivalence on a multiplicative ternary hyperring, we obtain three isomorphism theorems(The first, the second and the third) on a multiplicative ternary hyperring.

## 2. PRELIMINARIES

**Definition 2.1.** Let  $S$  be a non-empty set endowed with a binary operation, namely, the addition operation and ternary multiplication. We denote the ternary multiplication on  $S$  by juxtaposition and the system  $S$  endowed with the above two operations is said to be a ternary ring if  $S$  forms an additive commutative group satisfying the following conditions:

- (i):  $(abc)de = a(bcd)e = ab(cde)$ ;
- (ii):  $(a+b)cd = acd+bcd$ ;
- (iii):  $a(b+c)d = abd+acd$ ;
- (iv):  $ab(c+d) = abc+abd$

for all  $a, b, c, d, e \in S$ .

**Definition 2.2.** A ternary ring  $S$  is said to admit an identity provided that there exist elements  $\{(e_i, f_i) \in S \times S (i = 1, 2, \dots, n)\}$  such that  $\sum_{i=1}^n e_i f_i x = \sum_{i=1}^n e_i x f_i = \sum_{i=1}^n x e_i f_i = x$  for all  $x \in S$ . In this case, the ternary ring  $S$  is said to be a ternary ring with identity  $\{(e_i, f_i) : i = 1, 2, \dots, n\}$ . In particular, if there exists an element  $e \in S$  such that  $eex = exe = xee = x$  for all  $x \in S$ , then the element  $e \in S$  is called a unital element of the ternary ring  $S$ .

It is easy to see that  $xye = (exe)ye = ex(eye) = exy$  and  $xye = x(eye)e = xe(yee) = xey$ , for all  $x, y \in S$ . Hence, the following proposition follows.

**Proposition 2.3.** *If  $e$  is a unital element of a ternary ring  $S$ , then  $exy = xey = xye$ , for all  $x, y \in S$ .*

**Definition 2.4.** Let  $S$  and  $T$  be two ternary rings. Then a mapping  $f : S \rightarrow T$  is called a ternary ring homomorphism of  $S$  into  $T$  if

- (i):  $f(a+b) = f(a)+f(b)$
- (ii):  $f(abc) = f(a)f(b)f(c)$

A bijective ternary ring homomorphism is called a ternary ring isomorphism and in this case we write  $S \cong T$ .

### 3. MULTIPLICATIVE TERNARY HYPERRING

**Definition 3.1.** By a ternary hyperoperation ‘ $\circ$ ’ on a nonempty set  $H$ , we shall mean a mapping  $\circ: H \times H \times H \rightarrow P^*(H)$  when  $P^*(H)$  is the set of all nonempty subsets of  $H$ . For  $x, y, z \in H$ , the image of the element  $(x, y, z) \in H \times H \times H$  under the mapping ‘ $\circ$ ’ will be denoted by  $x \circ y \circ z$  (which is called the ternary hyperproduct of  $x, y, z$ ).

**Definition 3.2.** A multiplicative ternary hyperring  $(S, +, \circ)$  is an additive commutative group  $(S, +)$  endowed with a ternary hyper operation ‘ $\circ$ ’ such that the following conditions hold

- (i):  $(a \circ b \circ c) \circ d \circ e = a \circ (b \circ c \circ d) \circ e = a \circ b \circ (c \circ d \circ e)$
- (ii):  $(a + b) \circ c \circ d \subseteq a \circ c \circ d + b \circ c \circ d$ ;
- (iii):  $a \circ (b + c) \circ d \subseteq a \circ b \circ d + a \circ c \circ d$ ;
- (iv):  $a \circ b \circ (c + d) \subseteq a \circ b \circ c + a \circ b \circ d$ ;

for all  $a, b, c, d, e \in S$ , where if the inclusions in (ii) – (iv) are replaced by equalities, then the multiplicative ternary hyperring is called a strongly distributive multiplicative ternary hyperring.

We have the following remark.

*Remark 3.3.* It is immediate to see that the notion of multiplicative ternary hyperring coincides with the notion of a ternary ring if and only if  $|a \circ b \circ c| = 1$  for all  $a, b, c \in S$ .

On the other hand if  $(S, +, \circ)$  is a ternary ring then  $(S, +, \cdot)$  can be regarded as a strongly distributive multiplicative ternary hyperring if we take  $a \circ b \circ c = \{a \cdot b \cdot c\}$  for all  $a, b, c \in S$ .

Thus the above notion of a multiplicative ternary hyperring is a generalization of the notion of ternary ring.

The following definitions are some basic definitions in this paper.

**Definition 3.4.** Let  $(S, +, \circ)$  be a multiplicative ternary hyperring. For a nonempty set  $A$  we define  $A \circ x \circ y = \cup\{\sum a \circ x \circ y : a \in A\}$ , for any  $x, y \in S$ .

**Definition 3.5.** A multiplicative ternary hyperring  $(S, +, \circ)$  is called commutative if  $a_1 \circ a_2 \circ a_3 = a_{\sigma(1)} \circ a_{\sigma(2)} \circ a_{\sigma(3)}$ , where  $\sigma$  is a permutation of  $\{1, 2, 3\}$  for all  $a_1, a_2, a_3 \in S$ .

**Definition 3.6.** A multiplicative ternary hyperring  $(S, +, \circ)$  is called weakly commutative if  $a_1 \circ a_2 \circ a_3 \cap a_{\sigma(1)} \circ a_{\sigma(2)} \circ a_{\sigma(3)} \neq \phi$  where  $\sigma$  is a permutation of  $\{1, 2, 3\}$  for all  $a_1, a_2, a_3 \in S$ .

**Definition 3.7.** The additive identity '0' of a multiplicative ternary hyperring  $(S, +)$  is said to be a zero (strong zero) of  $(S, +, \circ)$  if  $0 \in a \circ b \circ 0 = a \circ 0 \circ b = 0 \circ a \circ b$  (resp.  $\{0\} = a \circ 0 \circ b = 0 \circ a \circ b = a \circ b \circ 0$ ) for all  $a, b \in S$ .

Unless otherwise stated by an ternary hyperring we shall mean an ternary hyperring with zero.

We give below some examples of multiplicative ternary hyperrings.

*Example 3.8.* Consider the ring  $(Z, +, \cdot)$  of the set of all integers with respect to the usual addition and multiplication of integers. Corresponding to any subset  $A$  of the set of integers there exists a multiplicative ternary hyperring  $(Z_A, +, \circ)$ , where  $Z_A = Z$  and for any  $x, y, z \in Z_A, +$

is the usual addition of integers and  $x \circ y \circ z = \{x \cdot a \cdot y \cdot b \cdot z : a, b \in A\}$ . The above multiplicative ternary hyperring is called the multiplicative ternary hyperring induced by  $A$ .

*Example 3.9.* Let  $S$  be the set of all integers. Then  $(S, +)$  is a commutative group with respect to the usual addition of integers. On  $S$  we define a ternary hyperoperation ‘ $\circ$ ’ as follows:

$a \circ b \circ c = [0, x]$  for all  $a, b, c \in S$ , where  $x = \max(a, b, c)$ . Then  $(S, +, \circ)$  is a multiplicative ternary hyperring.

*Example 3.10.* Let  $Z$  be the set of all integers. Suppose that  $n \in Z$  is arbitrarily chosen but fixed integer. Define a ternary hyperoperation ‘ $\circ$ ’ on  $Z$  by  $a \circ b \circ c = \{abc + nk : k \in Z \text{ for all } a, b, c \in Z\}$ . Then with respect to the usual addition of integers and defined ternary hyperoperation, the system  $(Z, +, \circ)$  forms a commutative multiplicative ternary hyperring with a zero 0.

#### 4. REGULAR AND STRONGLY REGULAR EQUIVALENCES ON A MULTIPLICATIVE TERNARY HYPERRING

Let  $\rho$  be an equivalence on a non-empty set  $S$  and  $P(S)$  denote the power set of  $S$ . Let  $P^*(S) = P(S) - \{\emptyset\}$ . Then, we define two relations  $\bar{\rho}$  and  $\bar{\bar{\rho}}$  on  $P^*(S)$  as follows:

- (i): For any  $A, B \in P^*(S)$ ,  $A\bar{\rho}B$  holds if and only if for each  $a \in A$  there exists  $b \in B$  such that  $a\rho b$  holds and also for each  $b' \in B$  there exists  $a' \in A$  such that  $a'\rho b'$  holds.
- (ii):  $A\bar{\bar{\rho}}B$  holds if and only if  $a\rho b$  holds for all  $a \in A$  and  $b \in B$ .

**Definition 4.1.** An equivalence  $\rho$  defined on a multiplicative ternary hyperring  $(S, +, \circ)$  is called

- (i) regular if  $\rho$  is a congruence on the commutative group  $(S, +)$  i.e.  $a\rho b \Rightarrow (a+c)\rho(b+c)$  for  $a, b, c \in S$  and  $a\rho b, c\rho d, e\rho f \Rightarrow (a \circ c \circ e)\bar{\rho}(b \circ d \circ f)$  for  $a, b, c, d, e, f \in S$ ;

(ii) strongly regular if  $\rho$  is a congruence on the commutative group  $(S, +)$  i.e.  $a\rho b \Rightarrow (a + c)\rho(b + c)$  for  $a, b, c \in S$  and  $a\rho b, c\rho d, e\rho f \Rightarrow (a \circ c \circ e)\bar{\rho}(b \circ d \circ f)$  for  $a, b, c, d, e, f \in S$ .

*Remark 4.2.* The second condition stated in (i) and (ii) of the Definition 4.1. are equivalent to the following conditions respectively:

$a\rho b \Rightarrow (a \circ c \circ d)\bar{\rho}(b \circ c \circ d), (c \circ a \circ d)\bar{\rho}(c \circ b \circ d)$  and  $(c \circ d \circ a)\bar{\rho}(c \circ d \circ b)$  for all  $a, b, c, d \in S$  and  $a\rho b \Rightarrow (a \circ c \circ d)\bar{\rho}(b \circ c \circ d), (c \circ a \circ d)\bar{\rho}(c \circ b \circ d)$  and  $(c \circ d \circ a)\bar{\rho}(c \circ d \circ b)$  for all  $a, b, c, d \in S$ .

It is clear that the strongly regular equivalence is a regular equivalence on a multiplicative ternary hyperring.

To give an example of a regular equivalence on a multiplicative ternary hyperring which is not a strongly regular equivalence, we first define homomorphism of multiplicative ternary hyperrings.

**Definition 4.3.** Let  $(S, +, \circ)$  and  $(S', +, \circ)$  be two multiplicative ternary hyperrings. Then a mapping  $f : S \rightarrow S'$  is called a homomorphism (a good homomorphism) if  $f(a + b) = f(a) + f(b)$  and  $f(a \circ b \circ c) \subseteq f(a) \circ f(b) \circ f(c)$  (resp.  $f(a \circ b \circ c) = f(a) \circ f(b) \circ f(c)$ ).

**Definition 4.4.** Let  $f$  be a good homomorphism from a multiplicative ternary hyperring  $(S, +, \circ)$  to a multiplicative ternary hyperring  $(T, +, \circ)$ . Then the relation  $\rho_f$  on  $S$  defined by  $a\rho_f b$  if and only if  $f(a) = f(b)$ , for  $a, b \in S$ , is called the relation on  $S$  induced by  $f$ .

For multiplicative ternary hyperrings, we have the following propositions.

**Proposition 4.5.** *The relation  $\rho_f$  induced by a good homomorphism  $f$  from a multiplicative ternary hyperring  $(S, +, \circ)$  to a multiplicative ternary hyperring  $(T, +, \circ)$  is a regular equivalence on  $(S, +, \circ)$ .*

*Proof.* Obviously  $\rho_f$  is an equivalence on  $S$ . Let  $a, b, c, d \in S$  be such that  $a\rho_f b$  and  $c\rho_f d$ . Then  $f(a) = f(b)$  and  $f(c) = f(d)$  and hence we have  $f(a + c) = f(a) + f(c) = f(b) + f(d) = f(b + d)$ . Thus  $(a + c)\rho_f(b + d)$ .

Hence  $\rho_f$  is a congruence on the additive commutative group  $(S, +)$ . Let  $a\rho_f a'$ ,  $b\rho_f b'$  and  $c\rho_f c'$ , where  $a, b, c, a', b', c' \in S$ . Then we have  $f(a) = f(a')$ ,  $f(b) = f(b')$  and  $f(c) = f(c')$ . Now  $f(a \circ b \circ c) = f(a) \circ f(b) \circ f(c)$  (Since  $f$  is a good homomorphism)  $= f(a') \circ f(b') \circ f(c') = f(a' \circ b' \circ c')$ . Now, for any  $x \in a \circ b \circ c$  there exists an element  $y \in a' \circ b' \circ c'$  such that  $f(x) = f(y)$  i.e.  $x\rho_f y$ . Also for any  $y' \in a' \circ b' \circ c'$  there exists an element  $x' \in a \circ b \circ c$  such that  $f(x') = f(y')$  i.e.  $x'\rho_f y'$ . This implies that  $(a \circ b \circ c)\overline{\rho_f}(a' \circ b' \circ c')$ . Hence we have shown that  $\rho_f$  is a regular equivalence on  $(S, +, \circ)$ . Obviously  $\rho_f$  is not a strongly regular relation on  $(S, +, \circ)$ .  $\square$

**Proposition 4.6.** *An equivalence  $\rho$  on a multiplicative ternary hyperring  $(S, +, \circ)$  is regular if and only if  $(S/\rho, +, \circ)$  is a multiplicative ternary hyperring, where  $S/\rho = \{a_\rho : a \in S\}$  and  $a_\rho$  is the equivalence class containing  $a$ ,  $a_\rho + b_\rho = (a + b)_\rho$  and  $a_\rho \circ b_\rho \circ c_\rho = \{x_\rho : x \in a \circ b \circ c\}$  for any  $a, b, c \in S$ .*

*Proof.* Let  $\rho$  be a regular equivalence on  $(S, +, \circ)$ . Then  $\rho$  is a congruence on the additive commutative group  $(S, +)$  and hence we have the quotient group  $(S/\rho, +)$ . Now we define a multiplicative ternary hyperoperation on  $S/\rho$  by  $a_\rho \circ b_\rho \circ c_\rho = \{x_\rho : x \in a \circ b \circ c\}$ . Let  $a_\rho = a'_\rho, b_\rho = b'_\rho$  and  $c_\rho = c'_\rho$ , where  $a, b, c, a', b', c' \in S$ . Then  $a\rho a', b\rho b'$  and  $c\rho c'$ . Since  $\rho$  is regular,  $(a \circ b \circ c)\overline{\rho}(a' \circ b' \circ c')$ . Let  $x_\rho \in (a_\rho \circ b_\rho \circ c_\rho)$ . Then  $x \in a \circ b \circ c$ . Since  $(a \circ b \circ c)\overline{\rho}(a' \circ b' \circ c')$ , there exists  $y \in (a' \circ b' \circ c')$  such that  $x\rho y$ . Then  $x_\rho = y_\rho \in a'_\rho \circ b'_\rho \circ c'_\rho$ . Hence  $a_\rho \circ b_\rho \circ c_\rho \subseteq a'_\rho \circ b'_\rho \circ c'_\rho$ . Similarly, we obtain  $a'_\rho \circ b'_\rho \circ c'_\rho \subseteq a_\rho \circ b_\rho \circ c_\rho$ . Hence  $a'_\rho \circ b'_\rho \circ c'_\rho = a_\rho \circ b_\rho \circ c_\rho$ . Thus ' $\circ$ ' is well-defined. Next let  $x_\rho \in (a_\rho \circ b_\rho \circ c_\rho) \circ d_\rho \circ e_\rho$ , where  $a, b, c, d, e \in S$ . Then  $x \in p \circ d \circ e$ , where  $p_\rho \in (a_\rho \circ b_\rho \circ c_\rho)$ . This implies  $p \in (a \circ b \circ c)$ . Thus  $x \in (a \circ b \circ c) \circ d \circ e = a \circ b \circ (c \circ d \circ e)$ . So  $x_\rho \in (a_\rho \circ b_\rho \circ y_\rho)$ , where  $y \in c \circ d \circ e$ . So  $y_\rho \in (c_\rho \circ d_\rho \circ e_\rho)$ . Thus  $x_\rho \in a_\rho \circ b_\rho \circ (c_\rho \circ d_\rho \circ e_\rho)$ . Thus  $(a_\rho \circ b_\rho \circ c_\rho) \circ d_\rho \circ e_\rho \subseteq a_\rho \circ b_\rho \circ (c_\rho \circ d_\rho \circ e_\rho)$ . Similarly we can prove the converse. Hence  $(a_\rho \circ b_\rho \circ c_\rho) \circ d_\rho \circ e_\rho = a_\rho \circ b_\rho \circ (c_\rho \circ d_\rho \circ e_\rho)$ . Similarly we can prove that  $a_\rho \circ (b_\rho \circ c_\rho \circ d_\rho) \circ e_\rho = a_\rho \circ b_\rho \circ (c_\rho \circ d_\rho \circ e_\rho)$ . So ' $\circ$ ' is associative. Lastly let  $x_\rho \in a_\rho \circ b_\rho \circ (c_\rho + d_\rho) = a_\rho \circ b_\rho \circ (c + d)_\rho$ , where  $a, b, c, d \in S$ . This



implies  $x \in a \circ b \circ (c+d) \subseteq a \circ b \circ c + a \circ b \circ d$ . So  $x = y+z$ , where  $y \in a \circ b \circ c$  and  $z \in a \circ b \circ d$ . Hence  $x_\rho = (y+z)_\rho = y_\rho + z_\rho$ , where  $y_\rho \in a_\rho \circ b_\rho \circ c_\rho$  and  $z_\rho \in a_\rho \circ b_\rho \circ d_\rho$ . Thus  $a_\rho \circ b_\rho \circ (c_\rho + d_\rho) \subseteq a_\rho \circ b_\rho \circ c_\rho + a_\rho \circ b_\rho \circ d_\rho$ . Similarly we can prove  $(a_\rho + b_\rho) \circ c_\rho \circ d_\rho \subseteq a_\rho \circ c_\rho \circ d_\rho + b_\rho \circ c_\rho \circ d_\rho$  and  $a_\rho \circ (b_\rho + c_\rho) \circ d_\rho \subseteq a_\rho \circ b_\rho \circ d_\rho + a_\rho \circ c_\rho \circ d_\rho$ , for  $a, b, c, d \in S$ . Thus  $(S/\rho, +, \circ)$  is a multiplicative ternary hyperring. Conversely, suppose that  $(S/\rho, +, \circ)$  is a multiplicative ternary hyperring. Since  $(S/\rho, +)$  is an additive commutative group,  $+$  is well defined. Let  $a_\rho b_\rho$  and  $c_\rho d_\rho$ . Then  $a_\rho = b_\rho$  and  $c_\rho = d_\rho$ . Hence  $(a+c)_\rho = a_\rho + c_\rho = b_\rho + d_\rho = (b+d)_\rho$ . Thus  $(a+c)\rho(b+d)$ . So  $\rho$  is a congruence on  $(S, +)$ . Again ' $\circ$ ' is well-defined. Let  $a_\rho a'_\rho, b_\rho b'_\rho$  and  $c_\rho c'_\rho$ , where  $a, b, c, a', b', c' \in S$ . Then  $a_\rho = a'_\rho, b_\rho = b'_\rho$  and  $c_\rho = c'_\rho$ . Hence  $a_\rho \circ b_\rho \circ c_\rho = a'_\rho \circ b'_\rho \circ c'_\rho$ . Now  $x \in a \circ b \circ c \Rightarrow x_\rho \in a_\rho \circ b_\rho \circ c_\rho = a'_\rho \circ b'_\rho \circ c'_\rho \Rightarrow x_\rho = y_\rho$ , where  $y \in a' \circ b' \circ c' \Rightarrow x\rho y$ , where  $y \in a' \circ b' \circ c'$ . Similarly, we can prove the converse. Hence  $(a \circ b \circ c)\bar{\rho}(a' \circ b' \circ c')$  and hence  $\rho$  is regular.  $\square$

**Definition 4.7.** For a regular equivalence relation  $\rho$  on a multiplicative ternary hyperring  $(S, +, \circ)$  the multiplicative ternary hyperring  $(S/\rho, +, \circ)$  is called the quotient ternary hyperring of  $(S, +, \circ)$  by  $\rho$ .

**Proposition 4.8.** Let  $f$  be a good homomorphism from a multiplicative ternary hyperring  $(S, +, \circ)$  to a multiplicative ternary hyperring  $(T, +, \circ)$ . Then  $(f(S), +, \circ)$  is also a multiplicative ternary hyperring.

*Proof.* Let  $f(a), f(b), f(c) \in f(S)$ , where  $a, b, c \in S$ . Then  $f(a) + f(b) = f(a+b) \in f(S)$ ,  $-f(a) = f(-a) \in f(S)$  and  $f(a) \circ f(b) \circ f(c) = f(a \circ b \circ c) \in f(S)$ . Hence  $(f(S), +, \circ)$  is a multiplicative ternary hyperring.  $\square$

**Proposition 4.9.** Let  $f$  be a homomorphism from a multiplicative ternary hyperring  $(S, +, \circ)$  to a multiplicative ternary hyperring  $(T, +, \circ)$ . Then  $(f^{-1}(T), +, \circ)$  is also a multiplicative ternary hyperring, where  $f^{-1}(T) = \{s \in S \mid f(s) \in T\}$ .

*Proof.* Obviously  $f^{-1}(T) \neq \phi$ , since  $0_s \in f^{-1}(T)$ . Let  $s_1, s_2 \in f^{-1}(T)$ . Then  $f(s_1), f(s_2) \in T$ . This implies  $f(s_1 - s_2) = f(s_1) - f(s_2) \in$

$T$ . Hence  $s_1 - s_2 \in f^{-1}(T)$ . Again let  $s_1, s_2, s_3 \in f^{-1}(T)$ . Then  $f(s_1), f(s_2), f(s_3) \in T$ . This implies that  $f(s_1 \circ s_2 \circ s_3) \subseteq f(s_1) \circ f(s_2) \circ f(s_3) \subseteq T$ . Thus if  $x \in s_1 \circ s_2 \circ s_3$  then  $f(x) \in f(s_1 \circ s_2 \circ s_3) \subseteq T$ . Hence  $x \in f^{-1}(T)$ . So  $s_1 \circ s_2 \circ s_3 \subseteq f^{-1}(T)$ . The other properties required to define a multiplicative ternary hyperring are obviously satisfied. So  $(f^{-1}(T), +, \circ)$  is a multiplicative ternary hyperring.  $\square$

*Remark 4.10.* Let  $f$  be a good homomorphism from a multiplicative ternary hyperring  $(S, +, \circ)$  to a multiplicative ternary hyperring  $(T, +, \circ)$ . Then from Proposition 4.5. it follows that  $\rho_f$  is regular and hence from Proposition 4.6. it follows that  $(S/\rho_f, +, \circ)$  is a multiplicative ternary hyperring.

We now state some theorems of multiplicative ternary hyperrings. The following theorem is the first isomorphism theorem of multiplicative ternary hyperrings.

**Theorem 4.11.** *Let  $f$  be a good homomorphism from a multiplicative ternary hyperring  $(S, +, \circ)$  to a multiplicative ternary hyperring  $(T, +, \circ)$ . Then the multiplicative ternary hyperring  $(f(S), +, \circ)$  is isomorphic to the multiplicative ternary hyperring  $(S/\rho_f, +, \circ)$ .*

*Proof.* Define a map  $\phi : f(S) \rightarrow S/\rho_f$  by  $\phi(f(a)) = a_{\rho_f}$  for all  $a \in S$ . Now  $f(a) = f(b), a, b \in S \Leftrightarrow a_{\rho_f} = b_{\rho_f} \Leftrightarrow \phi(f(a)) = \phi(f(b))$ . Hence  $\phi$  is well-defined and injective. Obviously  $\phi$  is surjective. Let  $a, b \in S$ . Now  $\phi(f(a) + f(b)) = \phi(f(a + b)) = (a + b)_{\rho_f} = a_{\rho_f} + b_{\rho_f}$  (since  $\rho_f$  is a congruence on  $(S, +)$ )  $= \phi(f(a)) + \phi(f(b))$ . Further  $\phi(f(a) \circ f(b) \circ f(c)) = \phi(f(a \circ b \circ c))$  (since  $f$  is a good homomorphism)  $= \phi(f(x) : x \in a \circ b \circ c) = \{x_{\rho_f} : x \in a \circ b \circ c\} = a_{\rho_f} \circ b_{\rho_f} \circ c_{\rho_f} = \phi(f(a)) \circ \phi(f(b)) \circ \phi(f(c))$  for  $a, b, c \in S$ . Thus  $\phi$  is an isomorphism. Hence  $(f(S), +, \circ)$  is isomorphic to  $(S/\rho_f, +, \circ)$ .  $\square$

The notion of strongly regular equivalence relation plays an important role in the theory of multiplicative ternary hyperring. In fact, starting

with a multiplicative ternary hyperring and applying a strongly regular equivalence relation on it, we can easily construct a ternary ring structure on the quotient set.

**Theorem 4.12.** *An equivalence relation  $\rho$  on a multiplicative ternary hyperring  $(S, +, \circ)$  is strongly regular if and only if  $(S/\rho, +, \circ)$  is a ternary ring.*

*Proof.* Suppose that  $\rho$  is a strongly regular equivalence relation on  $(S, +, \circ)$ . Since a strongly regular equivalence relation is regular, by Proposition 4.6. it follows that  $(S/\rho, +, \circ)$  is a multiplicative ternary hyperring. Now we proceed to show that  $|a_\rho \circ b_\rho \circ c_\rho| = 1$  for all  $a, b, c \in S$ . Let  $x_\rho, y_\rho \in a_\rho \circ b_\rho \circ c_\rho$ . Then  $x, y \in a \circ b \circ c$ . Since  $a\rho a, b\rho b$  and  $c\rho c$  hold, we have  $(a \circ b \circ c)\bar{\rho}(a \circ b \circ c)$ . Since  $\rho$  is strongly regular,  $x\rho y$  which implies that  $x_\rho = y_\rho$ . Thus  $|a_\rho \circ b_\rho \circ c_\rho| = 1$ . This shows that  $(S/\rho, +, \circ)$  is a ternary ring. Conversely let  $(S/\rho, +, \circ)$  be a ternary ring. Obviously  $\rho$  is a congruence on  $(S, +)$ . Let  $a\rho a', b\rho b'$  and  $c\rho c'$ . Since the hyperoperation ‘ $\circ$ ’ on  $S/\rho$  is well defined. So  $a_\rho \circ b_\rho \circ c_\rho = a'_\rho \circ b'_\rho \circ c'_\rho$ . Since  $|x_\rho \circ y_\rho \circ z_\rho| = 1$  for all  $x, y, z \in S$ , it follows that  $|a_\rho \circ b_\rho \circ c_\rho| = 1 = |a'_\rho \circ b'_\rho \circ c'_\rho|$  (1). Let  $x \in a \circ b \circ c$ . Then  $x_\rho \in a_\rho \circ b_\rho \circ c_\rho$ . Since  $|a_\rho \circ b_\rho \circ c_\rho| = 1$ ,  $a_\rho \circ b_\rho \circ c_\rho = x_\rho$ . Let  $y \in a' \circ b' \circ c'$ . Then, as above,  $a'_\rho \circ b'_\rho \circ c'_\rho = y_\rho$ . Now by (1) we have  $x_\rho = y_\rho$  i.e.  $x\rho y$ . This is true for all  $x \in a \circ b \circ c$  and for all  $y \in a' \circ b' \circ c'$ . So  $(a \circ b \circ c)\bar{\rho}(a' \circ b' \circ c')$ . This shows that  $\rho$  is strongly regular.  $\square$

**Theorem 4.13.** *Let  $f$  be a good homomorphism from a multiplicative ternary hyperring  $(S, +, \circ)$  to a multiplicative ternary hyperring  $(T, +, \circ)$ . Then the equivalence relation  $\rho_f$  induced by  $f$  on  $S$  is strongly regular if and only if the multiplicative ternary hyperring  $(f(S), +, \circ)$  becomes a ternary ring.*

*Proof.* The proof follows from Theorem 4.11. and Theorem 4.12.  $\square$

For strongly regular equivalences, we have the following theorem.

**Lemma 4.14.** *Let  $\{\rho_i : i \in I\}$  be a set of strongly regular equivalence relations on a multiplicative ternary hyperring  $(S, +, \circ)$ . Then  $\cap \rho_i$  is a strongly regular equivalence relation on  $(S, +, \circ)$ .*

*Proof.* First we notice that  $\rho = \cap \rho_i$  is a congruence on  $(S, +)$ . Let  $a\rho a', b\rho b', c\rho c'$  for  $a, b, c, a', b', c' \in S$ . Then  $a\rho_i a', b\rho_i b', c\rho_i c'$  for all  $i \in I$ . Since each  $\rho_i$  is strongly regular,  $(a \circ b \circ c) \overline{\rho_i}(a' \circ b' \circ c')$ . This implies that  $x\rho_i y$  for all  $i \in I$ , for all  $x \in a \circ b \circ c$  and for all  $y \in a' \circ b' \circ c'$ . Hence  $x(\bigcap_{i \in I} \rho_i)y$  i.e.  $x\rho y$ . This leads to  $(a \circ b \circ c) \overline{\rho}(a' \circ b' \circ c')$ . Hence  $\rho = \cap \rho_i$  is strongly regular.  $\square$

In the following theorem, we consider the lattice structure of the set of strongly regular equivalences on a multiplicative ternary hyperring.

**Theorem 4.15.** *The set of all strongly regular equivalence relations on a multiplicative ternary hyperring  $(S, +, \circ)$  forms a complete lattice w.r.t. the set inclusion.*

*Proof.* Let  $T = \{\rho_i : i \in I\}$  be the set of all strongly regular equivalence relations on  $(S, +, \circ)$ . Obviously  $T$  is a poset w.r.t the set-inclusion with the greatest element  $S \times S$ . Let  $T' = \{\rho_j : j \in J \subseteq I\}$  be a nonempty subset of  $T$ . Then by the above lemma  $\bigcap_{j \in J} \rho_j$  is a strongly regular equivalence relation on  $(S, +, \circ)$  and it is the glb of  $T'$ . Consequently  $T = \{\rho_i : i \in I\}$  forms a complete lattice w.r.t. the set-inclusion.  $\square$

**Definition 4.16.** Let  $\theta$  and  $\phi$  be two regular equivalence relations on a multiplicative ternary hyperring  $(S, +, \circ)$  such that  $\theta \subseteq \phi$ . We define a relation  $\phi/\theta$  on  $S/\theta$  as follows:  $(a_\theta)\phi/\theta(b_\theta)$  if and only if  $a\phi b$  for  $a, b \in S$ .

For regular equivalence, we have the following lemma.

**Lemma 4.17.**  *$\phi/\theta$  is a regular equivalence relation on the quotient ternary hyperring  $(S/\theta, +, \circ)$ .*

*Proof.* Obviously  $\phi/\theta$  is an equivalence relation on the quotient set  $S/\theta$ . Now  $(a_\theta)\phi/\theta(a'_\theta)$  and  $(b_\theta)\phi/\theta(b'_\theta)$ , where  $a, b, a', b' \in S$

$$\begin{aligned} &\Rightarrow a\phi a' \text{ and } b\phi b' \\ &\Rightarrow (a+b)\phi(a'+b') \text{ since } \phi \text{ is a congruence on } (S, +) \\ &\Rightarrow (a+b)_\theta(\phi/\theta)(a'+b')_\theta \\ &\Rightarrow (a_\theta + b_\theta)(\phi/\theta)(a'_\theta + b'_\theta) \\ &\Rightarrow \phi/\theta \text{ is a congruence on } (S/\theta, +) \end{aligned}$$

Again  $a_{i_\theta}(\phi/\theta)b_{i_\theta}, i = 1, 2, 3$  and  $a_i, b_i \in S$ .

$$\begin{aligned} &\Rightarrow a_i\phi b_i, i = 1, 2, 3 \\ &\Rightarrow (a_1 \circ a_2 \circ a_3)\bar{\phi}(b_1 \circ b_2 \circ b_3) \quad (1) \end{aligned}$$

since  $\phi$  is a regular equivalence on  $S$ . Let  $x_\theta \in a_{1_\theta} \circ a_{2_\theta} \circ a_{3_\theta}$ .

Then  $x \in a_1 \circ a_2 \circ a_3$ . So by (1) there exists an element  $y \in b_1 \circ b_2 \circ b_3$  such that  $x\phi y$ . This implies that  $(x_\theta)\phi/\theta(y_\theta)$ . So for each element  $x_\theta \in a_{1_\theta} \circ a_{2_\theta} \circ a_{3_\theta}$  there exists an element  $y_\theta \in b_{1_\theta} \circ b_{2_\theta} \circ b_{3_\theta}$  such that  $(x_\theta)\phi/\theta(y_\theta)$ . Similar is the converse. So  $(a_{1_\theta} \circ a_{2_\theta} \circ a_{3_\theta})\bar{\phi}/\theta(b_{1_\theta} \circ b_{2_\theta} \circ b_{3_\theta})$ . So  $\phi/\theta$  is a regular equivalence relation on  $(S/\theta, +, \circ)$ .  $\square$

The following theorem is the second isomorphism theorem of multiplicative ternary hyperrings.

**Theorem 4.18.** *Let  $(S, +, \circ)$  be a multiplicative ternary hyperring and  $\theta, \phi$  be two regular equivalence relations on  $S$  such that  $\theta \subseteq \phi$ . Then the quotient ternary hyperrings  $((S/\theta)/(\phi/\theta), +, \circ)$  and  $(S/\phi, +, \circ)$  are isomorphic.*

*Proof.* We define a mapping  $f : S/\theta \rightarrow S/\phi$  by  $f(a_\theta) = a_\phi$  for all  $a \in S$ . Now  $a_\theta = b_\theta \Rightarrow a\theta b \Rightarrow a\phi b$  (since  $\theta \subseteq \phi$ )  $\Rightarrow a\phi = b\phi \Rightarrow f(a_\theta) = f(b_\theta)$ . So  $f$  is well defined. Obviously the mapping  $f$  is surjective. Now, we consider the following equality.

$$\begin{aligned}
f(a_\theta \circ b_\theta \circ c_\theta) &= f(\{x_\theta : x_\theta \in a_\theta \circ b_\theta \circ c_\theta\}) \\
&= f(\{x_\theta : x \in a \circ b \circ c\}) \\
&= \{f(x_\theta) : x \in a \circ b \circ c\} \\
&= \{x_\phi : x \in a \circ b \circ c\} \\
&= a_\phi \circ b_\phi \circ c_\phi \\
&= f(a_\theta) \circ f(b_\theta) \circ f(c_\theta)
\end{aligned}$$

Thus  $f$  is a good homomorphism. Hence by Theorem 4.11.  $((S/\theta)/\rho_f, +, \circ) \cong (S/\phi, +, \circ)$ , where  $\rho_f$  is the regular equivalence relation on  $S/\theta$ , defined by  $(a_\theta)\rho_f(b_\theta)$  iff  $f(a_\theta) = f(b_\theta)$ . Now  $(a_\theta)\rho_f(b_\theta) \Leftrightarrow f(a_\theta) = f(b_\theta) \Leftrightarrow a_\phi = b_\phi \Leftrightarrow a\phi b \Leftrightarrow (a_\theta)(\phi/\theta)(b_\theta)$ . Hence  $\rho_f = \phi/\theta$ . Thus  $((S/\theta)/(\phi/\theta), +, \circ) \cong (S/\phi, +, \circ)$ .  $\square$

**Proposition 4.19.** *Let  $(T, +, \circ)$  be a multiplicative subternary hyperring of a multiplicative ternary hyperring  $(S, +, \circ)$  and  $\theta$  be a regular equivalence relation on  $(S, +, \circ)$ . Then the following hold.*

- (i)  $T^\theta = \{a \in S : \text{there exists } b \in T \text{ such that } a\theta b\}$  is a subternary hyperring of  $(S, +, \circ)$ ;
- (ii)  $\theta_T = \theta \cap T^2$  is a regular equivalence relation on  $(T, +, \circ)$ .

*Proof.* (i) Obviously  $T \subseteq T^\theta$  and hence  $T^\theta \neq \phi$ . Let  $a_1, a_2 \in T^\theta$ . Then there exist  $b_1, b_2 \in T$  such that  $a_1\theta b_1$  and  $a_2\theta b_2$ . So  $(a_1 - a_2)\theta(b_1 - b_2)$ . Hence  $a_1 - a_2 \in T^\theta$ . Let  $a_1, a_2, a_3 \in T^\theta$ . Then  $a_i\theta b_i$  for  $i = 1, 2, 3$  for some  $b_1, b_2, b_3 \in T$ . So  $(a_1 \circ a_2 \circ a_3)\bar{\theta}(b_1 \circ b_2 \circ b_3)$  (since  $\theta$  is a regular equivalence relation on  $S$ ). Let  $x \in a_1 \circ a_2 \circ a_3$ . By (1) there exists an element  $y \in b_1 \circ b_2 \circ b_3$  such that  $x\theta y$ , where  $y \in T$ . So  $x \in T^\theta$ . Thus  $a_1 \circ a_2 \circ a_3 \in P^*(T^\theta)$ . Hence  $T^\theta$  is a subternary hyperring of  $(S, +, \circ)$ .  
(ii) Obviously  $\theta_T$  is an equivalence relation on  $T$ . Now  $a_1\theta_T b_1$  and  $a_2\theta_T b_2, a_1, a_2, b_1, b_2 \in T$

$$\begin{aligned}
&\Rightarrow a_1\theta b_1, a_2\theta b_2, (a_1, b_1), (a_2, b_2) \in T^2 \\
&\Rightarrow (a_1 + a_2)\theta(b_1 + b_2), (a_1 + a_2, b_1 + b_2) \in T^2 \\
&\Rightarrow (a_1 + a_2)\theta_T(b_1 + b_2).
\end{aligned}$$

Let  $a_i\theta_T b_i$  for  $i = 1, 2, 3$ . Then  $a_i\theta b_i$  and  $(a_i, b_i) \in T^2$   
 $\Rightarrow (a_1 \circ a_2 \circ a_3)\overline{\theta}(b_1 \circ b_2 \circ b_3)$  and  $((a_1 \circ a_2 \circ a_3), (b_1 \circ b_2 \circ b_3)) \in (P^*(T))^2$ .  
 $\Rightarrow (a_1 \circ a_2 \circ a_3)\overline{\theta_T}(b_1 \circ b_2 \circ b_3)$ .  
 Thus  $\theta_T$  is a regular equivalence relation on  $(T, +, \circ)$ .  $\square$

Finally, we prove the third isomorphism theorem of multiplicative ternary hyperrings.

**Theorem 4.20.** *Let  $(T, +, \circ)$  be a multiplicative subternary hyperring of the multiplicative ternary hyperring  $(S, +, \circ)$ . Let  $\theta$  be a regular equivalence relation on  $(S, +, \circ)$ . Then  $(T^\theta/\theta_{T^\theta}, +, \circ) \cong (T/\theta_T, +, \circ)$ .*

*Proof.* We define a mapping  $f : T \rightarrow T^\theta/\theta_{T^\theta}$  by  $f(x) = x\theta_{T^\theta}$ . Let  $y\theta_{T^\theta} \in T^\theta/\theta_{T^\theta}$ . Then  $y \in T^\theta$ . This implies that there exists  $x \in T(\subseteq T^\theta)$  such that  $(x, y) \in \theta$ . Again  $(x, y) \in (T^\theta)^2$ . So  $(x, y) \in \theta \cap (T^\theta)^2 = \theta_{T^\theta}$ . So  $y\theta_{T^\theta} = x\theta_{T^\theta} = f(x)$ . Thus  $f$  is surjective. Let  $x_1, x_2 \in T$ . Then  $(f(x_1 + x_2) = (x_1 + x_2)\theta_{T^\theta} = x_1\theta_{T^\theta} + x_2\theta_{T^\theta} = f(x_1) + f(x_2)$ . Let  $x_1, x_2, x_3 \in T$ . Now  $f(x_1 \circ x_2 \circ x_3) = f(\{y : y \in x_1 \circ x_2 \circ x_3\}) = \{f(y) : y \in x_1 \circ x_2 \circ x_3\} = \{y\theta_{T^\theta} : y \in x_1 \circ x_2 \circ x_3\} = x_1\theta_{T^\theta} \circ x_2\theta_{T^\theta} \circ x_3\theta_{T^\theta} = f(x_1) \circ f(x_2) \circ f(x_3)$ . Thus  $f$  is a good homomorphism. Hence by Theorem 4.11,  $T/\rho_f \cong T^\theta/\theta_{T^\theta}$ . Now for  $x, y \in T, x\rho_f y \Leftrightarrow f(x) = f(y) \Leftrightarrow x\theta_{T^\theta} = y\theta_{T^\theta} \Leftrightarrow (x, y) \in \theta_{T^\theta} \Leftrightarrow (x, y) \in \theta$  and  $(x, y) \in T^2(x, y \in T) \Leftrightarrow x\theta_T = y\theta_T \Leftrightarrow x\theta_T y$ . So  $\rho_f = \theta_T$ . Hence  $T^\theta/\theta_{T^\theta}, +, \circ) \cong (T/\theta_T, +, \circ)$ .  $\square$

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