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ON GENERAL n-ARY HYPERSTRUCTURE SEMILATTICES

AKBAR DEHGHAN NEZHAD AND NAJMEH KHAJUEE

Abstract. In this paper, the n-ary hyperstructure will be applied to some aspects of lattice theory. We introduce the concepts of general n-ary hyperstructure semilattice (or **gnh**-semilattice) and **Gnh**-subsemilattice, ideal of **gnh**-semilattice, **gno**-order, **Gno**-order, multiplier of type α on **gnh**-semilattice, F-quasi invariant subset of **gnh**-semilattice and so on. We also study some of their related properties.

Key Words: **gnh**-semilattice, **Gnh**-semilattice, **Gno**-order, multiplier of type α on **gnh**-semilattice.

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1. Introduction and preliminaries

Hyperstructure theory was born in 1934 when Marty [17] defined hypergroups as a generalization of groups. Eighty years have elapsed since Martys pioneer paper. During this period, numerous papers on algebraic hyperstructures have been published, the field has experimented an enormous growth. A recent book [6] contains a wealth of applications. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, and probabilistic. H_v-structures were for the first time introduced by Vougiouklis in Fourth AHA congress (1990) [22]. The concept of H_v-structures constitute a generalization of the well-known algebraic hyperstructures (hypergroup,

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∗Address correspondence to Akbar Dehghan Nezhad; E-mail: dehghannezhad@iust.ac.ir

hyperring, hypermodule and so on). Actually some axioms concerning the above hyperstructures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. In [7], Davvaz surveyed the theory of H_v-structures. Hyperlattices were for the first time introduced by Konstantinidou and Mittas [15]. The concept of hyperlattice is a generalization of the concept of lattice [2]. Other contributor to the development of hyperlattice theory were Konstantinidou [11, 12, 14, 16, 15, 13], Ashrafi [1], Rahnamai-Barghi [19, 20], Xiao and Zhao [23]. In [8] Dehghan Nezhad and Davaze introuced the concept of H_v-semilattice and study some of their related properties. In 2011, K.H. Kim [10] introduced and studied the properties of multipliers in BE-algebras and in [21] M. Sambasiva Rao introduced the notation multiplier of hypersemilattice and also studied some properties of multipliers.

The paper is organized as follows. In section 1, we present definitions weak associative and intersection commutative for general n-ary hyperstructure. In section 2, firstly, we define quasi **gnh**-semilattice. It is shown that if $(L, *_n)$ be a **gnh**-semilattice then $(P^*(L), \bigotimes_n)$ is a **Gnh**-semilattice. Also we introduce **gnh** on cartesian product of **gnh**-semilattice. We prove the direct product of n **gnh**-semilattice is a **gnh**-semilattice on their cartesian products. In sections 3 and 4, we define **gnh**-subsemilattice and ideal of **gnh**-semilattice. The image and inverse images of subsemilattice (ideal) under a strong homomorphism are studied. In section 5, we define a general n-order (**gno**) on $(L, *_n)$ and a **Gno** (General n-order) on $(P^*(L), \bigotimes_n)$. We investigate connections between **gno** and **Gno**. Finally in section 6, we introduce and study the properties of multipliers of type α in **gnh**-semilattice (**Gnh**-semilattice). We will study image and inverse images of ideals under a multiplier of type α of **gnh**-semilattice.
Let ω be the smallest infinite countable ordinal. We consider the smallest infinite ordinal ω as the set of all smaller ordinals, i.e. as the domain of all finite ordinals (non-negative integers).

Definition 1.1. [5, 6]. Let $\{X_k; k \in \omega\}$ be a system of non-empty sets. The *general* $\omega -$ *hyperstructure*, we mean the pair $(\{X_k; k \in \omega\}, *_\omega)$, where $*_\omega : \prod_{k\in\omega} X_k \longrightarrow P^*(\bigcup_{k\in\omega} X_k)$ is a mapping assigning to any sequence $\{x_k\}_{k\in\omega} \in \prod_{k\in\omega} X_k$ a non-empty subset $*_\omega(\{x_k\}_{k\in\omega}) \subset \bigcup_{k\in\omega} X_k$. Similarly as above, with hyperoperation is associted a mapping of power sets

$$\bigotimes_{\omega} : \prod_{k\in\omega} P^*(X_k) \longrightarrow P^*(\bigcup_{k\in\omega} X_k)$$

defined by

$$\bigotimes_{\omega}(\{A_k\}_{k\in\omega}) =$$

$$\bigcup\{*_\omega(\{x_k\}_{k\in\omega}); (\{x_k\}_{k\in\omega}) \in \prod_{k\in\omega} A_k, A_k \in P^*(X_k), k \in \omega\}.$$

Let us formulate the special case:

Definition 1.2. [4, 5, 6]. Let $n \in \omega$ be an arbitrary positive integer, $n \geq 1$. Let $\{X_k; k = 1, ..., n\}$ be a system of non-empty sets. By a *general* $n - ary$ *hyperstructure* we mean the pair $(\{X_k; k = 1, ..., n\}, *_n)$, where

$$*_n : \prod_{k=1}^{n} X_k \longrightarrow P^*(\bigcup_{k=1}^{n} X_k)$$

is a mapping assigning to any n-tuple $(x_1, ..., x_n) \in \prod_{k=1}^{n} X_k$ a non-empty subset $*_n(x_1, ..., x_n) \subset \bigcup_{k=1}^{n} X_k$.
Similarly as above, with this hyperoperation there is associated a mapping of power sets
$\prod_{k=1}^{n} P^*(X_k) \longrightarrow P^*(\bigcup_{k=1}^{n} X_k)$ defined by

$$\bigotimes_{n}(A_1, ..., A_n) =$$

$$\bigcup\{*_n(x_1, ..., x_n); (x_1, ..., x_k) \in \prod_{k=1}^{n} A_k, A_k \in P^*(X_k), k = 1, ..., n\}.$$

This construction is based on an idea of Nezhad and Hashemi [9] for $n = 2$. Hyperstructures with n-ary hyperoperations are investigated among others in [3, 18].
We shall use the following abbreviated notation, the sequence $x_i, x_{i+1}, ..., x_j$ will be denoted by x_i^j, $x_i^i = x_i$ for $j < i$, x_i^j is the empty set. In this convention $*_n(x_1, ..., x_i, y_{i+1}, ..., y_j, z_{j+1}, ..., z_n)$ will be written as $*_n(x_1^i, y_{i+1}^j, y_{j+1}^n)$.
We called $*_n$ **gnh** (general n-ary hyperstructure) and $\bigotimes_n$ **Gnh** (General n-ary hyperstructure). When $X_1 = ... = X_n = X$, $*_n$ is called **gnh** on X and $\bigotimes_n$ is called **Gnh** on $P^*(X)$.

Let $\bigcap_{k=1}^{n} X_k \neq \varnothing$ and S_n be permutation group of order n. We say **gnh** is intersection commutative if for all $\sigma \in S_n$ and $x_1^n \in \bigcap_{k=1}^{n} X_k$

$$*_n(x_1^n) = *_n(x_{\sigma(1)}^{\sigma(n)}).$$

If $*_n$ is a **gnh** on X, we say that $*_n$ is commutative. Let $x \in X$ and we denote $*_n(x, ..., x)$ with $*_n(x)$. In the following definition X_k^l means $X_k, ..., X_l$ for $k, l \in \mathbb{N}, k < l$.

Definition 1.3. A **gnh** is associative whenever for any $x_1, ..., x_{2n-1}$ that $x_1 \in X_1$, $x_i \in X_1^i$ for $i = 2, ..., n$, $x_{n+j} \in X_{j+1}^n$ for $j = 1, ..., n-1$ and $x_{2n-1} \in X_n$.
For any $i, j \in \{1, 2, ..., n\}$, we have

$$*_n\big(x_1^{i-1}, *_n(x_i^{n+i-1}), x_{n+i}^{2n-1}\big) = *_n\big(x_1^{j-1}, *_n(x_j^{n+j-1}), x_{n+j}^{2n-1}\big).$$

A **gnh** is week associative if $\bigcap_{i=1}^{2n-1} *_n(x_1^{i-1}, *_n(x_i^{n+i-1}), x_{n+i}^{2n-1}) \neq \varnothing$.

Remark 1.4. We have similar definition for **gnh** on X.

2. Quasi gnh-semilattice

Definition 2.1. Let $L_1, ..., L_n$ be non-empty sets with a **gnh** $*_n$. Let $\bigcap_{k=1}^{n} L_k = L$ such that the following conditions hold:
i) $a \in *_n(a)$ for all $a \in \prod_{k=1}^{n} L_k$ (idempotent),
ii) intersection commutative (or commutative),
iii) weak associative.
Then $(L, *_n)$ is called a quasi **gnh**-semilattice (or **gnh**-semilattice).

Example 2.2. Let L_m denote the space of $m \times m$ real matrices. We define a **gnh**, $*_n$ on L_m as follows:
$*_n(A_1^n) = \{A_1, ..., A_n, (A_1)^2, ..., (A_n)^2\}$ for all $A_1^n \in L_m$. For any $A_1^{2n-1} \in L_m$, we have
i) $A_1 \in *_n(A_1) = \{A_1, (A_1)^2\}$,
ii) it is clear that $*_n$ is commutative,
iii) it is easy to check that

$$A_1^n \in \bigcap_{i=1}^{2n-1} *_n\big(A_1^{i-1}, *_n(A_i^{n+i-1}), A_{n+i}^{2n-1}\big) \neq \varnothing.$$

Therefore $(L_m, *_n)$ is a **gnh**-semilattice.

Example 2.3. Let K be a non-empty set. We define a **gnh**, $*_n$ on K as follows: $*_n(k_1^n) = \{k_1^n\}$ for all $k_1^n \in K$. Then $(K, *_n)$ is **gnh**-semilattice.

Example 2.4. Consider the classical differential ring of real functions $g \in C^\infty(I)$, $I = (a, b) \subseteq \mathbb{R}$ (not excluding the case $I = \mathbb{R}$) with the usual differentiation. For any $g_1^n \in C^\infty(I)$ we define a **gnh**, $*_n$ on the ring $C^\infty(I)$ by, for all $x \in I$, $*_n(g_1^n) = \{g_1, ..., g_n, g'_1, ..., g'_n\}$. It is obvious that $*_n$ is idempotent and commutative. For any $g_1^{2n-1} \in C^\infty(I)$, we have

$$g_1^n \in \bigcap_{i=1}^{2n-1} *_n(g_1^{i-1}, *_n(g_i^{n+i-1}), g_{n+i}^{2n-1}) \neq \varnothing.$$

Therefore $(C^\infty(I), *_n)$ is a **gnh**-semilattice.

Proposition 2.5. *Let* $(L, *_n)$ *be a* **gnh***-semilattice then* $(P^*(L), \bigotimes_n)$ *is a* **Gnh***-semilattice*

Proof. For all $A \in P^*(L)$ we have $A \in \bigotimes_n A = \bigcup_{a\in A} *_n(a)$. So $\bigotimes_n$ is idempotent.
For all $A_1^{2n-1} \in P^*(L)$ and $a_i \in A_i,\ i = 1, ..., 2n-1$,

$$\bigotimes_n (A_1^n) = \bigcup_{a_i \in A_i} *_n(a_1^n) = \bigcup *_n\big(a_{\sigma(1)}^{\sigma(n)}\big) = \bigotimes_n \big(A_{\sigma(1)}^{\sigma(n)}\big)$$

thus $\bigotimes_n$ is commutative.

$$\bigcap_{i=1}^{2n-1} \bigotimes_n (A_1^{i-1}, \bigotimes_n (A_i^{n+i-1}), A_{n+i}^{2n-1}) =$$

$$\bigcap_{i=1}^{2n-1} \bigcup_{a_i \in A_i} *_n(a_1^{i-1}, *_n(a_i^{n+i-1}), a_{n+i}) \neq \varnothing.$$

Therefore $\bigotimes_n$ is weak associative. □

Definition 2.6. Let $*_n$ and $*'_n$ be two **gnh** on L. We call $*_n$ the dual of $*'_n$ if and only if for all $l_1^n \in L$, $*_n(l_1^n) = *'_n(l_n, ..., l_1)$. Similarly let $\bigotimes_n$ and $\bigotimes'_n$ be two **Gnh** on $P^*(L)$ then $\bigotimes_n$ is dual $\bigotimes'_n$ if and only if for all $A_1^n \in P^*(L)$, $\bigotimes_n(A_1^n) = \bigotimes'_n(A_n, ..., A_1)$.

Proposition 2.7. *I)* $(L, *_n)$ *is a* **gnh***-semilattice if and only if* $(L, *'_n)$ *is a* **gnh***-semilattice.*
II) $(P^*(L), \bigotimes_n)$ *is a* **Gnh***-semilattice if and only if* $(P^*(L), \bigotimes'_n)$ *is a* **Gnh***-semilattice.*

Proof. By Definitions 2.1, 2.6 and Proposition 2.5, the proof is clear. □

Let $(L, *_n)$, $(L', *'_n)$ be two **gnh**-semilattices and for all $l_1^n \in L$ a map $f : L \longrightarrow L'$ is called a weak homomorphism if $f(*_n(l_1^n)) \cap *'_n(f(l)_1^n) \neq \varnothing$ wherever $f(l)_i = f(l_i)$, $i = 1, .., n$. To shorten notation, we write, $*'_n(f(l)_1^n) = *'_n(f(l)_1, ..., f(l_n)) = *'_n(f(l_1), .., f(l_n))$.
The map f is called a inclusion homomorphism if $f(*_n(l_1^n)) \subseteq *'_n(f(l)_1^n)$. Finally, the function f is called a strong homomorphism (preserving **gnh**)

if $f(*_n(l_1^n)) = *'_n(f(l)_1^n)$. If f is a onto, one to one and strong homomorphism, then it is called isomorphism. If moreover f defined on the same **gnh**-semilattice then it is called automorphism. It is verification that the set of all automorphism in L, written $Aut(L)$, is a group. If f is injective as a map of sets, then f is said to be a monomorphism. If f is surjective, then f is called an epimorphism.
If $f : L \longrightarrow L'$ and $g : L' \longrightarrow L''$ are homomorphisms of **gnh**-semilattices, it easy to see that $g \circ f : L \longrightarrow L''$, is also a homorphism. Likewise the composition of monomorphism is also a monomorphism, similarly to epimorphisms and isomorphisms.

Proposition 2.8. *I) Let $(L, *_n)$ be a* **gnh***-semilattice and L' be a non-empty set with a* **gnh** *$*'_n$. If a function $f : L \longrightarrow L'$ is surjective and strong homomorphism, then $(L', *'_n)$ is a* **gnh***-semilattice.*
II) Let ψ_1 and ψ_2 be two strong homomorphism of **gnh***-semilattice L upon* **gnh***-semilattice $(L', *'_n)$ and $(L'', *''_n)$ respectively, such that $\psi_1^{-1} \circ \psi_1 \subseteq \psi_2^{-1} \circ \psi_2$. Then, a unique strong homomorphism φ of L' upon L'' such that $\varphi \circ \psi_1 = \psi_2$, exists.*

Proof. I) For all $a_1^{2n-1} \in L'$ we have
i) $a_1 = f(a) \in f(*_n(a)) = *'_n(f(a)) = *'_n(a_1)$, i.e. $a_1 \in *'_n(a_1)$.
ii) $*'_n(a_1^n) = *'_n((f(b)_1^n)) = f(*_n(b_1^n)) = f\big(*_n (b_{\sigma(1)}^{\sigma(n)})\big) = *'_n\big(f(b)_{\sigma(1)}^{\sigma(n)}\big) = *'_n\big(a_{\sigma(1)}^{\sigma(n)}\big)$.
iii)

$$\bigcap_{i=1}^{2n-1} *'_n(a_1^{i-1}, *'_n(a_i^{n+i-1}), a_{n+i}^{2n-1}) =$$

$$\bigcap_{i=1}^{2n-1} *'_n\big(f(b)_1^{i-1}, *'_n(f(b)_i^{n+i-1}), f(b)_{n+i}^{2n-1}\big) =$$

$$\bigcap_{i=1}^{2n-1} f((*_n(b_1^{i-1}, *_n(b_i^{n+i-1}), b_{n+i}^{2n-1})) \neq \varnothing.$$

Since we have $*_n(b_1^{i-1}, *_n(b_i^{n+i-1}), b_{n+i}^{2n-1}) \neq \varnothing$ for all $b_1^{2n-1} \in L$.
II) We show that φ is a strong homomorphism of L' upon L''. For all $a_1^n \in L'$ we have $\varphi(*'(a_1^n)) = \varphi(*'_n(\psi_1(l)_1^n)) = \varphi(\psi_1(*_n(l_1^n))) = \psi_2(*_n(l_1^n)) = *''_n(\psi_2(l)_1^n) = *''_n(\varphi(\psi_1(l)_1^n)) = *''_n(\varphi(a)_1^n)$. □

Definition 2.9. Let $(L_k, *_{n_k}), k = 1, ..., n$ be n **gnh**-semilattices and $(l_{ik})_{k=1}^n \in L_i$, $i = 1, ..., n$. The map $\Pi_n : \prod_{k=1}^n L_k \longrightarrow P^*(\prod_{k=1}^n L_k)$ is a **gnh** on cartesian product $\prod_{k=1}^n L_k$ as follows:

$$\Pi_n\Big((l_{k1})_{k=1}^n, ..., (l_{kn})_{k=1}^n\Big) = \{(c_1^n) | c_i \in *_{n_i}(l_{ik})_{k=1}^n, i = 1, ..., n\}.$$

Definition 2.10. Let $(L, *_n)$ be a **gnh**-semilattice ρ an equivalence relation on L and $\rho(l)$ the ρ-equivalence class of the element $l \in L$. In L/ρ consider the **gnh** $*'_n$ defined on the usual manner:

$$*'_n(\rho(l)_1^n) = \{\rho(\xi) | \xi \in *_n(l_1^n)\}$$

for all $l_1^n \in L$.

Proposition 2.11. *I) $(L/\rho, *'_n)$ is a* **gnh**-*semilattice on L/ρ.*
II) The direct product of n **gnh**-*semilattices is a* **gnh**-*semilattice on $\prod_{k=1}^n L_k$.*

Proof. *I*) For all $l \in L$ we have $\rho(l) = \{t \in L | (l, t) \in \rho\}$. It is easy to verify that the $*'_n$ is idempotent and commutative. We show that this is weak associative. For all $l_1^{2n-1} \in L$, we have

$$*_n\big(l_1^{i-1}, *_n(l_i^{n+i-1}), l_{n+i}^{2n-1}\big) \in *'_n\big(\rho(l)_1^{i-1}, *'_n(\rho(l)_i^{n+i-1}), \rho(l)_{n+i}^{2n-1}\big).$$

Since $\bigcap_{i=1}^{2n-1} *_n\big(l_1^{i-1}, *_n(l_i^{n+i-1}), l_{n+i}^{2n-1}\big) \neq \varnothing$.
So $\bigcap_{i=1}^{2n-1} *'_n\big(\rho(l)_1^{i-1}, *'_n(\rho(l)_i^{n+i-1}), \rho(l)_{n+i}^{2n-1}\big) \neq \varnothing$, therefore $*'_n$ is weak associative.
II) *i*) For each $(d_1^n) \in \prod_{k=1}^n L_k$, we have that $d_k \in *_{n_k}(d_k)$, $k = 1, ..., n$, thus

$$(d_1^n) \in \{(c_1^n) | c_k \in *_{n_k}(d_k)\} = \Pi_n\Big((d_1^n), ..., (d_1^n)\Big).$$

ii) For all $(l_{k1})_{k=1}^n, ..., (l_{kn})_{k=1}^n \in \prod_{k=1}^n L_k$ and for $i = 1, ..., n$ we have

$$*_{n_i}(l_{ik})_{k=1}^n = *_{n_i}(l_{ik})_{k=\sigma(1)}^{\sigma(n)}$$

consequently Π_n is commutative.
iii) For all $(l_{k1})_{k=1}^n, ..., (l_{k(2n-1)})_{k=1}^n \in \prod_{k=1}^n L_k$ let $(l_{ki})_{k=1}^n = f_i$, $i = 1, ..., 2n-1$ for $j = 1, ..., n$ since $*_{n_j}$ are associative therefore

$$\bigcap_{i=1}^{2n-1} *_{n_j}\Big(l_{j1}^{j(i-1)}, *_{n_j}(l_{ji}^{j(n+i-1)}), l_{j(n+i)}^{j(2n-1)}\Big) \neq \varnothing$$

by Definition 2.9 as a result

$$\bigcap_{m=1}^{2n-1} \Pi_n\Big(f_1^{m-1}, \Pi_n(f_m^{n+m-1}), f_{n+m}^{2n-1}\Big) \neq \varnothing.$$

Thus Π_n is weak associative. □

Definition 2.12. Let $(L, *_n)$ be a **gnh**-semilattice. An element $a \in L$ is called an absorbent element of L if it satisfies $c_i \in *_n(a, c_1^{n-1})$, $i = 1, ..., n-1$ for all $c_1^{n-1} \in L$. An element $b \in L$ is called a fixed element of L if it satisfies $*_n(c_1^{n-1}, b) = \{b\}$ for all $c_1^{n-1} \in L$.

Definition 2.13. Let $(L, *_n)$ be a **gnh**-semilattice. A subset $A \in P^*(L)$ is called a fixed subset of $P^*(L)$ if it satisfies $\bigotimes_n(A_1^{n-1}, A) = A$ for all $A_1^{n-1} \in P^*(L)$.

Proposition 2.14. *I) Let $(L_k, *_{n_k})$ be* **gnh***-semilattices (where $k = 1, ..., n$). If a_1^n are absorbent elements of L_1^n respectively, then (a_1^n) is an absorbent element of $(\prod_{k=1}^n L_k, \Pi_n)$.*
II) Let $(L, *_n)$ and $(L', *'_n)$ be two **gnh***-semilattices and $f : L \longrightarrow L'$ be an epimorphism of* **gnh***-semilattices. If a is an absorbent element of L, then $f(a)$ is also an absorbent element of L'. Also image a fixed element of L is fixed element of L'.*

Proof. (I) Since a_1^n are absorbent elements, for $i = 1, ..., n-1$ and $k = 1, ..., n$, we have $c_{ki} \in *_{n_k}(a_k, c_{k1}^{k(n-1)})$ and $c_{ki} \in Ł_k$. Then $(c_{ki})_{k=1}^n \in \prod_{k=1}^n L_k$ and $(c_{ki})_{k=1}^n \in \{(f_1^n) | f_k \in *_{n_k}(a_k, c_{k1}^{k(n-1)}\}$. Thus (a_1^n) is an absorbent element of $(\prod_{k=1}^n L_k, \Pi_n)$.
II) There exists $c_1^{n-1} \in L$ such that $f(c_i) = s_i$ for any $s_1^{n-1} \in L'$ because f is surjective since a is an absorbent element provided $c_i \in *_n(a, c_1^{n-1})$ for any $c_1^{n-1} \in L$. Then we have
$s_i = f(c_i) \in f(*_n(a, c_1^{n-1})) = *'_n(f(a), f(c)_1^{n-1}) = *'_n(f(a), s_1^{n-1})$ for all $s_1^{n-1} \in L'$. Therefore $f(a)$ is an absorbent element of L'. The proof for a fixed element is analogous. □

3. gnh-SUBSEMILATTICE

Let $(L, *_n)$ be a **gnh**-semilattice, and M be a non-empty subset of L. Then M is called **gnh**-subsemilattice of $(L, *_n)$ if $*_n(m_1^n) \in P^*(M)$ for all $m_1^n \in M$. That is to say, M is an **gnh**-subsemilattice $(L, *_n)$ if and only if M is closed under the **gnh** on L. A **gnh**-subsemilattice M is a single point **gnh**-subsemilattice if $|M| = 1$ and a **gnh**-subsemilattice M such that $M \neq L$ is called a proper subsemilattice. We may easily get the conclusion as follows: M is a **gnh**-semilattice of $(L, *_n)$ if and only if $\bigotimes_n(M) = M$.

Proposition 3.1. *Let $f : L \longrightarrow L'$ be a strong homomorphism of* **gnh***-semilattices. Then the following conditions hold:*
I) If M is a **gnh***-subsemilattice of $(L, *_n)$. Then $f(M)$ is a* **gnh***-subsemilattice of $(L', *'_n)$.*
II) If f is surjective and N is a **gnh***-subsemilattice of $(L', *'_n)$, then $f^{-1}(N)$, which is defined by $f^{-1}(N) = \{l \in L | f(l) \in N\}$, is also a* **gnh***-subsemilattice of $(L, *_n)$.*

Proof. (I) Since f is a strong homomorphism of **gnh**-semilattice, there exists $m_1^n \in M$ such that $f(m_i) = b_i$ for all $b_i \in f(M), i = 1, ..., n$. By the definition of **gnh**-subsemilattice $*_n(m_1^n) \subseteq M$ holds. Hence, we have $*'_n(b_1^n) = *'_n(f(m)_1^n) = f(*_n(m_1^n)) \subseteq f(M)$. Consequently, $f(M)$ is a **gnh**-subsemilattice of $(L', *'_n)$.
(II) Since f is a surjective function, $f^{-1}(N)$ always exists. For all $m_1^n \in f^{-1}(N)$, $f(m_i) \in N$, $i = 1, ..., n$ we have $f(*_n(m_1^n)) = *'_n(f(m)_1^n) \subseteq N$. So $f^{-1}(N)$ is a **gnh**-subsemilattice of $(L, *_n)$. □

Proposition 3.2. *Let $(L, *_n)$ be a* **gnh**-*semilattice and let M and N be* **gnh**-*subsemilattices of $(L, *_n)$. Then $M \cap N$ is also a* **gnh**-*subsemilattice of $(L, *_n)$ if $M \cap N$ is non-empty.*

Proof. The proof is obvious according to subsemilattice definition which have been mentioned above. □

Example 3.3. Consider K_m denote the space of $m \times m$ real idempotent matrices. Then $(K_m, *_n)$ is a **gnh**-subsemilattice of $(L_m, *_n)$.

Example 3.4. In example 2.3, each non-empty subset of K is a **gnh**-subsemilattice of $(K, *_n)$.

4. The ideal of gnh-semilattice

The concept ideal play a vital role in the study of algebra structure. In this section, we introduce the definition of ideal of **gnh**-semilattice and discuss some basic properties of it.

Definition 4.1. Let $(L, *_n)$ be a **gnh**-semilatice, and I be non-empty subset of L. We say I is an ideal of $(L, *_n)$ if $\bigotimes_n(L, ..., L, I) \subseteq I$. If $I \neq L$, then I is called a proper ideal of $(L, *_n)$.

Proposition 4.2. *Let $(L, *_n)$ be a* **gnh**-*semilattice and let I be a non-empty subset of L. Then the following condition are equivalent:*
I) I is an ideal of $(L, *_n)$,
II) $*_n(l_1^{n-1}, i) \in P^*(I)$ for all $l_1^{n-1} \in L$ and $i \in I$,
III) $\bigotimes_n(l_1^{n-1}, I) \subseteq I$.

Proof. By Definitions 2.1, 4.1 and Proposition 2.5 the proof is clear □

Obviously, any **gnh**-semilattice is a **gnh**-subsemilattice and ideal of itself. If I is an ideal of $(L, *_n)$, then I is a **gnh**-subsemilattice of $(L, *_n)$.

Proposition 4.3. *Let I, J be ideals and M a* **gnh**-*subsemilattice of a* **gnh**-*semilattice $(L, *_n)$.*
I) Then $I \cap M$ is an ideal of M, $I \cup M$ is a **gnh**-*subsemilattice of L.*
II) (i) $I \cap J$ is an ideal of $(L, *_n)$ and $I \cap J = \bigotimes_n(L, ..., L, I, J)$,
(ii) $I \cup J$ is also an ideal of $(L, *_n)$.
III) If a is an absorbent element L, then the following condition hold:
(i) $I = L$ if and only if $a \in I$,
(ii) I is a proper ideal of $(L, *_n)$ if and only if $a \notin I$.

Proof. The proofs (I) and (III) are obvious according to Proposition 4.2, subsemilattice and 2.12 Definitions . We only give the main ideas of the prove (II).
(i) Let us prove $I \cap J \neq \varnothing$. Suppose that $i \in I$, $j \in J$, $l_1^{n-2} \in L$. Then $*_n(l_1^{n-2}, i, j) \subseteq I$, $*_n(l_1^{n-2}, i, j) \subseteq J$ by item (II) of proposition 4.2, that is

$*_n(l_1^{n-2}, j, i) \subseteq I \cap J$. So, we have $I \cap J \neq \varnothing$.
For all $i \in I \cap J$, i.e., $i \in I$ and $i \in J$, and for all $l_1^{n-1} \in L$, we have $*_n(l_1^{n-1}, i) \subseteq I$ and $*_n(l_1^{n-1}, i) \subseteq J$, i.e., $*_n(l_1^{n-1}, i) \in P^*(I \cap J)$. Thus $I \cap J$ is an ideal of $(L, *_n)$. By Definition 4.1, we can easily get that $\bigotimes_n(L, ..., L, I, J) \subseteq I \cap J$. For all $i \in I \cap J$ we have $i \in *_n(i) \subseteq I \cap J$, i.e., $I \cap J \subseteq \bigotimes_n(L, ..., L, I, J)$. So $I \cap J = \bigotimes_n(L, ..., L, I, J)$.
(ii) For all $i \in I \cup J$ and for all $l_1^{n-1} \in L$, we have $*_n(l_1^{n-1}, i) \subseteq I$ or $*_n(l_1^{n-1}, i) \subseteq J$. Hence $*_n(l_1^{n-1}, i) \subseteq I \cup J$ i.e., $*_n(l_1^{n-1}, i) \in P^*(I \cup J)$. Consequently $I \cup J$ is an ideal of $(L, *_n)$. □

Proposition 4.4. *I) Let $f : L \longrightarrow L'$ be a strong homomorphism of* **gnh**-*semilattices. If a is a fixed element of L', then $f^{-1}(a) = \{l \in L | f(l) = a\}$ is an ideal of $(L, *_n)$,*
II) If f be an epimorphism then we can get the following results:
i) If I is an ideal of $(L, *_n)$, then $f(I)$ is also an ideal of $(L', *'_n)$ and
ii) If J is an ideal of $(L', *'_n)$, then $f^{-1}(J)$, which is denoted by $f^{-1}(J) = \{l \in L | f(l) \in J\}$ is also an ideal of $(L, *_n)$.

Proof. (I) For all $l_1^{n-1} \in L$ and all $l \in f^{-1}(a)$,
$f(*_n(l_1^{n-1}, l)) = *'_n(f(l)_1^{n-1}, f(l)) = *'_n(f(l)_1^{n-1}, a) = \{a\}$,
i.e., $*_n(l_1^{n-1}, l) \subseteq f^{-1}(a)$. Therefore $f^{-1}(a)$ is an ideal of $(L, *_n)$.
(II) It follows easily that proposition 4.2. □

5. General n-order on a **Gnh**-semilattice

In this section we define a general n-order (**gno**) on $(L, *_n)$ and a **Gno** (General n-order) on $(P^*(L), \bigotimes_n)$.

Definition 5.1. i) Let $(L, *_n)$, be a **gnh**-semilattice and $a, b \in L$. We say that $a \leq_L b$ if $*_n(a, c_1^{n-1}) \subseteq *_n(b, c_1^{n-1})$ for all $c_1^{n-1} \in L$, and $\leq_L$ is called the **gno** on **gnh**-semilattice L.
ii) Let $A, B \in P^*(L)$. We say that $A \leq_{P^*(L)} B$ if $\bigotimes_n(A, C_1^{n-1}) \subseteq \bigotimes_n(B, C_1^{n-1})$ for all $C_1^{n-1} \in P^*(L)$, and $\leq_{P^*(L)}$ is called the **Gno** on **Gnh**-semilattice $P^*(L)$.

Definition 5.2. i) Let $(L, *_n)$, be a **gnh**-semilattice and $a, b \in L$. If $a \leq_L b$ and $b \leq_L a$, then we say a is gn-equal to b which is denoted be $a =_L b$.
ii) Let $A, B \in P^*(L)$. If $A \leq_{P^*(L)} B$ and $B \leq_{P^*(L)} A$, then we say A is Gn-equal to B which is denoted be $A =_{P^*(L)} B$.

Remark 5.3. Let $(L, *_n)$, be a **gnh**-semilattice and $a, b \in L$. Also let $A, B \in P^*(L)$ then $a =_L b$ if and only if $*_n(a, c_1^{n-1}) =_L *_n(b, c_1^{n-1})$ for all $c_1^{n-1} \in L$ and $A =_{P^*(L)} B$ if and only if $\bigotimes_n(A, C_1^{n-1}) =_{P^*(L)} \bigotimes_n(B, C_1^{n-1})$ for all $C_1^{n-1} \in P^*(L)$

Proposition 5.4. *Let $(L, *_n)$, be a* **gnh**-*semilattice. Then $=_L$ and $=_{P^*(L)}$ are respectively equivalence relation on L and $P^*(L)$.*

Proof. The proof is immediate. □

Proposition 5.5. *Let $(L, *_n)$, be a* **gnh***-semilattice and A, B be non-empty subsets of L. If $a \leq_L b$ ($a =_L b$), for all $a \in A$ and $b \in B$, then $A \leq_{P^*(L)} B$ ($A =_{P^*(L)} B$).*

Proof. The proof is straightforward. □

Proposition 5.6. *Let $(L, *_n)$, be a* **gnh***-semilattice and let I be a ideal of $(L, *_n)$. If $b \in I$ and $a \leq_L b$, then $a \in I$.*

Proof. If $a \leq_L b$ we have $*_n(a, c_1^{n-1}) \cap *_n(b, c_1^{n-1}) \neq \varnothing$ for all $c_1^{n-1} \in L$. Let $c_1 = ... = c_{n-1} = a$. Then $a \in *_n(a) \subseteq *_n(b, a, ..., a) \subseteq I$, then $a \in I$. □

Proposition 5.7. *I) Let $(L, *_n)$ be a* **gnh***-semilattice, $[k] = \{x \in L | x =_L k\}$ and $\Delta_L = \{[k] | k \in L\}$. We may define a* **gnh** *on Δ_L by $*'_n([l_1], ..., [l_n]) = \{[l] | l \in *_n(l_1^n)\}$, then $(\Delta_L, *'_n)$ is also a* **gnh***-semilattice.*
II) Let $(P^*(L), \bigotimes_n)$ be a **Gnh***-semilattice, $[K] = \{X \in P^*(L) | X =_{P^*(L)} K\}$ and $\Delta_{P^*(L)} = \{[K] | K \in P^*(L)\}$. We may define a* **Gnh** *on $\Delta^*_P(L)$ by $\bigotimes'_n([A_1], ..., [A_n]) = \{[N] | N \in \bigotimes_n(A_1^n)\}$, then $(\Delta_{P^*(L)}, \bigotimes'_n)$ is also a* **Gnh***-semilattice.*

Proof. I) For all $[l_1], ..., [l_{2n-1}] \in \Delta_L$, we have:
$I-i$) Since $l_1 \in *_n(l_1)$, then $[l_1] \in \{[l] | l \in *_n(l_1)\} = *'_n([l_1])$.
$I-ii$) Since $*_n(l_1^n) = *_n(l_{\sigma(1)}^{\sigma(n)})$, then $*'_n([l_1], ..., [l_n]) = \{[l] | l \in *_n(l_1^n)\} = \{[l] | l \in *_n(l_{\sigma(1)}^{\sigma(n)})\} = *'_n([l_{\sigma(1)}], ..., [l_{\sigma(n)}])$.
$I-iii$) Since $\bigcap_{i=1}^{2n-1} *_n(l_1^{i-1}, (l_i^{n+i-1}), l_{n+i}^{2n-1}) \neq \varnothing$. By Definition 1.2, **gnh** $*'_n$ is weak associative.
In the same manner we can see that;
II) For all $[A_1], ..., [A_{2n-1}] \in \Delta_{P^*(L)}$ we have:
$II-i$) Since $A_1 \in \bigotimes_n(A_1)$, we see that $[A_1] \in \{[N] | N \in \bigotimes_n(A_1)\} = \bigotimes'_n([A_1])$.
$II-ii$) Since $\bigotimes_n(A_1^n) = \bigotimes_n(A_{\sigma(1)}^{\sigma(n)})$, then $\bigotimes'_n([A_1], ..., [A_n]) = \{[N] | N \in \bigotimes_n(A_1^n)\} = \{[N] | N \in \bigotimes_n(A_{\sigma(1)}^{\sigma(n)})\} = \bigotimes'_n([A_{\sigma(1)}], ..., [A_{\sigma(n)}])$.
$II-iii$) By Definition 1.2 and Proposition 2.5 **Gnh** $\bigotimes'_n$ is weak associative. □

6. Mutipliers of gnh-semilattices

In this section notation multipliers of **gnh**-semilattice of type α and multipliers of **Gnh**-semilattice of type α will be introduced and some properties of multipliers are studied. In addition, a set of equivalent conditions are established for two multipliers of a **gnh**-semilattice of type α to be equal in the sense of mappings. Further, we introduce the multipliers of direct products of n **gnh**-semilattices of type $\alpha = n - 1$. In addition, the properties of quasi invariance subsets are studied with respect to multipliers of **gnh**-semilattices of type α.

Definition 6.1. Let $(L, *_n)$ be a **gnh**-semilattice and $f : L \longrightarrow L$ be a self mapping. Also let $F : P^*(L) \longrightarrow P^*(L)$ be given by $F(A) = \cup_{a\in A} f(a)$ for all $A \in P^*(L)$.
i) Then F is called a multiplier of $P^*(L)$ of type α (where $\alpha = 1, ..., n$) if for all $A_1^n \in P^*(L)$ is satisfies

$$F\big(\bigotimes_n (A_1^\alpha, A_{\alpha+1}^n)\big) = \bigotimes_n \big(A_1^\alpha, F(A)_{\alpha+1}^n\big),$$

ii) a self mapping f is called multiplier of L of type α (where $\alpha = 1, ..., n$) if for all $l_1^n \in L$ is satisfies

$$F\big(*_n (l_1^\alpha, l_{\alpha+1}^n)\big) = *_n(l_1^\alpha, f(l)_{\alpha+1}^n).$$

Proposition 6.2. *Let* $(P^*(L), \bigotimes_n)$ *be a* **Gnh**-*semilattice and* F *multiplier of* $P^*(L)$ *of type* α. *Then for any* $A_1^n \in P^*(L)$, *we have the following:*
I) If A_1 *is a fixed subset of* $P^*(L)$ *then* $F(A_1) = A_1$.
II) If $\alpha = n - 1$ *then* $\bigotimes_n \big(A_1^{n-1}, F(A_n)\big) = \bigotimes_n \big(A_1^{i-1}, F(A_i), A_{i+1}^n\big)$ *for* $i = 1, ..., n - 1$.
III) If $\alpha = 1$ *then* $\bigotimes_n \big(A_1, F(A)_2^n\big) = \bigotimes_n \big(F(A)_1^{i-1}, A_i, F(A)_{i+1}^n\big)$ *for* $i = 2, ..., n$.

Proof. I) Let A_1 be a fixed subset of $P^*(L)$. Then $\bigotimes_n(A_1^n) = A_1$ for all $A_2^n \in P^*(L)$. Since $A_1 \in \bigotimes_n(A_1, ..., A_1)$ we get
$F(A_1) \in F\big(\bigotimes_n(A_1, ..., A_1)\big) = \big(\underbrace{A_1, ..., A_1}_{\alpha-time}, F(A_1), ..., F(A_1)\big) = A_1$. So $F(A_1) = A_1$.
II) For any $A_1^n \in P^*(L)$ and $i = 1, ..., n - 1$.

$$\bigotimes_n \big(A_1^{n-1}, F(A_n)\big) =$$

$F\big(\bigotimes_n(A_1^n)\big) = F\big(\bigotimes_n(A_{\sigma(1)}^{\sigma(n)})\big) = \bigotimes_n \big(A_1^{i-1}, F(A_i), A_{i+1}^n\big)$. The proof ($III$) is similar to ($II$). □

Proposition 6.3. *Let* $(L, *_n)$ *be a* **gnh**-*semilattice,* f *be a multiplier of* L *of type* α *and* F *be idempotent of* $P^*(L)$. *If* $f^2 = Id_L$ *then* $F\big(*_n (l_1^n)\big) = *_n(l_1^n)$.

Proof. Since f^2 is identity on L we have $F\big(*_n (l_1^n)\big) = F^2\big(*_n (l_1^n)\big) = F\big(F\big(*_n (l_1^n)\big)\big) = F\big(*_n (l_1^\alpha, f(l)_{\alpha+1}^n)\big) = *_n(l_1^\alpha, f^2(l)_{\alpha+1}^n) = *_n(l_1^n)$. □

Lemma 6.4. *Let* $(L, *_n)$ *be a* **gnh**-*semilattice and* f *a multiplier of* L *of type* α. *Then for any* $l_1^n \in L$, *we have the following:*
I) If l_1 *is a fixed element then* $f(l_1) = l_1$.
II) If $\alpha = 1$ *then* $*_n(l_1, f(l)_2^n) = *_n(f(l)_1^{i-1}, l_i, f(l)_{i+1}^n)$ *for* $i = 2, ..., n$.
II) If $\alpha = n - 1$ *then* $*_n(l_1^{n-1}, f(l_n)) = *_n(l_1^{i-1}, f(l_i), l_{i+1}^n)$ *for* $i = 1, ..., n - 1$.

Proof. I) Let l_1 be a fixed element of L. Then $*_n(l_1^n) = \{l_1\}$ for all $l_2^n \in L$. Since $l_1 \in *_n(l_1,...,l_1)$ we get $F(l_1) \in F(*_n(l_1,...,l_1)) = *_n(\underbrace{l_1,...,l_1}_{\alpha-time}, f(l_1),...,f(l_1)) = \{l_1\}$.
So $f(l_1) = F(l_1) = l_1$.
II) For any $l_1^n \in L$ and for $i = 2,...,n$.

$$*_n\big(l_1, f(l)_2^n\big) = F\big(*_n (l_1^n)\big) = F\big(*_n (l_{\sigma(1)}^{\sigma(n)}\big) = *_n\big(f(l)_1^{i-1}, l_i, f(l)_{i+1}^n\big).$$

The proof (III) is clear. □

Remark 6.5. I) Let $(L, *_n)$ be **gnh**-semilattice and n be even.
If f be multiplier of L of type $\alpha = n/2$ and F be multiplier of $P^*(L)$ of type $\alpha = n/2$, then
$*_n(l_1^{n/2}, f(l)_{n/2+1}^n) = *_n(f(l)_1^{n/2}, l_{n/2+1}^n)$ and $\bigotimes_n(A_1^{n/2}, F(A)_{n/2+1}^n)$
$= \bigotimes_n(F(A)_1^{n/2}, A_{n/2+1}^n)$.
II) $F = Id_{P^*(L)}$ if and only if F be multiplier of type n on $P^*(L)$.

Proposition 6.6. *Let $(\prod_{k=1}^n, \Pi_n)$ is direct product of* **gnh***-semilattices $(L_k, *_{n_k})$, $k = 1,...,k$ and e_k be a fixed element of L_k. Define self mapping $f_i : \prod_{k=1}^n L_k \longrightarrow \prod_{k=1}^n L_k$ by*

$$f_i(l_1^{i-1}, l_i, l_{i+1}^n) = (l_1^{i-1}, e_i, l_{i+1}^n)$$

for all $(l_1^n) \in \prod_{k=1}^n L_k$, $i = 1,...,n$ i.e., the functions f_i are multipliers of the direct product $\prod_{k=1}^n L_k$ of type $\alpha = n-1$.

Proof. Let $(l_{k1})_{k=1}^n, ..., (l_{kn})_{k=1}^n \in \prod_{k=1}^n L_k$. Then we get

$$f_i\Big(\Pi_n\big((l_{k1})_{k=1}^n, ..., (l_{kn})_{k=1}^n\big)\Big) = f_i\Big(\{(c_1^n) | c_j \in *_{n_j}(l_{jk})_{k=1}^n,\ j = 1,...,n\}\Big)$$

$= \{(c_1^{i-1}, e_i, c_{i+1}^n) | c_j \in *_{n_j}(l_{jk})_{k=1}^n$, if $j \neq i,\ j = 1,...,n$ and for $j = i, e_i \in *_{n_i}(l_{i1}^{i(i-1)}, e_i, l_{i(i+1)}^{in})\}$

$$= \Pi_n\Big((l_{k1})_{k=1}^n, ..., (l_{k(i-1)})_{k=1}^n, (l_{1i}^{(i-1)i}, e_i, l_{(i+1)i}^{ni}), (l_{k(i+1)})_{k=1}^n, ..., (l_{kn})_{k=1}^n\Big)$$

$$= \Pi_n\Big((l_{k1})_{k=1}^n, ..., (l_{k(i-1)})_{k=1}^n, f_i(l_{ki})_{k=1}^n, ..., (l_{kn})_{k=1}^n\Big).$$

Thus, for $i = 1,...,n$. The self mappings f_i are multipliers of the direct product $\prod_{k=1}^n L_k$ of type $\alpha = n-1$. □

Proposition 6.7. *Let f be a muliplier of a* **gnh***-semilattice $(L, *_n)$ of type $\alpha = n-1$. Then we have the following:*
I) If a is a fixed element of L, then $f^{-1}(a)$ is an ideal of L.
II) If I is an ideal of L, $F(I) = f(I)$ then $f(I)$ is an ideal of L.
III) If I is an ideal of L, then $f^{-1}(I)$ is an ideal of L.

Proof. *I*) Let a be a fixed element of L. Let $l \in f^{-1}(a)$. Then $f(l) = a$. For any $l_1^{n-1} \in L$, we have $F(*_n(l_1^{n-1}, l)) = *_n(l_1^{n-1}, f(l)) = *_n(l_1^{n-1}, a) = \{a\}$. So $*_n(l_1^{n-1}, l) \subseteq F^{-1}(a) = f^{-1}(a)$. Therefore $f^{-1}(a)$ is an ideal of L.
II) Let I be an ideal of L. Let $l_1^{n-1} \in L$ and $y \in f(I)$. Then $y = f(a)$ for some $a \in I$. Since I is an ideal, we get that $*_n(l_1^{n-1}, a) \subseteq I$. Now $*_n(l_1^{n-1}, f(a)) = F\big(*_n (l_1^{n-1}, a)\big) \subseteq F(I) = f(I)$. So $f(I)$ is an ideal of L.
III) Let I be an ideal L. Let $l_1^{n-1} \in L$ and $a \in f^{-1}(I)$. Then $f(a) \in I$. Since I is an ideal, we get that $F(*_n(l_1^{n-1}, a)) = *_n(l_1^{n-1}, f(a)) \subseteq \bigotimes_n(l_1^{n-1}, I) \subseteq I$. Thus $f^{-1}(I)$ is an ideal of L □

Remark 6.8. Let f be a multiplier of a **gnh**-semilattice $(L, *_n)$ of type α. If a is a fixed element of L, then $f^{-1}(a)$ is an ideal of L.

Definition 6.9. Let f be a multiplier of $(L, *_n)$ of type α. Define $\Omega_f(L) = \{l \in L | f(l) = l\}$.

By Lemma 6.4, every fixed element of L is a member of $\Omega_f(L)$. If f is an idempotent multiplier, then obviously $f(l) \in \Omega_f(L)$ for all $l \in L$.

Proposition 6.10. *Let f and g be two idempotent multipliers of a* **gnh***-semilattice L of type α such that $f \circ g = g \circ f$. Then the following conditions are equivalent.*
I) $f = g$.
II) $f(L) = g(L)$.
III) $\Omega_f(L) = \Omega_g(L)$.

Proof. The proof is similar to the proof of Theorem 2.10 [**?**]. □

Definition 6.11. Let $(L, *_n)$ be **gnh**-semilattice and F be a multiplier of a **Gnh**-semilattice $P^*(L)$ of type α. A subset M of L is called F-quasi invariant if for at least a non-emty subset K of M. $K \subseteq M$ implies $F(K) \subseteq M$.

Note that $\varnothing$ and L are F-quasi invariant subset of L. Also $\Omega_f(L)$ is a F-quasi invariant subset of L. Let us denoted the set of all F-quasi invariant subsets of a **Gnh**-semilattice by $\Lambda_F(L)$.

Theorem 6.12. *Let F be a multiplier of a* **Gnh***-semilattice $(P^*(L), \bigotimes_n)$ of type α. Then $(\Lambda_F(L), \bigotimes_n)$ is a* **Gnh***-semilattice.*

Proof. Let $A_1^n \in \Lambda_F(L)$ and F be a multiplier of a **Gnh**-semilattice $(P^*(L), \bigotimes_n)$ of type α. Let $X \subseteq \bigotimes_n(A_1^n)$. Then $X = *_n(a_1^n)$ for some $a_i \in A_i$, $i = 1, ..., n$. Thus $F(X) = F(*_n(a_1^n)) = *_n(a_1^\alpha, F(a)_{\alpha+1}^n) \subseteq \bigotimes_n(A_1^n)$.
Hence $\bigotimes_n(A_1^n)$ is F-quasi invariant. According to Proposition 2.5, the pair $(\Lambda_F(L), \bigotimes_n)$ is a **Gnh**-semilattice. □

Definition 6.13. Let $(L, *_n)$ be a **gnh**-semilattice. By a congruence on L we means an equivalence relation ρ such that $(l, t) \in \rho$ if and only if for every

$l_1^{n-1} \in L$ and for every $u \in *_n(l, l_1^{n-1})$, there exists $v \in *_n(t, l_1^{n-1})$ such that $(u, v) \in \rho$.

We now introduce a congruence on L in terms of multipliers of type $\alpha = n-1$.

Proposition 6.14. *Let f be a multiplier of a* **gnh**-*semilattice $(L, *_n)$ of type $\alpha = n-1$. Define a relation ρ_f on L by $(l, t) \in \rho_f$ if and only if $f(l) = f(t)$ for all $l, t \in L$. Then ρ_f is a congruence on L.*

Proof. Clearly ρ_f is an equivalence relation on L. Let $l_1^{n-1} \in L$ and $u \in *_n(l, l_1^{n-1})$. Then $F(u) \in F(*_n(l, l_1^{n-1})) = *_n(f(l), l_1^{n-1}) = *_n(f(t), l_1^{n-1}) = F(*_n(t, l_1^{n-1})$. So there exists $v \in *_n(t, l_1^{n-1})$. Thus $f(u) = F(v) = f(v)$ as a result $(u, v) \in \rho_f$. □

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Akbar Dehghan Nezhad
School of Mathematics, Iran University of Science and Technology, P.O.Box 16846 -13114, Tehran, Iran
Email: dehghannezhad@iust.ac.ir

Najmeh Khajuee
Department of Mathematics, Shahid Bahonar University of Kerman, P.O.Box 76169-133, Kerman, Iran
Email: n.khajuee@math.uk.ac.ir