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A NUMERICAL APPROACH BASED ON THE REPRODUCING KERNEL HILBERT METHOD ON NON-UNIFORM GIRDS FOR SOLVING SYSTEM OF FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we develop a numerical approach based on the reproducing kernel Hilbert (RKHS) method on non-uniform girds for solving the linear Fredholm integro-differential equations with variable coefficients. Furthermore, convergence of the proposed method is presented providing the theoretical basis of this method. Finally, we test our method on one example to demonstrate the efficiency and applicability of the proposed method.

Key Words: Reproducing kernel Hilbert space method; Fredholm integro-differential equations; variable coefficients

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1. Introduction

Integro-differential equations system have an important role in the fields of science and engineering [1–4]. Some boundary value problems arising in electromagnetic theory lead to the problem of solving integro-differential equations system [5].

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This paper investigates the exact and approximate solutions of the following linear Fredholm integro-differential equations system with variable coefficients using RKHSM

$$
(1.1)\begin{cases} \sum_{j=0}^{N} a_{ij}(s)u_1^{(j)}(s) + \lambda_i \int_0^1 k_i(s,t)u_1(t)\,dt \\ +\sum_{j=0}^{N} b_{ij}(s)u_2^{(j)}(s) + \mu_i \int_0^1 h_i(s,t)u_2(t)\,dt = f_i(s), i = 1,2, \\ \sum_{j=0}^{N-1} u_r^{(j)}(0) = p_r, r = 1,2, \\ \sum_{j=0}^{N-1} u_r^{(j)}(1) = q_r, r = 1,2. \end{cases}
$$

where $a_{ij}(s),\ b_{ij}(s),\ f_i(s)$ are arbitrary known smooth functions defined on the interval $[0,1],\ k_i(s,t),\ h_i(s,t)$ are given continuous functions on region $[0,1]\times[0,1]$, unknown functions $u_1(s),\ u_2(s)$ are continuous on the interval $[0,1],\ u_1^{(j)}(s),\ u_2^{(j)}(s)$ are the j^{th} order derivatives of functions $u_1(s),\ u_2(s)$, respectively, $\lambda_i,\ \mu_i,\ p_r,\ q_r$ are given constants and N belongs to $\mathbb{N}$.

The integro-differential equation arises in many physical applications, such as potential theory and Dirichlet problems, electrostatics, mathematical problems of radiative equilibrium, the particle transport problems of astrophysics and reactor theory, and radiative heat transfer problems. Recently, a huge amount of research work has been motivated by the concept of a system of integro-differential equations. Several powerful mathematical methods such as Galerkin method [6], Petrov Galerkin method [7], Tau method [8], collocation method [9], block pulse functions method [10], Chebyshev polynomial method [11], Legendre wavelets [12], Taylor series [13], Adomain's method [14], He's homotopy perturbation method [15] and others [16–22] have been proposed to obtain exact and approximate solution of linear Fredholm integro-differential equations system. The application of RKHSM in linear and nonlinear problems has been developed by many researchers [23–25]. This method obtains the exact solution in series form and provides approximate solution with high precision [26–32].

The rest of the paper is organized as follows. Section 2 introduces some reproducing kernel spaces. Section 3 is devoted to solve Eqs. (1.1) by RKHSM. An numerical example is presented in Section 4. The last section is a brief conclusion.

2. Reproducing kernel spaces

In this section, several reproducing kernel spaces are introduced.

Definition 2.1. ([32]) Let E be a nonempty abstract set and H be a real Hilbert space of functions $\varphi : E \longrightarrow \mathbb{R}$. A function $k : E \times E \longrightarrow \mathbb{R}$ is called reproducing kernel of the real Hilbert space H if and only if

(1) $k(s,.) \in H$ for all $s \in E$,
(2) $\varphi(s) = (\varphi(.), k(s,.))_H$ for all $\varphi \in H$ and all $s \in E$.

Definition 2.2. ([32]) A real Hilbert space H of functions on a set is called a reproducing kernel Hilbert space (RKHS) if there exists a reproducing kernel k of H.

It is known that the reproducing kernel of a Hilbert space is unique, and that existence of a reproducing kernel is due to the Riesz representation theorem. The reproducing kernel k of a Hilbert space H completely determines the space H. Every sequence of functions $\{\varphi_i\}_{i=1}^{\infty}$ which converges strongly to a function φ in H, converges also in the pointwise sense. Further, this convergence is uniform on every subset of E on which $s \mapsto k(s,s)$ is bounded.

2.1. **The reproducing space $W_2^{N+1}[0,1]$.**

Definition 2.3. The space $W_2^{N+1}[0,1]$ is defined by

$$W_2^{N+1}[0,1] = \{u(s)|u(s), u^{'}(s), \dots, u^{(N)}(s), \tag{2.1}$$

which are absolute continuous real valued functions on the interval $[0,1]$ and

$$u^{(N+1)} \in L^2[0,1],\ \sum_{j=0}^{N-1} u_r^{(j)}(0) = 0,\ \sum_{j=0}^{N-1} u_r^{(j)}(1) = 0,\ r = 1,2. \tag{2.2}$$

It is equipped with the inner product

$$\begin{aligned} < u(s), v(s) >_{W_2^{N+1}} &= \sum_{j=0}^{N} u^{(j)}(0)v^{(j)}(0) + \int_0^1 u^{(N+1)}(s)v^{(N+1)}(s)ds, \\ u(s), v(s) &\in W_2^{N+1}[0,1], \end{aligned} \tag{2.3}$$

and the norm

$$||u||_{W_2^{N+1}} = \sqrt{< u(s), u(s) >_{W_2^{N+1}}}, u(s) \in W_2^{N+1}[0,1]. \tag{2.4}$$

In [32], it is proved that space $W_2^{N+1}[0,1]$ normed by $||u||_{W_2^{N+1}}$, is a Hilbert space.

Theorem 2.4. *Suppose that*

$$R_s(t) = \begin{cases} \sum_{j=0}^{2N+1} \alpha_j(s)t^j, & t \le s, \\ \sum_{j=0}^{2N+1} \beta_j(s)t^j, & t > s, \end{cases} \tag{2.5}$$

satisfies the following generalized differential equations

$$\begin{cases} R_s(0) - \frac{\partial^j R_s(0)}{\partial t^j} + (-1)^{N-j}\frac{\partial^{2N+1-j} R_s(0)}{\partial t^{2N+1-j}} - (-1)^N \frac{\partial^{2N+1} R_s(0)}{\partial t^{2N+1}} = 0, \\ \frac{\partial^N R_s(0)}{\partial t^N} - \frac{\partial^{N+1} R_s(0)}{\partial t^{N+1}} = 0, \\ \frac{\partial^{N+1} R_s(1)}{\partial t^{N+1}} = 0, \\ \frac{\partial^{2N+1} R_s(1)}{\partial t^{2N+1}} - (-1)^j \frac{\partial^{2N+1-j} R_s(1)}{\partial t^{2N+1-j}} = 0, \quad j = 1, \ldots, N-1, \end{cases} \tag{2.6}$$

and

$$(-1)^{N+1}\frac{\partial^{2N+2} R_s(t)}{\partial t^{2N+2}} = \delta(t-s). \tag{2.7}$$

then the Hilbert space $W_2^{N+1}[0,1]$ is a reproducing kernel space with the reproducing kernel function $R_s(t)$, that is, for each $u \in W_2^{N+1}[0,1]$ and a fixed $s \in [0,1]$, it follows that

$$< u(t), R_s(t) >_{W_2^{N+1}} = u(s).$$

Proof. Applying integration by parts several times, we have

$$\begin{aligned} < u(t), R_s(t) >_{W_2^{N+1}} &= \sum_{j=0}^{N} u^{(j)}(0)(\frac{\partial^j R_s(0)}{\partial t^j}) + \int_0^1 u^{(N+1)}(t)(\frac{\partial^{N+1} R_s(t)}{\partial t^{N+1}})\, dt \\ &= \sum_{j=0}^{N}[u^{(j)}(0)(\frac{\partial^j R_s(0)}{\partial t^j}) + (-1)^j u^{(N-j)}(t)(\frac{\partial^{N+1+j} R_s(t)}{\partial t^{N+1+j}})|_0^1] \\ &+ (-1)^{N+1}\int_0^1 u(t)(\frac{\partial^{2N+2} R_s(t)}{\partial t^{2N+2}})\, dt. \end{aligned} \tag{2.8}$$

Since

$$\sum_{j=0}^{N}(-1)^j u^{(N-j)}(t)(\frac{\partial^{N+1+j} R_s(t)}{\partial t^{N+1+j}}) = \sum_{j=0}^{N}(-1)^{N-j} u^{(j)}(t)(\frac{\partial^{2N+1-j} R_s(t)}{\partial t^{2N+1-j}}), \tag{2.9}$$

then

$$\begin{aligned} < u(t), R_s(t) >_{W_2^{N+1}} &= \sum_{j=1}^{N-1} -u^{(j)}(0)[R_s(0) - \frac{\partial^j R_s(0)}{\partial t^j} + (-1)^{N-j}\frac{\partial^{2N+1-j}R_s(0)}{\partial t^{2N+1-j}} \\ &- (-1)^N \frac{\partial^{2N+1}R_s(0)}{\partial t^{2N+1}}] + u^{(N)}(0)[\frac{\partial^N R_s(0)}{\partial t^N} - \frac{\partial^{N+1}R_s(0)}{\partial t^{N+1}}] \\ &+ \sum_{j=1}^{N-1}(-1)^{N+1}u^{(j)}(1)[\frac{\partial^{2N+1}R_s(1)}{\partial t^{2N+1}} - (-1)^j\frac{\partial^{2N+1-j}R_s(1)}{\partial t^{2N+1-j}}] \\ &+ u^{(N)}(1)[\frac{\partial^{N+1}R_s(1)}{\partial t^{N+1}}] + (-1)^{N+1}\int_0^1 u(t)(\frac{\partial^{2N+2}R_s(t)}{\partial t^{2N+2}})\, dt. \end{aligned}$$

Since $R_s(t) \in W_2^{N+1}[0,1]$, it follows that

$$\sum_{j=0}^{N-1}\frac{\partial^j R_s(0)}{\partial t^j} = 0, \sum_{j=0}^{N-1}\frac{\partial^j R_s(1)}{\partial t^j} = 0. \tag{2.10}$$

Then Eqs. (2.6) and (2.7) imply that

$$< u(t), R_s(t) >_{W_2^{N+1}} = \int_0^1 u(t)\delta(t-s)\, dt = u(s).$$

Characteristic equation of Eq. (2.7) is given by $\lambda^{2N+2} = 0$, then we can obtain characteristic values $\lambda = 0$ (a $2N+2$ multiple root). On the other hand, for Eq. (2.7), let $R_s(t)$ satisfy

$$\frac{\partial^l R_s(s-0)}{\partial t^l} = \frac{\partial^l R_s(s+0)}{\partial t^l}, \quad l = 0, 1, \dots, 2N. \tag{2.11}$$

Integrating Eq. (2.7) from $s-\varepsilon$ to $s+\varepsilon$ with respect to t and let $\varepsilon \to 0$, we have the jump degree of $\frac{\partial^{2N+1}R_s(t)}{\partial t^{2N+1}}$ at $t = s$,

$$\frac{\partial^{2N+1}R_s(s-0)}{\partial t^{2N+1}} - \frac{\partial^{2N+1}R_s(s+0)}{\partial t^{2N+1}} = 1. \tag{2.12}$$

Applying Eqs. (2.6), (2.10), (2.11), (2.12), the unknown coefficients of Eq. (2.5) has obtained a unique in Appendix A. □

Theorem 2.5. *Let* $\{s_i\}_{i=1}^{\infty}$ *be a dense subset of interval* $[0,1]$, *then* $\{R_{s_i}(t)\}_{i=1}^{\infty}$ *is a complete system of the space* $W_2^{N+1}[0,1]$.

Proof. For each fixed $u = u(s) \in W_2^{N+1}[0,1]$; if $< u(t), R_{s_i}(t) >_{W_2^{N+1}[0,1]} = 0$; then $u(s_i) = 0$.
Taking into account the density of $\{s_i\}_{i=1}^{\infty} \subset [0,1]$, It follows that $u(s) = 0$.
So, the proof of the theorem is complete. □

2.2. **The reproducing kernel space** $W_2^1[0,1]$**.**

Definition 2.6. The space $W_2^1[0,1]$ is defined by
$W_2^1[0,1] = \{u(s)|u(s)$ is absolute continuous real valued function on the interval $[0,1]$ and $u' \in L^2[0,1]\}$.
It is equipped with the inner product

$$< u(s), v(s) >_{W_2^1} = u(0)v(0) + \int_0^1 u'(s)v'(s)\, ds\ ,u(s), v(s) \in W_2^1[0,1],$$

and the norm $||u||_{W_2^1} = \sqrt{(u(s), u(s))_{W_2^1}}$, $u(s) \in W_2^1[0,1]$.

In [32], it is proved that space $W_2^1[0,1]$ normed by $||u||_{W_2^1}$, is a Hilbert space.

Theorem 2.7. *Hilbert space* $W_2^1[0,1]$ *is a reproducing kernel space with the reproducing kernel function*

$$r_s(t) = \begin{cases} 1+t, & t \leq s, \\ 1+s, & t > s, \end{cases} \tag{2.13}$$

that is, for each $u(s) \in W_2^1[0,1]$ *and a fixed* $s \in [0,1]$, *it follows that*

$$(u(t), r_s(t))_{W_2^1[0,1]} = u(s).$$

Theorem 2.8. *Let* $\{s_i\}_{i=1}^{\infty}$ *be a dense subset of interval* $[0,1]$, *then* $\{r_{s_i}(t)\}_{i=1}^{\infty}$ *is a complete system of* $W_2^1[0,1]$.

Proof. The proof is similar to proof of Theorem 2.5. □

3. The reproducing kernel method

In this paper, we shall give the exact or approximate solution of Eqs. (1.1) in a reproducing kernel space. We assume that Eqs. (1.1) have a unique solution.
To deal with the system, we consider Eqs. (1.1) as

$$Au(s) = f(s), \tag{3.1}$$

where

$$A : W_2^{N+1}[0,1] \bigoplus W_2^{N+1}[0,1] \longrightarrow W_2^1[0,1] \bigoplus W_2^1[0,1],$$
$$A = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right].$$

with

$$\begin{aligned} A_{i1}u_1(s) &= \textstyle\sum_{j=0}^{N} a_{ij}u_1^{(j)}(s) + \lambda_i \int_0^1 k_i(s,t)u_1(t)\, dt, i = 1,2, \\ A_{i2}u_2(s) &= \textstyle\sum_{j=0}^{N} b_{ij}u_2^{(j)}(s) + \mu_i \int_0^1 h_i(s,t)u_2(t)\, dt, i = 1,2, \end{aligned} \tag{3.2}$$

$$f(s)=(f_1(s),f_2(s))^T\in W_2^1[0,1]\bigoplus W_2^1[0,1], \tag{3.3}$$

$$u(s)=(u_1(s),u_2(s))^T\in W_2^{N+1}[0,1]\bigoplus W_2^{N+1}[0,1]. \tag{3.4}$$

The inner product space $W_2^{N+1}[0,1]\bigoplus W_2^{N+1}[0,1]$ is defined as

$$W_2^{N+1}[0,1]\bigoplus W_2^{N+1}[0,1]=\{u(s)=(u_1(s),u_2(s))^T|u_1(s),u_2(s)\in W_2^{N+1}[0,1]\}.$$

The inner product and norm are defined by

$$<u(s),v(s)>=\sum_{i=1}^2<u_i(s),u_i(s)>_{W_2^{N+1}[0,1]},u(s),v(s)\in W_2^{N+1}[0,1]\bigoplus W_2^{N+1}[0,1],$$

$$||u(s)||=\sqrt{\sum_{i=1}^2||u_i(s)||^2_{W_2^{N+1}[0,1]}},u(s)\in W_2^{N+1}[0,1]\bigoplus W_2^{N+1}[0,1].$$

It is easy to verify that $W_2^{N+1}[0,1]\bigoplus W_2^{N+1}[0,1]$ is a Hilbert space. Also, $W_2^1[0,1]\bigoplus W_2^1[0,1]$ is a Hilbert space in a similar manner.

Lemma 3.1. *If $A_{ij}:W_2^{N+1}[0,1]\to W_2^1[0,1]$ are bounded linear operators, then $A:W_2^{N+1}[0,1]\bigoplus W_2^{N+1}[0,1]\to W_2^1[0,1]\bigoplus W_2^1[0,1]$ is a bounded linear operator.*

Proof. Clearly, A is a linear operator. For each

$$u\in W_2^{N+1}[0,1]\bigoplus W_2^{N+1}[0,1],$$

we have

$$\begin{aligned}||Au(s)|| &= \sqrt{\sum_{i=1}^2||\sum_{j=1}^2A_{ij}u_j(s)||^2_{W_2^{N+1}[0,1]}}\\ &\le \sqrt{\sum_{i=1}^2(\sum_{j=1}^2||A_{ij}||||u_j(s)||_{W_2^{N+1}[0,1]})^2}\\ &\le \sqrt{\sum_{i=1}^2(\sum_{j=1}^2||A_{ij}||^2)(\sum_{j=1}^2||u_j(s)||^2_{W_2^{N+1}[0,1]})}\\ &= (\sum_{i=1}^2\sum_{j=1}^2||A_{ij}||^2)^{\frac{1}{2}}||u(s)||_{W_2^{N+1}[0,1]}. \end{aligned} \tag{3.5}$$

Thus A is a bounded operator. □

It is easy to see that the adjoint operator of A is $A^* = [\begin{array}{cc} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{array}]$, where A_{ij}^* is the adjoint operator of A_{ij}.

3.1. **The representation of the solution of system (3.1).** In this section, we will give the representation of analytical solution of system (3.1) in the space $W_2^{N+1}[0,1] \bigoplus W_2^{N+1}[0,1]$. In system (3.1), in view of Lemma 3.1, it is clear that $A : W_2^{N+1}[0,1] \bigoplus W_2^{N+1}[0,1] \to W_2^1[0,1] \bigoplus W_2^1[0,1]$ is a bounded linear operator. Put

$$\varphi_{ij}(s) = r_{s_i}(s)\vec{e}_j = \begin{cases} (r_{s_i}(s), 0)^T, & j = 1, \\ \\ (0, r_{s_i}(s))^T, & j = 2, \end{cases} \tag{3.6}$$

and $\psi_{ij}(s) = A^*\varphi_{ij}(s), \quad i = 1, 2, ..., \quad j = 1, 2$, where $r_{s_i}(s)$ is the reproducing kernel of $W_2^1[0,1]$ and A^* is the adjoint operator of A. The orthonormal system $\{\bar{\psi}_{i1}(s), \bar{\psi}_{i2}(s)\}_{i=1}^{\infty}$ of $W_2^{N+1}[0,1] \bigoplus W_2^{N+1}[0,1]$ can be derived from Gram-Schmidt orthogonalization process of

$$\{\psi_{i1}(s), \psi_{i2}(s)\}_{i=1}^{\infty},$$

i.e.

$$\bar{\psi}_{ij}(s) = \textstyle\sum_{l=1}^{i} \sum_{k=1}^{j} \beta_{lk}^{ij} \psi_{lk}(s).$$

Theorem 3.2. *For system (3.1), if $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_{i1}(s), \psi_{i2}(s)\}_{i=1}^{\infty}$ is a complete system for $W_2^{N+1}[0,1] \bigoplus W_2^{N+1}[0,1]$ and $\psi_{ij}(s) = A_t R_s(t)\vec{e}_j|_{t=s_i}$.*

Proof. Note that

$$\psi_{ij}(s) = A^*\varphi_{ij}(s) =< A^*\varphi_{ij}(t), R_s(t)\vec{e}_j >=< \varphi_{ij}(t), A_t R_s(t)\vec{e}_j >=< r_{s_i}(t)\vec{e}_j, A_t R_s(t)\vec{e}_j >= A_t R_s(t)\vec{e}_j|_{t=s_i}.$$

For each fixed $u(s) \in W_2^{N+1}[0,1] \bigoplus W_2^{N+1}[0,1]$, let $< u(s), \psi_{ij}(s) >= 0, \quad i = 1, 2, \ldots$ which means that,

$$< Au(s), \varphi_{ij}(s) >= 0. \tag{3.7}$$

Note that

$$u(s) = \textstyle\sum_{j=1}^{2} u_j(s)\vec{e}_j = \sum_{j=1}^{2} < u(t), R_s(t)\vec{e}_j > \vec{e}_j.$$

From Eq. (3.7), we have

$$Au(s_i) = \textstyle\sum_{j=1}^{2} < Au(s), \varphi_{ij}(s) > \vec{e}_j = 0, \quad i = 1, 2, \ldots.$$

Since $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, we must have $Au(s) = 0$. Since system (3.1) has a unique solution, it follows that $u = 0$ from the existence of A^{-1}. Therefore, $\{\psi_{i1}(s), \psi_{i2}(s)\}_{i=1}^{\infty}$ is the complete system of $W_2^{N+1}[0,1] \bigoplus W_2^{N+1}[0,1]$. $\square$

Theorem 3.3. *If $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, and the solution of system (3.1) is unique, then the solution of system (3.1) is*

$$u(s) = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \beta_{lk}^{ij} f_k(s_l)\bar{\psi}_{lk}(s). \tag{3.8}$$

Proof. Theorem 3.2, it is easy to show that $\{\bar{\psi}_{i1}(s), \bar{\psi}_{i2}(s)\}_{i=1}^{\infty}$ is a complete orthonormal basis for $W_2^{N+1}[0,1] \bigoplus W_2^{N+1}[0,1]$.
Note that $< v(s), \varphi_{ij}(s) >= v_j(s_i)$ for each $v(s) \in W_2^1[0,1] \bigoplus W_2^1[0,1]$. Hence, we have

$$\begin{aligned} u(s) &= \sum_{i=1}^{\infty}\sum_{j=1}^{2} < u(s), \bar{\psi}_{ij}(s) > \bar{\psi}_{ij}(s) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2} < u(s), \sum_{l=1}^{i}\sum_{k=1}^{j} \beta_{lk}^{ij}\psi_{lk}(s) > \bar{\psi}_{ij}(s) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \beta_{lk}^{ij} < u(s), A^*\varphi_{lk}(s) > \bar{\psi}_{lk}(s) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \beta_{lk}^{ij} < Au(s), \varphi_{lk}(s) > \bar{\psi}_{lk}(s) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \beta_{lk}^{ij} f_k(s_l)\bar{\psi}_{lk}(s). \qquad (3.9) \end{aligned}$$

$\square$

Lemma 3.4. $|u(s)| \leq (N+2)||u||_{W_2^{N+1}[0,1]}$, *for* $u(s) \in W_2^{N+1}[0,1]$.

Proof. Suppose that $u(s) \in W_2^{N+1}[0,1]$ then

$$u^{(N)}(s) = u^{(N)}(0) + \int_0^s u^{(N+1)}(t)\,dt. \tag{3.10}$$

Subsequently, we can obtain

$$\begin{aligned}
u^{(N-1)}(s) &= u^{(N-1)}(0) + su^{(N)}(0) + \int_0^s \int_0^{t_1} u^{(N+1)}(t)\, dtdt_1,\\
u^{(N-2)}(s) &= \sum_{k=0}^{2} \frac{1}{k!} s^{(k)} u^{(N-2+k)}(0) + \int_0^s \int_0^{t_1} \int_0^{t_2} u^{(N+1)}(t)\, dtdt_1dt_2,\\
&\vdots\\
u(s) &= \sum_{k=0}^{N} \frac{1}{k!} s^k u^{(k)}(0) + \int_0^s \int_0^{t_1} \cdots \int_0^{t_N} u^{(N+1)}(t)\, dtdt_1 \dots dt_N,
\end{aligned}$$

and therefore, $|u(s)| \leq \sum_{k=0}^{N} |u^{(k)}(0)| + \int_0^1 |u^{(N+1)}(t)|\, dt$.
Note that

$$\begin{aligned}
|u(0)| &\leq \left[(u(0))^2 + \sum_{k=1}^{N}(u^{(k)}(0))^2 + \int_0^1 (u^{(N+1)}(t))^2 dt\right]^{\frac{1}{2}} = ||u||_{W_2^{N+1}}\\
|u^{'}(0)| &\leq \left[(u^{'}(0))^2 + \sum_{k=0,k\neq 1}^{N}(u^{(k)}(0))^2 + \int_0^1 (u^{(N+1)}(t))^2 dt\right]^{\frac{1}{2}} = ||u||_{W_2^{N+1}},\\
&\vdots\\
|u^{(N)}(0)| &\leq \left[\sum_{k=0}^{N-1}(u^{(k)}(0))^2 + (u^{N}(0))^2 + \int_0^1 (u^{(N+1)}(t))^2 dt\right]^{\frac{1}{2}} = ||u||_{W_2^{N+1}}.
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 |u^{N+1}(t)|\, dt &\leq \left[\int_0^1 dt\right]^{\frac{1}{2}} \left[\int_0^1 (u^{N+1}(t))^2 dt\right]^{\frac{1}{2}} = \left[\int_0^1 (u^{N+1}(t))^2 dt\right]^{\frac{1}{2}}\\
&\leq \left[\sum_{k=0}^{N}(u^{(k)}(0))^2 + \int_0^1 (u^{(N+1)}(t))^2 dt\right]^{\frac{1}{2}}, \qquad (3.11)
\end{aligned}$$

thus, $|u(s)| \leq (N+2)||u||_{W_2^{N+1}}$. □

Theorem 3.5. *Truncating n-term of the infinite series in Eq. (3.8), we obtain the approximate solution of Eqs. (1.1)*

$$u_n(s) = (u_{1,n}(s), u_{2,n}(s))^T = \sum_{i=1}^{n} \sum_{j=1}^{2} \sum_{l=1}^{i} \sum_{k=1}^{j} \beta_{lk}^{ij} f_k(s_l) \bar{\psi}_{lk}(s), \qquad (3.12)$$

which converges uniformly to the exact solution $u(s) = (u_1(s), u_2(s))^T$ as $n \to \infty$.

Proof. It suffices to prove that $||u-u_n||^2 \to 0$ as $n \to \infty$.
Notice that $||u-u_n||^2 \to \sum_{i=1}^{2} ||u_i - u_{i,n}||^2_{W_2^{N+1}[0,1]} \to 0$. By the expression of $R_s(t)$, we have

$$
\begin{aligned}
|u_i(s) - u_{i,n}(s)| &= |< u_i(t) - u_{i,n}(t), R_s(t) >| \\
&\leq ||R_s(t)|| ||u_i - u_{i,n}||_{W_2^{N+1}[0,1]} \\
&= \sqrt{R_s(s)} ||u_i - u_{i,n}||_{W_2^{N+1}[0,1]} \\
&\leq \sqrt{N+2} ||u_i - u_{i,n}||_{W_2^{N+1}[0,1]}. \qquad (3.13)
\end{aligned}
$$

This argument shows that $u_{i,n}(s)$ converges uniformly to $u_i(s)$ on the interval $[0,1]$ as $n \to \infty$. □

4. Numerical example

Table 2. The error $\|e_n(s)\|$ for $n = 30, 40, 50$.

s	$\|e_{30}(s)\|$	$\|e_{40}(s)\|$	$\|e_{50}(s)\|$
0.1	$2.4563e-4$	$1.3612e-4$	$8.8451e-5$
0.2	$2.3196e-4$	$1.0241e-4$	$8.5206e-5$
0.3	$1.9270e-4$	$8.4289e-5$	$6.8210e-5$
0.4	$1.6043e-4$	$6.5873e-5$	$5.4135e-5$
0.5	$7.9323e-5$	$6.3517e-5$	$5.2324e-5$
0.6	$6.1109e-5$	$6.3641e-5$	$5.0136e-5$
0.7	$5.1956e-5$	$5.6710e-5$	$3.5107e-5$
0.8	$2.7836e-5$	$2.2457e-5$	$2.7234e-5$
0.9	$2.4374e-5$	$2.5376e-5$	$2.0538e-5$
1.0	$1.7321e-5$	$1.7901e-5$	$1.6206e-5$

To illustrate the effectiveness of the proposed method , a test example is carried out in this section. For comparing the solution series given by RKHS method with exact solution, we report the sum of absolute errors which is defined by

$$\|e_n(s)\| = |u_1(s) - u_{1,n}(s)| + |u_2(s) - u_{2,n}(s)|. \qquad (4.1)$$

Example 4.1. Consider a system of second-order linear Fredholm integro-differential equations [8]

$$(4.2)\quad \begin{cases} u_1''(s) + 2\int_0^1 stu_1(t)\,dt + u_2'(s) - 6\int_0^1 stu_2(t)\,dt = 3s^2 + \frac{3}{10}s + 8, \\ u_1'(s) + 3\int_0^1 (2s+t^2)u_1(t)\,dt + u_2''(s) - 6\int_0^1 (2s+t^2)u_2(t)\,dt = 21s + \frac{4}{5}, \\ u_1(0) + u_1'(0) = 1, u_1(1) + u_1'(1) = 10, \\ u_2(0) + u_2'(0) = 1, u_2(1) + u_2'(1) = 7. \end{cases}$$

The exact solutions are $u_1(s) = 3s^2 + 1$ and $u_2(s) = s^3 + 2s - 1$. Applying the RKHS method, we obtain the approximate solution for $n = 30, 40, 50$. We choose $\{s_i = \frac{i-1}{n-1}\}_{i=1}^n$ to construct the orthonormal system $\{\bar{\psi}_{i1}(s), \bar{\psi}_{i2}(s)\}_{i=1}^n$ in the space reproducing Hilbert

$$(4.3)\qquad W_2^{N+1}[0,1] \bigoplus W_2^{N+1}[0,1].$$

The numerical results are given in Table 1 and Fig. 1. We see that the approximation solution obtained by the present method has good agreement with the exact solution.

5. Conclusions

In this paper, it is shown that the RKHS method is quiet efficient and well suited for finding the exact or approximate solution for system of linear Fredholm integro-differential equations with variable coefficients. The applicability and accuracy of the approaches were checked by calculating the approximate solution at selected grid points. Based on obtained results of the proposed method for illustrative example, we have the following remarkable conclusions:

- The proposed method provides the solution in a convergent series with components that can be simply computed.
- The results obtained by using the proposed method are very attractive and contributed to the good agreement between approximate and exact values in the numerical example.
- The proposed method can be easily implemented and its algorithm is simple and efficient to the approximate solution.

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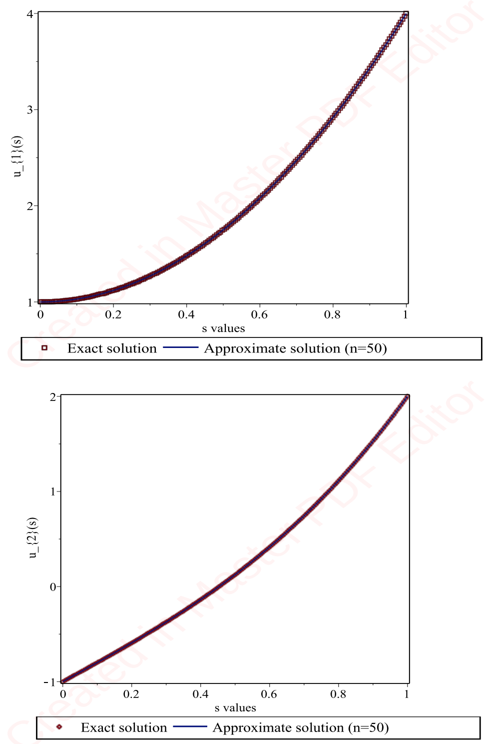

Plot of $u_1(s)$ and $u_{1,50}(s)$ Plot of $u_2(s)$ and $u_{2,50}(s)$

FIGURE 1. Comparison of the exact solution with approximate solution given by RKHS method.

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