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RIGHT DERIVATIONS ON ORDERED SEMIGROUPS

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Abstract. Over the last few decades, several authors have investigated the relationship between the commutativity of ring R and the existence of certain specified derivations of R. In this paper, we introduce the concept of right derivation on semigroups and we study some of the properties of right derivation of semigroups. We prove that if d be a non-zero right derivation of a cancellative ordered semigroup M, then M is a commutative ordered semigroup.

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1. Introduction

Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. The first result in this derivation is due to Posner [15] in 1957. In the year 1990, Bresar and Vukman [1] established that a prime ring must be commutative if it admits a nonzero left derivation. Kim [2,3,4] studied right derivation and generalized derivation of incline algebra. The notion of derivation of algebraic structures is useful for characterization of algebraic structures. The notion of derivation has also been generalized in various directions such as right derivation, left derivation, f-derivation, reverse derivation, orthogonal derivation, (f, g)-derivation, generalized right derivation, etc. Murali Krishna Rao and Venkateswarlu[7,8] introduced the notion of generalized right derivation of $\Gamma-$ incline and right derivation of ordered $\Gamma-$semiring. Murali Krishna Rao [9-14] studied ideals of various aigebraic structures. In this paper, we introduce the concept of right derivation on semigroups and we study some of the properties of right derivation of semigroups.

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2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions necessary for this paper.

Definition 2.1. A semigroup is an algebraic system $(S, .)$ consisting of a non-empty set S together with an associative binary operation $'' \cdot ''$.

Definition 2.2. A non-empty subset A of a semigroup M is called

(i) a subsemigroup of Mif $AA \subseteq A$.
(ii) a quasi ideal of M if A is $AM \cap MA \subseteq A$.
(iii) a bi-ideal of M if A is a subsemigroup of M and $AMA \subseteq A$.
(iv) an interior ideal of M if A is a subsemigroup of M and $MAM \subseteq A$.
(v) a left (right) ideal of M if $MA \subseteq A(AM \subseteq A)$.
(vi) an ideal if $AM \subseteq A$ and $MA \subseteq A$.
(vii) a bi-quasi-interior ideal of M if A is a subsemigroup of M and $AMAMA \subseteq A$.
((viii) a bi-interior ideal of M if A is a subsemigroup of M and $MAM \cap AMA \subseteq A$.
((ix) a left bi-quasi ideal (right bi-quasi ideal) of M if A is a subsemigroup and $MA \cap AMA \subseteq A$ $(AM \cap AMA \subseteq A)$.
((x) a left quasi-interior ideal (right quasi-interior ideal) of M if A is a subsemigroup of $(M, +)$ and $MAMA \subseteq A$ $(AMAM \subseteq A)$.

Definition 2.3. Let M be a semigroup. If there exists $1 \in M$ such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in M$, is called an unity element of M then M is said to be semigroup with unity.

Definition 2.4. An element a of a semigroup S is called a regular element if there exists an element b of S such that $a = aba$.

Definition 2.5. A semigroup S is called a regular semigroup if every element of S is a regular element.

Definition 2.6. An element a of a semigroup S is called an idempotent element if $aa = a$.

Definition 2.7. An element b of a semigroup M is called an inverse element of a of M if $ab = ba = 1$.

Definition 2.8. A non-empty subset A of a semigroup M is called

(i) a subsemigroup of M if $AA \subseteq A$.
(ii) a left(right) ideal of M if A is an subsemigroup of M and $MA \subseteq A(AM \subseteq A)$.
(iii) an ideal if $MA \subseteq A$ and $AM \subseteq A$.

Definition 2.9. A semigroup M is called a group if for each non-zero element of M has multiplication inverse.

Definition 2.10. A semigroup M is called an ordered semigroup if it admits a compatible relation $\leq$. i.e. $\leq$ is a partial ordering on M satisfies the following conditions. If $a \leq b$ and $c \leq d$ then

(i) $ac \leq bd$
(ii) $ca \leq db$, for all $a, b, c, d \in M$

Definition 2.11. An ordered semigroup M is said to have zero element if there exists an element $0 \in M$ such that $0x = x0 = 0$, for all $x \in M$.

An ordered semigroup M is said to be commutative semigroup if $xy = yx$, for all $x, y \in M$

Definition 2.12. A non zero element a in an ordered semigroup M is said to be a zero divisor if there exists non zero element $b \in M$, such that $ab = ba = 0$.

Definition 2.13. An ordered semigroup M with unity 1 and zero element 0 is called an integral ordered semigroup if it has no zero divisors.

Definition 2.14. An ordered semigroup M is said to be totally ordered semigroup if any two elements of M are comparable.

Definition 2.15. In an ordered semigroup M

(i) the semigroup M is said to be positively ordered, if $a \leq ab$ and $b \leq ab$, for all $a, b \in M$.
(ii) the semigroup M is said to be negatively ordered if $ab \leq a$ and $ab \leq b$, for all $a, b \in M$.

Definition 2.16. A non-empty subset A of an ordered semigroup M is called a subsemigroup M if $ab \in A$ for all $a, b \in A$.

Definition 2.17. Let M be an ordered semigroup. A non-empty subset I of M is called a left (right) ideal of an ordered semigroup M if $MI \subseteq I$ $(IM \subseteq I)$ and if for any $a \in M,\ b \in I,\ a \leq b \Rightarrow a \in I$. I is called an ideal of M if it is both a left ideal and a right ideal of M.

Definition 2.18. Let M and N be ordered semigroups. A mapping $f : M \rightarrow N$ is called a homomorphism if $f(ab) = f(a)f(b)$, for all $a, b \in M$.

Definition 2.19. Let M be an ordered semigroup. A mapping $f : M \rightarrow M$ is called an endomorphism if

(i) f is an onto ,
(ii) $f(ab) = f(a)f(b)$, for all $a, b \in M$.

3. Right derivation of ordered semigroups

In this section, we introduce the concept of right derivation on semigroups and we study some of the properties of right derivation of semigroups

Definition 3.1. Let M be an ordered semigroup. If a mapping $d : M \rightarrow M$ satisfies the following conditions

(i) $d(xy) = d(x)yd(y)x$
(ii) If $x \leq y$ then $d(x) \leq d(y)$, for all $x, y \in M$.

then d is called a right derivation of M.

Example 3.2. Let N be a the set of all natural numbers.
Let $M = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mid a, b, c \in N \cup \{0\} \right\}$ be additive semigroup with respect to usual multiplication of matrices. Define $[a_{ij}] \leq [b_{ij}]$ if and only if $a_{ij} \leq b_{ij}$, for all i, j, where $a_{ij}, b_{ij} \in M$. Then M is an ordered semigroup.
Define $d : M \rightarrow M$ by $d \begin{pmatrix} c & a \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & o \end{pmatrix}$. Then d is a right derivation of M.

Theorem 3.3. *Let M be a commutative idempotent ordered semigroup. Then for a fixed element $a \in M$, the mapping $d_a : M \rightarrow M$ given by $d_a(x) = xa$, for all $x \in M$ is a right derivation of M.*

Proof. Let M be a commutative ordered semigroup and $a \in M$. Suppose $x, y \in M$.

$$\begin{aligned} d_a(xy) &= (xy)a \\ &= (xy)a(xy)a \\ &= (xa)y(ya)x \\ &= d_a(x)yd_a(y)x. \\ \text{Suppose } x \leq y &\Rightarrow xa \leq ya \\ &\Rightarrow d_a(x) \leq d_a(y). \end{aligned}$$

Hence d_a is a right derivation of M. □

Theorem 3.4. *Let d be a right derivation of an ordered semigroup M. Then $d(0) = 0$*

Proof. Let d be a right derivation of the ordered semigroup M. Then

$$\begin{aligned} d(0) &= d(00) \\ &= d(0)0d(0)0 \\ &= 00 \\ &= 0. \\ \text{Therefore } d(0) &= 0 \end{aligned}$$

□

Theorem 3.5. *Let d be a right derivation of an idempotent negatively ordered semigroup M. Then $d(x) \leq x$, for all $x \in M$.*

Proof. Let d be a right derivation of the idempotent negatively ordered semigroup M. Then

$$\begin{aligned} d(x) &= d(xx) \\ &= d(x)xd(x)x \\ &\leq d(x)x \\ \Rightarrow d(x) &\leq d(x)x \\ \Rightarrow d(x) &\leq x. \end{aligned}$$

Hence the theorem. □

Theorem 3.6. *Let M be a negatively ordered semigroup . Then $d(xy) \leq d(x)d(y))$.*

Proof. Let M be a negatively ordered semigroup. Suppose $x, y \in M$.
Then $d(x)y \leq d(x)$ and $d(y)x \leq d(y)$. Therefore

$$\begin{aligned} d(xy) &= d(x)yd(y)x \\ &\leq d(x)d(y) \end{aligned}$$

Hence the theorem. □

Theorem 3.7. *Let M be an idempotent negatively ordered semigroup. Then $d(xd(x)) \leq d(x)$, for all $x \in M$.*

Proof. Let M be an idempotent negatively ordered semigroup. Then

$$\begin{aligned} d(xd(x)) &= d(x)d(x)d(d(x))x \\ &\leq d(x)d(x) \\ &\leq d(x). \\ \text{Therefore } d(xd(x)) &\leq d(x). \end{aligned}$$

□

Let d be a right derivation of an ordered semigroup M. Define $kerd = \{x \in M \mid d(x) = 0\}$

Theorem 3.8. *Let d be a right derivation of an idempotent negatively ordered integral semigroup M. Then $kerd$ is an ideal of an ordered semigroup M.*

Proof. Let d be a right derivation of the idempotent negatively ordered integral semigroup M. Suppose $x, y \in kerd$. Then $d(x) = 0$ and $d(y) = 0 \Rightarrow d(xy) = 0. \Rightarrow xy \in kerd$. Therefore $kerd$ is a subsemigroup of M.

Let $x \in M$ and $y \in kerd$ such that $x \leq y$. Then $d(y) = 0$.

$$\begin{aligned} \Rightarrow d(xy) &= d(x)yd(y)x \\ &= d(x)y0x \\ &= 0 \\ \text{since } x \leq y \Rightarrow & xx \leq xy \\ \Rightarrow & d(xx) \leq d(xy) \\ \Rightarrow & d(xx) \leq 0. \\ \Rightarrow & d(xx) = 0. \\ \text{Therefore } & d(x) = 0 \end{aligned}$$

Hence $x \in kerd$. Then $kerd$ is an ideal of the ordered semigroup M. □

Theorem 3.9. *Let M be a negatively ordered semigroup. Then $d(xy) \leq d(x)$, for all $x, y \in M$ and $d(xy) \leq d(y)$.*

Proof. Let M be a negatively ordered semigroup.

$$\begin{aligned} d(xy) &= d(x)yd(y)x \\ &\leq d(x)d(y) \\ &\leq d(x). \end{aligned}$$

Similarly we can prove $d(xy) \leq d(y)$. □

Theorem 3.10. *Let d be a right derivation of a commutative idempotent -negatively ordered semigroup M. Define a set $Fix_d(M) = \{x \in M \mid d(x) = x\}$. Then $Fix_d(M)$ is an ideal of M.*

Proof. Let d be a right derivation of the commutative idempotent -negatively ordered semigroup M. Suppose $x, y \in Fix_d(M)$. Then

$$\begin{aligned} d(x) &= x, d(y) = y \\ d(xy) &= d(x)yd(y)x \\ &= xyyx \\ &= xyxy = xy. \text{Therefore } xy \in Fix_d(M) \end{aligned}$$

Therefore $Fix_d(M)$ is a subsemigroup of M. Suppose $x \leq y$ and $y \in Fix_d(M)$.

$$\begin{aligned} & x \leq y \\ & \Rightarrow xx \leq xy \\ & \Rightarrow x \leq xy \\ \text{Therefore } & xy = x \\ & \Rightarrow d(xy) = d(x) \\ & \Rightarrow = d(x)yd(y)x = d(x) \\ & \Rightarrow d(x)yyx = d(x) \\ & \Rightarrow d(x)yx = d(x). \\ \text{Therefore } & x \leq d(x) \end{aligned}$$

By Theorem 3.5 $d(x) \leq x$..Therefore $d(x) = x$. Hence $Fix_d(M)$ is an ideal of M. □

Theorem 3.11. *Let M be a negatively ordered semigroup with unity 1 and d be a right derivation of M. Then*

(i) $d(1)x \geq d(x)$
(ii) *If* $d(1) = 1$ *then* $x \geq d(x)$.

Proof. Let M be a negatively ordered semigroup with unity 1, d be a right derivation of M and $x \in M$. Then $x1 = x$ and $1x = x$.

$$\begin{aligned} d(x) &= d(x1) \\ &= d(x)1d(1)x \\ \Rightarrow d(x) &= d(x)1d(1)x \leq d(1)x \end{aligned}$$

Suppose $d(1) = 1$ then $d(1)x \geq d(x)$. Therefore $1x \geq d(x) \Rightarrow x \geq d(x)$. Hence the theorem. □

Theorem 3.12. *Let M be an idempotent negatively ordered semigroup with unity and d be a right derivation of M. Then $d(1) = 1$ if and only if $d(x) = x$.*

Proof. Let M be a negatively idempotent ordered semigroup with unity and d be a right derivation. Suppose $x \in M$.

$$\begin{aligned} d(x) &= d(xx) \\ &= d(x)xd(x)x \\ &= d(x)x \leq x \\ d(x) &\leq x. \end{aligned}$$

Suppose $d(1) = 1$. By Theorem [3.11], we have $x \leq d(x)$. Therefore $d(x) = x$. Converse is obvious. □

Theorem 3.13. *Let M be a cancellative ordered semgroup and d be a non-zero right derivation of M. Then M is a commutative ordered semigroup.*

Proof. Let M be a cancellative ordered semgroup and $d : M \to M$ be a non-zero right derivation of M. Suppose $a, b \in M$.

$$\begin{aligned} d(aba) &= d(a)bd(ba)a \\ &= d(a)bd(b)ad(a)ba \cdots \ (1) \\ d(aba) &= d(ab)ad(a)ab \\ &= d(a)bd(b)ad(a)ab \cdots \ (2). \end{aligned}$$

From (1) and (2) ,

$$\begin{aligned} d(a)bd(b)ad(a)ba &= d(a)bd(b)ad(a)ab \\ &\Rightarrow \ ba = ab, \text{ for all } \ a, b \in M. \end{aligned}$$

Hence M is a commutative ordered semigroup. □

4. Conclusion

Over the last few decades, several authors have investigated the relationship between the commutativity of ring R and the existence of certain specified derivations of R. In this paper, we introduced the concept of right derivation on semigroups, we studied some of the properties of right derivation of semigroups. We proved that if d is a non-zero right derivation of a cancellative ordered semigroup M, then M is a commutative ordered semigroup.

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