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(a, b)-FUZZY SUBRINGS AND (a, b)-FUZZY IDEALS OF A RING

SHRIRAM KHANDERAO NIMBHORKAR AND JYOTI ASHOK KHUBCHANDANI

Abstract. As an extension of the concept of a fuzzy subring and a fuzzy ideal, a new kind of a fuzzy subring and a fuzzy ideal called an (a, b)-fuzzy subring and an (a, b)-fuzzy ideal of a ring is defined and their properties are studied. We also investigate the preimage of an (a, b)-fuzzy subring and an (a, b)-fuzzy ideal under a ring homomorphism. Also, (a, b)-level fuzzy subrings (fuzzy ideals) are studied. A necessary and sufficient condition for two (a, b)-level fuzzy subrings (fuzzy ideals) to be equal is proved. We show that the set of cosets of an (a, b)-fuzzy ideal forms a ring.

Key Words: (a, b)-fuzzy subring, (a, b)-fuzzy ideal, (a, b)-fuzzy level subset.
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1. Introduction

In 1965, Zadeh [3] introduced the concept of a fuzzy set. Later in 1971, Rosenfeld [1] used this concept to define a fuzzy subgroupoid and a fuzzy subgroup. Liu [5] studied fuzzy invariant subgroups, fuzzy ideals and proved some fundamental properties. Sharma [4] introduced and studied the concept of an α-fuzzy subgroup. We extend this concept to form (a, b)-fuzzy subrings and (a, b)-fuzzy ideals of a ring R.

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∗Address correspondence to Jyoti A. Khubchandani; E-mail: khubchandani_jyoti@yahoo.com

2. **Preliminaries**

Throughout in this paper R denotes a commutative ring with identity. We recall some definitions and results.

Definition 2.1. [3] Let S be a nonempty set. A mapping $\omega : S \to [0, 1]$ is called a fuzzy subset of S.

Remark 2.2. [3] If ω and σ are two fuzzy subsets of R, then
(i) $\omega \subseteq \sigma$ if and only if $\omega(x) \leq \sigma(x)$;
(ii) $(\omega \cup \sigma)(x) = \max\{\omega(x), \sigma(x)\} = \omega(x) \vee \sigma(x)$;
(iii) $(\omega \cap \sigma)(x) = \min\{\omega(x), \sigma(x)\} = \omega(x) \wedge \sigma(x)$; for all $x \in R$.

Definition 2.3. [2] Let X and Y be two nonempty sets and $g : X \to Y$ be a mapping. Let $\omega \in [0, 1]^X$ and $\sigma \in [0, 1]^Y$. Then the image $g(\omega) \in [0, 1]^Y$ and the inverse image $g^{-1}(\sigma) \in [0, 1]^X$ are defined as follows:
for all $y \in Y$,

$$g(\omega)(y) = \begin{cases} \vee\{\omega(x) \mid x \in X, g(x) = y\}, & \text{if } g^{-1}(y) \neq \phi, \\ 0, & \text{otherwise.} \end{cases}$$

and $g^{-1}(\sigma)(x) = \sigma(g(x))$ for all $x \in X$.

Definition 2.4. [3] Let ω be a fuzzy subset of a set S and let $t \in [0, 1]$. The set $\omega_t = \{x \in R \mid \omega(x) \geqslant t\}$ is called a level subset of ω.

Clearly, $\omega_t \subseteq \omega_s$ whenever $t > s$.

Definition 2.5. [5] A fuzzy subset ω of R is called a fuzzy subring, if for all $x, y \in R$, the following conditions hold:
(i) $\omega(x - y) \geq \min(\omega(x), \omega(y))$;
(ii) $\omega(xy) \geq \min(\omega(x), \omega(y))$.

Definition 2.6. [5] A fuzzy subset ω of R is called a fuzzy ideal, if for all $x,\ y \in R$, the following conditions are satisfied:
(i) $\omega(x - y) \geq \min(\omega(x), \omega(y))$;
(ii) $\omega(xy) \geq \max(\omega(x), \omega(y))$.

3. (a, b)**-Fuzzy subsets and their properties**

Sharma [4] introduced the concept of an α-fuzzy subgroup. We extend this concept to a subring and an ideal of a ring. This notion is used to construct a fuzzy subring (ideal) from a fuzzy set.

Definition 3.1. Let ω be a fuzzy subset of R. Let $0 \leq b < a \leq 1$. Then the fuzzy set ω_b^a of R defined by $\omega_b^a(x) = \min\{\omega(x), 1 - a + b\}$, for all

$x \in R$, is called as the (a, b)-fuzzy subset of R with respect to the fuzzy set ω.

Lemma 3.2. *(i) Let ω and η be two fuzzy subsets of X. Then $(\omega \cap \eta)^a_b = \omega^a_b \cap \eta^a_b$.*
(ii) Let $g : X \longrightarrow Y$ be an onto mapping and η be a fuzzy subset of Y. Define $\eta \circ g : X \to [0, 1]$ by $(\eta \circ g)(x) = \eta(g(x))$. Then $\eta^a_b \circ g = (\eta \circ g)^a_b$.
(iii) Let $g : X \longrightarrow Y$ be a onto mapping and η be two fuzzy subsets of Y. Then $g^{-1}(\eta^a_b) = (g^{-1}(\eta))^a_b$.

Proof. (i): For all $x \in X$ we have

$$\begin{aligned}(\omega \cap \eta)^a_b(x) &= \min\{(\omega \cap \eta)(x), 1 - a + b\} \\ &= \min\{\min\{\omega(x), \eta(x)\}, 1 - a + b\} \\ &= \min\{\min\{\omega(x), 1 - a + b\}, \min\{\eta(x), 1 - a + b\}\} \\ &= \min\{\omega^a_b(x), \eta^a_b(x)\} \\ &= \omega^a_b(x) \cap \eta^a_b(x) \\ &= (\omega^a_b \cap \eta^a_b)(x).\end{aligned}$$

Hence, $(\omega \cap \eta)^a_b = \omega^a_b \cap \eta^a_b$.
(ii): For all $x \in X$, we have

$$\begin{aligned}(\eta^a_b \circ g)(x) &= \eta^a_b(g(x)) \\ &= \min\{\eta(g(x)), 1 - a + b\} \\ &= \min\{(\eta \circ g)(x), 1 - a + b\} \\ &= (\eta \circ g)^a_b(x).\end{aligned}$$

Hence, $\eta^a_b \circ g = (\eta \circ g)^a_b$.
(iii): Consider

$$\begin{aligned}g^{-1}(\eta^a_b)(x) &= \eta^a_b(g(x)) \\ &= \min\{\eta(g(x)), 1 - a + b\} \\ &= \min\{g^{-1}(\eta(x)), 1 - a + b\} \\ &= (g^{-1}(\eta))^a_b(x), \quad \text{for all } x \in X.\end{aligned}$$

Hence, $g^{-1}(\eta^a_b) = (g^{-1}(\eta))^a_b$.

□

4. (a, b)-Fuzzy subrings

Definition 4.1. Let ω be a fuzzy subset of R. Let $0 \leq b < a \leq 1$. Then ω is called an (a, b)-fuzzy subring of R if ω_b^a is a fuzzy subring of R, that is, if the following conditions hold:
(i) $\omega_b^a(x - y) \geq \min\{\omega_b^a(x), \omega_b^a(y)\}$;
(ii) $\omega_b^a(xy) \geq \min\{\omega_b^a(x), \omega_b^a(y)\}$, for all $x, y \in R$.

Proposition 4.2. *If ω is a fuzzy subring of R, then ω is also (a, b)-fuzzy subring of R.*

Proof. For $x, y \in R$ we have

$$\begin{aligned} \omega_b^a(x-y) &= \min\{\omega(x-y), 1-a+b\} \\ &\geq \min\{\min\{\omega(x), \omega(y)\}, 1-a+b\}, \\ &\quad \text{(since } \omega \text{ is a fuzzy subring of } R) \\ &= \min\{\min\{\omega(x), 1-a+b\}, \min\{\omega(y), 1-a+b\}\} \\ &= \min\{\omega_b^a(x), \omega_b^a(y)\}. \end{aligned} \tag{4.1}$$

Also,

$$\begin{aligned} \omega_b^a(xy) &= \min\{\omega(xy), 1-a+b\} \\ &\geq \min\{\min\{\omega(x), \omega(y)\}, 1-a+b\}, \\ &\quad \text{(since } \omega \text{ is a fuzzy subring of } R) \\ &= \min\{\min\{\omega(x), 1-a+b\}, \min\{\omega(y), 1-a+b\}\} \\ &= \min\{\omega_b^a(x), \omega_b^a(y)\}. \end{aligned} \tag{4.2}$$

It follows from (4.1) and (4.2), that ω is (a, b)-fuzzy subring of R. □

The following example shows that the converse of Proposition 4.2 need not hold.

Example 4.3. Consider the fuzzy subset of the ring $R = \mathbb{Z}_8$ defined as follows:

$$\omega(x) = \begin{cases} 0.4, & \text{if } x = \{0, 4\}, \\ 0.7, & \text{if } x = \{1, 2, 3, 5, 6, 7\}. \end{cases}$$

We note that for $x = 6$, $y = 2$, $\omega(6) = \omega(2) = 0.7$ and
$\omega(x - y) = \omega(6 - 2) = \omega(4) = 0.4$. Thus, $\omega(x - y) \ngeq \min\{\omega(x), \omega(y)\}$.
Hence, ω is not a fuzzy subring of R.
We note that if $a = 0.9$, $b = 0.2$, then $1 - a + b = 0.3$ and so

$\omega(x) > 1 - a + b = 0.3$ for all $x \in R$. Hence

$$\omega_{0.2}^{0.9}(x) = \min\{\omega(x), 0.3\} = 0.3, \text{ for all } x \in R.$$

Therefore,

$$\omega_{0.2}^{0.9}(x-y) \geq \min\{\omega_{0.2}^{0.9}(x), \omega_{0.2}^{0.9}(y)\}$$

and

$$\omega_{0.2}^{0.9}(xy) \geq \min\{\omega_{0.2}^{0.9}(x), \omega_{0.2}^{0.9}(y)\}.$$

Hence, ω is an $(0.9, 0.2)$-fuzzy subring of R.

Proposition 4.4. *The intersection of two (a,b)-fuzzy subrings of a ring R is again an (a,b)-fuzzy subring of R.*

Proof. Let ω and η be two (a,b)-fuzzy subrings of a ring R.
For $x, y \in R$, we have

$$\begin{aligned}(\omega \cap \eta)_b^a(x-y) &= (\omega_b^a \cap \eta_b^a)(x-y), \text{ by Lemma 3.2} \\ &= \min\{\omega_b^a(x-y), \eta_b^a(x-y)\}\} \\ &\geq \min\{\min\{\omega_b^a(x), \omega_b^a(y)\}, \min\{\eta_b^a(x), \eta_b^a(y)\}\} \\ &= \min\{\min\{\omega_b^a(x), \eta_b^a(x)\}, \min\{\omega_b^a(y), \eta_b^a(y)\}\} \\ &= \min\{(\omega_b^a \cap \eta_b^a)(x), (\omega_b^a \cap \eta_b^a)(y)\} \\ &= \min\{(\omega \cap \eta)_b^a(x)), (\omega \cap \eta)_b^a(y))\}. \end{aligned} \tag{4.3}$$

Also,

$$\begin{aligned}(\omega \cap \eta)_b^a(xy) &= (\omega_b^a \cap \eta_b^a)(xy), \text{ by Lemma 3.2} \\ &= \min\{\omega_b^a(xy), \eta_b^a(xy)\} \\ &\geq \min\{\min\{\omega_b^a(x), \omega_b^a(y)\}, \min\{\eta_b^a(x), \eta_b^a(y)\}\} \\ &= \min\{\min\{\omega_b^a(x), \eta_b^a(x)\}, \min\{\omega_b^a(y), \eta_b^a(y)\}\} \\ &= \min\{(\omega_b^a \cap \eta_b^a)(x), (\omega_b^a \cap \eta_b^a)(y)\} \\ &= \min\{(\omega \cap \eta)_b^a(x)), (\omega \cap \eta)_b^a(y))\}. \end{aligned} \tag{4.4}$$

It follows from (4.3) and (4.4), that $\omega \cap \eta$ is an (a,b)-fuzzy subring of R. □

The following example shows that the union of two (a,b)-fuzzy subrings of a ring R need not be an (a,b)-fuzzy subring of R.

Example 4.5. Define fuzzy subsets ω and η of the ring $R = \mathbb{Z}$ as follows:

$$\omega(x) = \begin{cases} 0.5, & \text{if } x \in 4\mathbb{Z}, \\ 0.1, & \text{otherwise.} \end{cases}$$

$$\eta(x) = \begin{cases} 0.25, & \text{if } x \in 5\mathbb{Z}, \\ 0.08, & \text{otherwise.} \end{cases}$$

Let $a = 0.5, b = 0.2$. Then $1 - a + b = 0.7$.
We note that ω and η are $(0.5, 0.2)$-fuzzy subrings of $\mathbb{Z}$.
We know that, $(\omega \cup \eta)(x) = \max\{\omega(x), \eta(x)\}$.
Therefore,

$$(\omega \cup \eta)(x) = \begin{cases} 0.5, & \text{if } x \in 4\mathbb{Z}, \\ 0.25, & \text{if } x \in 5\mathbb{Z}, \\ 0.1, & \text{if } x \notin 4\mathbb{Z} \cup 5\mathbb{Z}. \end{cases}$$

Let $x = 12, y = 5$. Then $(\omega \cup \eta)(x) = 0.5$, $(\omega \cup \eta)(y) = 0.25$ and $(\omega \cup \eta)(x - y) = 0.1$.
Also,

$$(\omega \cup \eta)^{0.5}_{0.2}(x) = \min\{(\omega \cup \eta)(x), 0.7\} = \min\{0.5, 0.7\} = 0.5.$$
$$(\omega \cup \eta)^{0.5}_{0.2}(y) = \min\{(\omega \cup \eta)(y), 0.7\} = \min\{0.25, 0.7\} = 0.25.$$
$$(\omega \cup \eta)^{0.5}_{0.2}(x - y) = \min\{(\omega \cup \eta)(x - y), 0.7\} = \min\{0.1, 0.7\} = 0.1.$$

Thus,

$$(\omega \cup \eta)^{0.5}_{0.2}(x - y) \ngeq \min\{(\omega \cup \eta)^{0.5}_{0.2}(x), (\omega \cup \eta)^{0.5}_{0.2}(y)\}.$$

Hence, $\omega \cup \eta$ is not a $(0.5, 0.2)$-fuzzy subring of R.

Theorem 4.6. *Let g be a homomorphism from a ring R onto a ring R'. If ω is an (a, b)-fuzzy subring of R', then $g^{-1}(\omega)$ is an (a, b)-fuzzy subring of R.*

Proof. Let $x, y \in R$. We have

$$\begin{aligned} (g^{-1}(\omega))^a_b(x - y) &= g^{-1}(\omega^a_b)(x - y), \text{ by Lemma 3.2} \\ &= \omega^a_b((g(x - y))) \\ &= \omega^a_b(g(x) - g(y)) \\ &\geq \min\{\omega^a_b(g(x)), \omega^a_b(g(y))\}, \\ &\quad \text{(since } \omega \text{ is an } (a, b)\text{-fuzzy subring of } R^{'}) \\ &= \min\{g^{-1}(\omega^a_b(x)), g^{-1}(\omega^a_b(y))\} \\ &= \min\{(g^{-1}(\omega))^a_b(x), (g^{-1}(\omega))^a_b(y)\} \qquad (4.5) \end{aligned}$$

We have

$$\begin{aligned}
&(g^{-1}(\omega))^a_b(xy)\\
&= g^{-1}(\omega^a_b)(xy), \text{ by Lemma 3.2}\\
&= \omega^a_b((g(xy)))\\
&= \omega^a_b(g(x)g(y))\\
&\geq \min\{\omega^a_b(g(x)), \omega^a_b(g(y))\},\\
&\quad \text{(since } \omega \text{ is an } (a,b)\text{-fuzzy subring of } R^{'})\\
&= \min\{g^{-1}(\omega^a_b)(x), g^{-1}(\omega^a_b)(y)\}\\
&= \min\{(g^{-1}(\omega))^a_b(x), (g^{-1}(\omega))^a_b(y)\}, \text{ by Lemma 3.2.} \qquad (4.6)
\end{aligned}$$

From (4.5) and (4.6), it follows that $g^{-1}(\omega)$ is an (a,b)-fuzzy subring of R. □

Definition 4.7. Let $\omega : R \to [0,1]$ be a fuzzy subset of R.
For $t \in [0,1]$, the (a,b)-level subset of ω is denoted by $(\omega^a_b)_t$ and is defined as $(\omega^a_b)_t = \{x \in R \mid \omega^a_b(x) \geq t\}$.

Example 4.8. Let $\omega : \mathbb{Z}_9 \to [0,1]$ be as follows:

$$\omega(x) = \begin{cases} 0.7, & if\ x = \{0,3,6\},\\ 0.1, & \text{otherwise.}\end{cases}$$

Let $a = 1, b = 0.5$ and $t = 0.4$. We have $1 - a + b = 0.5$.
Then

$$\omega^a_b(x) = \omega^1_{0.5}(x) = \begin{cases} 0.5, & \text{if } x = \{0,3,6\},\\ 0.1, & \text{otherwise.}\end{cases}$$

and $(\omega^1_{0.5})_{0.4} = \{x \in \mathbb{Z}_9 \mid \omega^1_{0.5}(x) \geq 0.4\} = \{0,3,6\}$.

Theorem 4.9. *Let R be a ring, $t \in [0,1]$ and $\omega : R \to [0,1]$ be an (a,b)-fuzzy subring of R. If the (a,b)-level subset is nonempty, then $(\omega^a_b)_t$ is a subring of R.*

Proof. We note that if $x, y \in (\omega^a_b)_t$, then $(\omega^a_b)(x) \geq t$ and $(\omega^a_b)(y) \geq t$.
We have $(\omega^a_b)(x - y) \geq \min\{\omega^a_b(x), \omega^a_b(y)\} = \min\{t,t\} = t$.
This implies that

$$x - y \in (\omega^a_b)_t. \qquad (4.7)$$

We have, $(\omega^a_b)(xy) \geq \min\{\omega^a_b(x), \omega^a_b(y)\} = \min\{t,t\} = t$.
This implies that

$$xy \in (\omega^a_b)_t. \qquad (4.8)$$

From (4.7) and (4.8), we conclude that $(\omega_b^a)_t$ is a subring of R. □

Theorem 4.10. *Let R be a ring and $\omega : R \to [0,1]$ be a fuzzy subset of R. Suppose that $(\omega_b^a)_t$ is a subring of R, for all $t \in [0,1]$. Then ω is an (a, b)-fuzzy subring of R.*

Proof. Let $x, y \in R$, $(\omega_b^a)(x) = t_1$ and $(\omega_b^a)(y) = t_2$ where $t_1, t_2 \in [0,1]$.
Then $(\omega_b^a)_{t_1}$ and $(\omega_b^a)_{t_2}$ are subrings of R.
Since, $t_1 \wedge t_2 \leq t_1$ and $t_1 \wedge t_2 \leq t_2$, we have $(\omega_b^a)_{t_1} \subseteq (\omega_b^a)_{t_1 \wedge t_2}$ and $(\omega_b^a)_{t_2} \subseteq (\omega_b^a)_{t_1 \wedge t_2}$.
Hence, $x \in (\omega_b^a)_{t_1}$ and $y \in (\omega_b^a)_{t_2}$ implies $x, y \in (\omega_b^a)_{t_1 \wedge t_2}$.
Then $x - y$ and $xy \in (\omega_b^a)_{t_1 \wedge t_2}$, since $(\omega_b^a)_t$ is a subring of R, for all $t \in [0,1]$.
This implies $(\omega_b^a)(x - y) \geq t_1 \wedge t_2 = \min\{(\omega_b^a)(x), (\omega_b^a)(y)\}$ and
$(\omega_b^a)(xy) \geq t_1 \wedge t_2 = \min\{(\omega_b^a)(x), (\omega_b^a)(y)\}$.
This proves that ω is an (a, b)-fuzzy subring of R. □

Definition 4.11. Let ω be an (a, b)-fuzzy subring of R and $t \in [0,1]$. Then the subring $(\omega_b^a)_t$ is said to be an (a, b)-level subring of ω.

Example 4.12. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$. Define a fuzzy subset ω as follows:

$$\omega(x) = \begin{cases} 0.75, & \text{if } x = \{(0,0), (0,2), (2,0), (2,2)\}, \\ 0.4, & \text{otherwise.} \end{cases}$$

We note that for $a = 0.9, b = 0.5, 1 - a + b = 0.6$, ω is an (a, b)-fuzzy subring of R.
Also,

$$\omega_{0.5}^{0.9}(x) = \begin{cases} 0.6, & \text{if } x = \{(0,0), (0,2), (2,0), (2,2)\}, \\ 0.4, & \text{otherwise.} \end{cases}$$

If $t = 0.5$, then $(\omega_{0.5}^{0.9})_t = \{(0,0), (0,2), (2,0), (2,2)\}$ is a subring of R and a $(0.9, 0.5)$-level subring of ω.

Theorem 4.13. *Let ω be an (a, b)-fuzzy subring of a ring R. Then two (a, b)-level subrings $(\omega_b^a)_{t_1}$, $(\omega_b^a)_{t_2}$ with $t_1 < t_2$ are equal if and only if there is no $x \in R$ such that $t_1 \leq \omega_b^a(x) < t_2$.*

Proof. Let $(\omega_b^a)_{t_1} = (\omega_b^a)_{t_2}$. If there exists $x \in R$ such that
$t_1 \leq \omega_b^a(x) < t_2$, then $x \in (\omega_b^a)_{t_1}$, but $x \notin (\omega_b^a)_{t_2}$ which is a contradiction.
Conversely, suppose there is no $x \in R$ such that $t_1 \leq \omega_b^a(x) < t_2$.
As $t_1 < t_2$ implies $(\omega_b^a)_{t_2} \subseteq (\omega_b^a)_{t_1}$.
Now, if $x \in (\omega_b^a)_{t_1}$, then $(\omega_b^a)(x) \geq t_1$.
Clearly, $\omega_b^a(x) \nleq t_2$. Since $\omega_b^a(x)$ and t_2 are real numbers, it follows that $\omega_b^a(x) \geq t_2$, i.e., $x \in (\omega_b^a)_{t_2}$. Hence, $(\omega_b^a)_{t_1} = (\omega_b^a)_{t_2}$. □

5. (a, b)-Fuzzy ideals

Definition 5.1. Let ω be a fuzzy subset of R and $0 \leq b < a \leq 1$. Then ω is called an (a, b)-fuzzy ideal of R if the following conditions hold:
(R_1) $\omega_b^a(x - y) \geq \min\{\omega_b^a(x), \omega_b^a(y)\}$;
(R_2) $\omega_b^a(xy) \geq \max\{\omega_b^a(x), \omega_b^a(y)\}$.

Remark 5.2. Let ω be an (a, b)-fuzzy subset of a commutative ring R. Then ω_b^a satisfies (R_2) if and only if $\omega_b^a(xy) \geq \omega_b^a(x)$, $\forall x, y \in R$.

Proposition 5.3. *If ω is a fuzzy ideal of R, then ω is also (a, b)-fuzzy ideal of R.*

Proof. For $x, y \in R$, we have

$$\begin{aligned}\omega_b^a(x-y) &= \min\{\omega(x-y), 1-a+b\}\\ &\geq \min\{\min\{\omega(x), \omega(y)\}, 1-a+b\},\\ &\quad \text{(since } \omega \text{ is a fuzzy ideal of } R)\\ &= \min\{\min\{\omega(x), 1-a+b\}, \min\{\omega(y), 1-a+b\}\}\\ &= \min\{\omega_b^a(x), \omega_b^a(y)\}. \end{aligned} \tag{5.1}$$

Also,

$$\begin{aligned}\omega_b^a(xy) &= \min\{\omega(xy), 1-a+b\}\\ &\geq \min\{\max\{\omega(x), \omega(y)\}, 1-a+b\},\\ &\quad \text{(since } \omega \text{ is a fuzzy ideal of } R)\\ &= \max\{\min\{\omega(x), \omega(y)\}, 1-a+b\}\\ &= \max\{\min\{\omega(x), 1-a+b\}, \min\{\omega(y), 1-a+b\}\}\\ &= \max\{\omega_b^a(x), \omega_b^a(y)\}. \end{aligned} \tag{5.2}$$

It follows from (5.1) and (5.2), that ω is a (a, b)-fuzzy ideal of ring R. □

The following example shows that the converse of Proposition 5.3 may not be true.

Example 5.4. Define a fuzzy subset ω of the ring $R = \mathbb{Z}_8$ as follows:

$$\omega(x) = \begin{cases} 0.45, & \text{if } x = \{0, 2, 4, 6\},\\ 0.75, & \text{otherwise.} \end{cases}$$

We note that for $x = 6$, $y = 3$, $\omega(6) = 0.45$, $\omega(3) = 0.75$, $xy = 18 = 2$, $\omega(xy) = 0.45$. Thus, $\omega(xy) \ngeq \max\{\omega(x), \omega(y)\}$.
Hence ω is not a fuzzy ideal of $\mathbb{Z}_8$.
But ω is a $(0.8, 0.1)$-fuzzy ideal of $\mathbb{Z}_8$.

Proposition 5.5. *If* $\omega : R \to [0, 1]$ *is an* (a, b)*-fuzzy ideal of* R*, then* $\omega_b^a(0) \geq \omega_b^a(x) \geq \omega_b^a(1)$*, for all* $x \in R$*.*

Proof. For any $x \in R$, we have

$$\begin{aligned}\omega_b^a(0) &= \omega_b^a(x - x)\\ &\geq \min\{\omega_b^a(x), \omega_b^a(x)\}, \text{ since } \omega \text{ is an } (a,b)\text{-fuzzy ideal of } R.\\ &= \omega_b^a(x).\\ &= \omega_b^a(x.1)\\ &\geq \omega_b^a(1).\end{aligned}$$

Hence, $\omega_b^a(0) \geq \omega_b^a(x) \geq \omega_b^a(1)$, for all $x \in R$. □

Proposition 5.6. *If* $\omega : R \to [0, 1]$ *is an* (a, b)*-fuzzy ideal of ring* R *with* $\omega_b^a(x - y) = \omega_b^a(0)$*, then* $\omega_b^a(x) = \omega_b^a(y)$*, for all* $x, y \in R$*.*

Proof. Since ω is an (a, b)-fuzzy ideal of R,

$$\begin{aligned}\omega_b^a(x) &= \omega_b^a(x - y + y)\\ &\geq \min\{\omega_b^a(x - y), \omega_b^a(y)\}\\ &= \min\{\omega_b^a(0), \omega_b^a(y)\}\\ &= \omega_b^a(y).\\ \omega_b^a(y) &= \omega_b^a(y - x + x)\\ &\geq \min\{\omega_b^a(y - x), \omega_b^a(x)\}\\ &= \min\{\omega_b^a(0), \omega_b^a(x)\}\\ &= \omega_b^a(x).\end{aligned}$$

Hence, $\omega_b^a(x) = \omega_b^a(y)$, for all $x, y \in R$. □

Proposition 5.7. *Let* $\omega : R \to [0, 1]$ *be an* (a, b)*-fuzzy ideal of* R*. If for some* $t \in [0, 1]$*, the* (a, b)*-level subset* $(\omega_b^a)_t$*, is nonempty, then it is an ideal of* R *where* $(\omega_b^a)_t = \{x \in R \mid \omega_b^a(x) \geq t\}$*.*

Proof. Let $x, y \in (\omega_b^a)_t$. Then $\omega_b^a(x) \geq t$ and $\omega_b^a(y) \geq t$.
As ω is an (a, b)-fuzzy ideal of R,

$$(\omega_b^a)(x - y) \geq \min\{\omega_b^a(x), \omega_b^a(y)\} = \min\{t, t\} = t.$$

Hence

$$x - y \in (\omega_b^a)_t. \tag{5.3}$$

Let $r \in R$ be arbitrary and $x \in (\omega_b^a)_t$, then $\omega_b^a(x) \geq t$.

$$(\omega_b^a)(rx) \geq \max\{\omega_b^a(r), \omega_b^a(x)\} \geq \omega_b^a(x) = t.$$

Hence,

$$rx \in (\omega_b^a)_t. \tag{5.4}$$

From (5.3) and (5.4), we conclude that $(\omega_b^a)_t$ is an ideal of R. □

Proposition 5.8. *Let $\omega : R \to [0,1]$ be an (a,b)-fuzzy subset of R. Suppose that $(\omega_b^a)_t$ is an ideal for all $t \in [0,1]$. Then ω is an (a,b)-fuzzy ideal of R.*

Proof. Let $x, y \in R$ and $\omega_b^a(x) = t_1$, $\omega_b^a(y) = t_2$, where $t_1, t_2 \in [0,1]$.
Then $(\omega_b^a)_{t_1}$ and $(\omega_b^a)_{t_2}$ are ideals of R.
Since, $t_1 \wedge t_2 \leq t_1$ and $t_1 \wedge t_2 \leq t_2$.
This implies that $(\omega_b^a)_{t_1} \subseteq (\omega_b^a)_{t_1\wedge t_2}$ and $(\omega_b^a)_{t_2} \subseteq (\omega_b^a)_{t_1\wedge t_2}$.
Hence, $x \in (\omega_b^a)_{t_1}$ and $y \in (\omega_b^a)_{t_2}$, which implies that $x, y \in (\omega_b^a)_{t_1\wedge t_2}$ and so $x - y \in (\omega_b^a)_{t_1\wedge t_2}$.
Thus,

$$\begin{aligned}\omega_b^a(x-y) &\geq t_1 \wedge t_2 = \min\{t_1, t_2\}, \\ &\qquad \text{as } t_1, t_2 \text{ are real numbers belonging to } [0,1] \\ &= \min\{\omega_b^a(x), \omega_b^a(y)\}. \end{aligned} \tag{5.5}$$

For $x, y \in R$, if $\omega_b^a(x) = t_1$, then $x \in (\omega_b^a)_{t_1}$.
Therefore, $xy \in (\omega_b^a)_{t_1}$ implies $\omega_b^a(xy) \geq t_1$.
Hence,

$$\omega_b^a(xy) \geq \omega_b^a(x). \tag{5.6}$$

Similarly,

$$\omega_b^a(xy) \geq \omega_b^a(y). \tag{5.7}$$

Hence, from (5.6) and (5.7),

$$\omega_b^a(xy) \geq \max\{\omega_b^a(x), \omega_b^a(y)\}. \tag{5.8}$$

Thus, from (5.5) and (5.8), we conclude that ω is an (a,b)-fuzzy ideal of R. □

Corollary 5.9. *If $\omega : R \to [0,1]$ is an (a,b)-fuzzy ideal of R, then $\{x \in R \mid \omega_b^a(x) = \omega_b^a(0)\}$ is an ideal of R, where 0 is the additive identity of R.*

Proof. Let $\tau = \{x \in R \mid \omega_b^a(x) = \omega_b^a(0)\}$.
Let $x, y \in \tau$. Then $\omega_b^a(x) = \omega_b^a(0)$ and $\omega_b^a(y) = \omega_b^a(0)$.

As ω is an (a, b)-fuzzy ideal, we have

$$\begin{aligned}\omega_b^a(x-y) &\geq \min\{\omega_b^a(x), \omega_b^a(y)\} \\ &= \min\{\omega_b^a(0), \omega_b^a(0)\} \\ &= \omega_b^a(0).\end{aligned}$$

By Proposition 5.5, we have $\omega_b^a(0) \geq \omega_b^a(x-y)$.
Thus, $\omega_b^a(x-y) = \omega_b^a(0)$, which implies that $x - y \in \tau$.
Let $r \in R$ and $x \in \tau$. Then $\omega_b^a(x) = \omega_b^a(0)$.
Also,

$$\begin{aligned}\omega_b^a(rx) &\geq \max\{\omega_b^a(r), \omega_b^a(x)\} \\ &= \max\{\omega_b^a(r), \omega_b^a(0)\} \\ &= \omega_b^a(0).\end{aligned}$$

Again by Proposition 5.5, $\omega_b^a(0) \geq \omega_b^a(rx)$
Thus $\omega_b^a(0) = \omega_b^a(rx)$ and so $rx \in \tau$.
Hence τ is an ideal of R. □

Proposition 5.10. *If $\omega : R \to [0, 1]$ is an (a, b)-fuzzy ideal of R, then $\{x \in R \mid \omega_b^a(x) > t\}$ is an ideal of R for all $t \in [0, 1]$.*

Proof. Let us write $(\omega_b^a)_t = \{x \in R \mid \omega_b^a(x) > t\}$.
Let $x, y \in (\omega_b^a)_t$. Then $\omega_b^a(x) > t$ and $\omega_b^a(y) > t$.
As ω is an (a, b)-fuzzy ideal of R, we have

$$\omega_b^a(x-y) \geq \min\{\omega_b^a(x), \omega_b^a(y)\} > \min\{t, t\} = t.$$

Hence, $x - y \in (\omega_b^a)_t$.
Now let $x \in \omega_b^a(x)$ and $r \in R$. Then

$$\omega_b^a(rx) \geq \max\{\omega_b^a(r), \omega_b^a(x)\} > \omega_b^a(x) > t.$$

Hence, $rx \in (\omega_b^a)_t$.
Thus, $\{x \in R \mid \omega_b^a(x) > t\}$ is an ideal of R for all $t \in [0, 1]$. □

Definition 5.11. Let ω be an (a, b)-fuzzy ideal of R. Then the ideals $(\omega_b^a)_t$ for $t \in [0, 1]$ are called (a, b)-level ideals of R.

Remark 5.12. Let ω be an (a, b)-fuzzy ideal of R and $t_1, t_2 \in [0, 1]$ be such that $t_1 \leq t_2$. We note that if $x \in (\omega_b^a)_{t_2}$, then $(\omega_b^a)(x) \geq t_2 \geq t_1$. Hence $x \in (\omega_b^a)_{t_1}$. Thus $(\omega_b^a)_{t_2} \subseteq (\omega_b^a)_{t_1}$.

Proposition 5.13. *Let $\omega : R \to [0, 1]$ be a (a, b)-fuzzy ideal of R. Two level ideals $(\omega_b^a)_{t_1}$, $(\omega_b^a)_{t_2}$ with $t_1 < t_2$ are equal if and only if there is no $x \in R$ such that $t_1 \leq \omega_b^a(x) < t_2$.*

Proof. Assume that $(\omega_b^a)_{t_1} = (\omega_b^a)_{t_2}$. If there exists $x \in R$ such that $t_1 \leq \omega_b^a(x) < t_2$, then $x \in (\omega_b^a)_{t_1}$ but $x \notin (\omega_b^a)_{t_2}$, a contradiction.

Conversely, suppose that there is no $x \in R$ such that $t_1 \leq \omega_b^a(x) < t_2$. Since, $t_1 < t_2$ we have $(\omega_b^a)_{t_2} \subseteq (\omega_b^a)_{t_1}$.
Now if $x \in (\omega_b^a)_{t_1}$, then $t_1 \leq \omega_b^a(x)$.
Hence, by the given condition it follows that $\omega_b^a(x) \nleq t_2$.
Since $\omega_b^a(x)$ and t_2 are real numbers belonging to $[0, 1]$, this implies that $\omega_b^a(x) \geq t_2$. Hence $x \in (\omega_b^a)_{t_2}$.
Therefore, $(\omega_b^a)_{t_1} = (\omega_b^a)_{t_2}$. □

Proposition 5.14. *The intersection of two (a, b)-fuzzy ideals of R is an (a, b)-fuzzy ideal.*

Proof. Let ω and η be two (a, b)-fuzzy ideals of R.
For $x, y \in R$, we have

$$\begin{aligned}(\omega \cap \eta)_b^a(x - y) &= (\omega_b^a \cap \eta_b^a)(x - y), \text{ by Lemma 3.2} \\ &= \min\{\omega_b^a(x - y), \eta_b^a(x - y)\} \\ &\geq \min\{\min\{\omega_b^a(x), \omega_b^a(y)\}, \min\{\eta_b^a(x), \eta_b^a(y)\}\} \\ &= \min\{\min\{\omega_b^a(x), \eta_b^a(x)\}, \min\{\omega_b^a(y), \eta_b^a(y)\}\} \\ &= \min\{\omega_b^a(x) \cap \eta_b^a(x), \omega_b^a(y) \cap \eta_b^a(y)\} \\ &= \min\{(\omega \cap \eta)_b^a(x), (\omega \cap \eta)_b^a(y)\}. \end{aligned} \tag{5.9}$$

Also, we have

$$\begin{aligned}(\omega \cap \eta)_b^a(xy) &= (\omega_b^a \cap \eta_b^a)(xy), \text{ by Lemma 3.2} \\ &= \min\{\omega_b^a(xy), \omega_b^a(xy)\} \\ &\geq \min\{\max\{\omega_b^a(x), \omega_b^a(y)\}, \max\{\eta_b^a(x), \eta_b^a(y)\}\}, \\ &\quad \text{as all the quantities involved belong to } [0, 1] \\ &= \max\{\min\{\omega_b^a(x), \omega_b^a(y)\}, \min\{\eta_b^a(x), \eta_b^a(y)\}\} \\ &= \max\{\min\{\omega_b^a(x), \eta_b^a(x)\}, \min\{\omega_b^a(y), \eta_b^a(y)\}\} \\ &= \max\{(\omega_b^a \cap \eta_b^a)(x)), (\omega_b^a \cap \eta_b^a)(y)\} \\ &= \max\{(\omega \cap \eta)_b^a(x), (\omega \cap \eta)_b^a(y)\}. \end{aligned} \tag{5.10}$$

It follows from (5.9) and (5.10), $\omega \cap \eta$ is an (a, b)-fuzzy ideal of R. □

The following example shows that the union of two (a, b)-fuzzy ideals may not be an (a, b)-fuzzy ideal.

Example 5.15. Let $R = \mathbb{Z}_{12}$. Define fuzzy subsets ω and η as follows:

$$\omega(x) = \begin{cases} 0.4, & \text{if } x = \{0, 2, 4, 6, 8, 10\}, \\ 0, \text{ otherwise.} \end{cases}$$

$$\eta(x) = \begin{cases} 0.2, & \text{if } x = \{0, 3, 6, 9\}, \\ 0.1, \text{ otherwise.} \end{cases}$$

It can be seen that ω and η are $(0.6, 0.3)$-fuzzy ideals of $\mathbb{Z}_{12}$.
We have

$$(\omega \cup \eta)(x) = \begin{cases} 0.4, & \text{if } x = \{0, 2, 4, 6, 8, 10\}, \\ 0.2, & \text{if } x = \{3, 9\}, \\ 0.1, \text{ otherwise.} \end{cases}$$

If we take $x = 9, y = 2$, then $x - y = 7$.
For $a = 0.6$ and $b = 0.3$, we have $1 - a + b = 0.7$.
Also, $(\omega \cup \eta)(x) = 0.2$, $(\omega \cup \eta)(y) = 0.4$ and $(\omega \cup \eta)(x - y) = 0.1$.
Now,

$$\begin{aligned} &(\omega \cup \eta)_b^a(x) = \min\{0.2, 0.7\} = 0.2, \\ &(\omega \cup \eta)_b^a(y) = \min\{0.4, 0.7\} = 0.4, \\ &(\omega \cup \eta)_b^a(x - y) = \min\{0.1, 0.7\} = 0.1. \\ &(\omega \cup \eta)_b^a(x - y) \ngeq \min\{(\omega \cup \eta)_b^a(x), (\omega \cup \eta)_b^a(y)\}. \end{aligned}$$

Thus, $\omega \cup \eta$ is not a $(0.6, 0.3)$-fuzzy ideal of $\mathbb{Z}_{12}$.

Proposition 5.16. *Let $g : R \to R'$ be an onto homomorphism of a ring R to a ring R'. If ω is an (a, b)-fuzzy ideal of R', then $g^{-1}(\omega)$ is an (a, b)-fuzzy ideal of R which is constant on $\ker g$.*

Proof. For $x, y \in R$. we have

$$\begin{aligned} &(g^{-1}(\omega))_b^a(x - y) \\ &= g^{-1}(\omega_b^a)(x - y), \quad \text{by Lemma 3.2} \\ &= \omega_b^a(g(x - y)) \\ &= \omega_b^a(g(x) - g(y)) \\ &\geq \min\{\omega_b^a(g(x)), \omega_b^a(g(y))\}, \\ &\qquad (\text{as } \omega \text{ is } (a, b)\text{-fuzzy ideal of } R') \\ &= \min\{g^{-1}(\omega_b^a)(x), g^{-1}(\omega_b^a)(y)\} \\ &= \min\{(g^{-1}(\omega))_b^a(x), (g^{-1}(\omega))_b^a(y)\}, \quad \text{by Lemma 3.2} \end{aligned} \tag{5.11}$$

Also, we have

$$\begin{aligned}
&(g^{-1}(\omega))^a_b(xy)\\
&= g^{-1}(\omega^a_b)(xy)\\
&= \omega^a_b(g(xy)) = \omega^a_b(g(x)g(y))\\
&\geq \max\{\omega^a_b(g(x)), \omega^a_b(g(y))\},\\
&\quad \text{(as } \omega \text{ is } (a,b)\text{-fuzzy ideal of } R')\\
&= \max\{g^{-1}(\omega^a_b)(x), g^{-1}(\omega^a_b)(y)\}\\
&= \max\{(g^{-1}(\omega))^a_b(x), (g^{-1}(\omega))^a_b(y)\}, \text{ by Lemma 3.2.} \qquad (5.12)
\end{aligned}$$

It follows from (5.11) and (5.12) that $g^{-1}(\omega)$ is an (a,b)-fuzzy ideal of R.
Next if $p \in kerg$, then $g(p) = 0'$, where $0'$ is the additive identity of R'. Therefore, $(g^{-1}(\omega))^a_b(p) = \omega^a_b(g(p)) = \omega^a_b(0')$ and so $g^{-1}(\omega)$ is constant on $kerg$. □

Now we consider the (a,b)-fuzzy quotient rings.

Definition 5.17. Let ω be an (a,b)-fuzzy ideal of R.
For $x \in R$, define a fuzzy set $x + \omega^a_b : R \to [0,1]$ by:
$(x + \omega^a_b)(y) = \min\{\omega(y - x), 1 - a + b\}$. The fuzzy set $x + \omega^a_b$ is called an (a,b)-fuzzy coset of the fuzzy ideal ω of R.

Proposition 5.18. *If ω is an (a,b)-fuzzy ideal of R, then*
(i) $0 + \omega^a_b = \omega^a_b$.
(ii) For any $t \in [0,1]$, $(x + \omega^a_b)_t = x + (\omega^a_b)_t$.
(iii) $\omega^a_b(x) = \omega^a_b(0) \Leftrightarrow x + \omega^a_b = \omega^a_b$.

Proof. (i): We have

$$\begin{aligned}
(0 + \omega^a_b)(x) &= \min\{\omega(x - 0), 1 - a + b\}\\
&= \min\{\omega(x), 1 - a + b\}\\
&= \omega^a_b(x).
\end{aligned}$$

Hence, $0 + \omega^a_b = \omega^a_b$.
(ii): Let $y \in R$. We have

$$\begin{aligned}
y \in (x + \omega^a_b)_t &\Leftrightarrow (x + \omega^a_b)(y) \geq t\\
&\Leftrightarrow \min\{\omega(y - x), 1 - a + b\} \geq t
\end{aligned}$$

$$\Leftrightarrow \{\min\{\omega(y),\omega(x)\},1-a+b\} \geq t$$
$$\Leftrightarrow \{\min\{\omega(y),1-a+b\},\min\{\omega(x),1-a+b\}\} \geq t$$
$$\Leftrightarrow \min\{\omega_b^a(y),\omega_b^a(x)\} \geq t$$
$$\Leftrightarrow \omega_b^a(y-x) \geq t$$
$$\Leftrightarrow y-x \in (\omega_b^a)_t$$
$$\Leftrightarrow y \in x+(\omega_b^a)_t.$$

Hence, $(x+\omega_b^a)_t = x+(\omega_b^a)_t$.
(iii): Assume that

$$\omega_b^a(x) = \omega_b^a(0). \tag{5.13}$$

Then for $y \in R$, we have

$$\begin{aligned}
(x+\omega_b^a)(y) &= \min\{\omega(y-x),1-a+b\} \\
&\geq \min\{\min\{\omega(y),\omega(x)\},1-a+b\} \\
&= \min\{\min\{\omega(y),1-a+b\},\min\{\omega(x),1-a+b\}\} \\
&= \min\{\omega_b^a(y),\omega_b^a(x)\} \\
&= \min\{\omega_b^a(y),\omega_b^a(0)\}, \text{ from } (5.13) \\
&= \omega_b^a(y), \text{ by Proposition } 5.5 \\
&= \omega_b^a(y-x+x) \\
&\geq \min\{\omega_b^a(y-x),\omega_b^a(x)\} \\
&= \min\{\omega_b^a(y-x),\omega_b^a(0)\}, \text{ from } (5.13) \\
&= \omega_b^a(y-x), \text{ by Proposition } 5.5 \\
&= \min\{\omega(y-x),1-a+b\} \\
&= (x+\omega_b^a)(y).
\end{aligned}$$

Thus, $x+\omega_b^a = \omega_b^a$.
Conversely, assume that $x+\omega_b^a = \omega_b^a$

$$\Rightarrow (x+\omega_b^a)(0) = \omega_b^a(0)$$
$$\Rightarrow \min\{\omega(0-x),1-a+b\} = \omega_b^a(0)$$
$$\Rightarrow \min\{\omega(-x),1-a+b\} = \omega_b^a(0)$$
$$\Rightarrow \min\{\omega(x),1-a+b\} = \omega_b^a(0)$$
$$\Rightarrow \omega_b^a(x) = \omega_b^a(0).$$

□

Theorem 5.19. *Let ω be a fuzzy ideal of R and τ be the collection of all fuzzy cosets of ω. Define,* $(x+\omega_b^a)+(y+\omega_b^a)=(x+y)+\omega_b^a$ *and* $(x+\omega_b^a)\cdot(y+\omega_b^a)=(x\cdot y)+\omega_b^a$, *for all* $x,y\in R$.
Then τ is a ring under these two operations.

Proof. First we shall show that these two operations are well–defined.
Let $x+\omega_b^a=x^{'}+\omega_b^a$ and $y+\omega_b^a=y^{'}+\omega_b^a$.
Then for $x^{'},y^{'}$,
$(x+\omega_b^a)(x')=(x^{'}+\omega_b^a)(x')$ and $(y+\omega_b^a)(y')=(y^{'}+\omega_b^a)(y')$.
Then by definition 5.17,
$\min\{\omega(x^{'}-x),1-a+b\}=\min\{\omega(x^{'}-x^{'}),1-a+b\}$ and
$\min\{\omega(y^{'}-y),1-a+b\}=\min\{\omega(y^{'}-y^{'}),1-a+b\}$.
Therefore, $\min\{\omega(x^{'}-x),1-a+b\}=\min\{\omega(0),1-a+b\}$ and
$\min\{\omega(y^{'}-y),1-a+b\}=\min\{\omega(0),1-a+b\}$.
Therefore, $\omega_b^a(x^{'}-x)=\omega_b^a(0)$ and $\omega_b^a(y^{'}-y)=\omega_b^a(0)$, by definition 3.1.
Therefore,

$$\omega_b^a(x^{'}-x)=\omega_b^a(0) \text{ and } \omega_b^a(y^{'}-y)=\omega_b^a(0). \tag{5.14}$$

For $z\in R$, we have

$$\begin{aligned}
&((x+y)+\omega_b^a)(z)\\
&=\min\{\omega(z-(x+y)),1-a+b\}\\
&=\min\{\omega(z-x-y),1-a+b\}\\
&=\min\{\omega(z-x^{'}-y^{'}+x^{'}-x+y^{'}-y),1-a+b\}\\
&\geq\min\{\{\omega(z-x^{'}-y^{'}),\omega(x^{'}-x),\omega(y^{'}-y)\},1-a+b\},\\
&\qquad\text{since } \omega \text{ is a fuzzy ideal of } R.\\
&=\min\{\min\{\omega(z-x^{'}-y^{'}),1-a+b\},\min\{\omega(x^{'}-x),1-a+b\},\\
&\qquad\min\{\omega(y^{'}-y),1-a+b\}\}\\
&=\min\{\omega_b^a(z-x^{'}-y^{'}),\omega_b^a(x^{'}-x),\omega_b^a(y^{'}-y)\}\\
&=\min\{\omega_b^a(z-x^{'}-y^{'}),\omega_b^a(0),\omega_b^a(0)\},\ \text{from (5.14).}\\
&=\omega_b^a(z-x^{'}-y^{'}),\ \text{by Proposition 5.5}\\
&=\min\{\omega(z-x^{'}-y^{'}),1-a+b\}\\
&=((x^{'}+y^{'})+\omega_b^a)(z).
\end{aligned}$$

Thus $((x+y)+\omega_b^a)(z)\geq((x^{'}+y^{'})+\omega_b^a)(z)$.
Similarly, we can show that $((x^{'}+y^{'})+\omega_b^a)(z)\geq((x+y)+\omega_b^a)(z)$.

Hence,

$$((x^{'}+y^{'})+\omega_b^a)(z)=((x+y)+\omega_b^a)(z). \tag{5.15}$$

We have

$$\begin{aligned}
&(xy+\omega_b^a)(z)\\
&=\min\{\omega(z-xy),1-a+b\}\\
&=\min\{\omega(z-x^{'}y^{'}+x^{'}y^{'}-xy),1-a+b\}\\
&\geq \min\{\min\{\omega(z-x^{'}y^{'}),\omega(x^{'}y^{'}-xy)\},1-a+b\},\\
&\qquad \text{since } \omega \text{ is a fuzzy ideal of } R\\
&=\min\{\min\{\omega(z-x^{'}y^{'}),1-a+b\},\min\{\omega(x^{'}y^{'}-xy),1-a+b\}\}\\
&=\min\{\omega_b^a(z-x^{'}y^{'}),\omega_b^a(x^{'}y^{'}-xy)\}.
\end{aligned} \tag{5.16}$$

We have

$$\begin{aligned}
&\omega_b^a(x^{'}y^{'}-xy)\\
&=\omega_b^a(x^{'}y^{'}-x^{'}y+x^{'}y-xy)\\
&=\omega_b^a(x^{'}(y^{'}-y)+(x^{'}-x)y)\\
&\geq \min\{\omega_b^a(x(y^{'}-y)),\omega_b^a((x^{'}-x)y)\},\ \text{by Proposition 5.3}\\
&\geq \min\{\max\{\omega_b^a(x),\omega_b^a(y^{'}-y)\},\max\{\omega_b^a(x^{'}-x),\omega_b^a(y)\}\}\\
&=\min\{\max\{\omega_b^a(x),\omega_b^a(0)\},\max\{\omega_b^a(0),\omega_b^a(y)\},\ \text{from (5.14)}.\\
&=\min\{\omega_b^a(0),\omega_b^a(0)\},\quad \text{by Proposition 5.5}\\
&=\omega_b^a(0).
\end{aligned} \tag{5.17}$$

Now, (5.16) becomes

$$\begin{aligned}
(xy+\omega_b^a)(z)&=\min\{\omega_b^a(z-x^{'}y^{'}),\omega_b^a(0)\}\\
&=\omega_b^a(z-x^{'}y^{'}),\quad \text{by Proposition 5.5}\\
&=\min\{\omega(z-x^{'}y^{'}),1-a+b\}\\
&=(x^{'}y^{'}+\omega_b^a)(z).
\end{aligned}$$

Similarly, we can show that $(x^{'}y^{'}+\omega_b^a)(z)\geq(xy+\omega_b^a)(z)$.
Hence, $(xy+\omega_b^a)(z)=(x^{'}y^{'}+\omega_b^a)(z)$.
Thus, the operations $+$ and $\cdot$ are well defined.

Further we have,

$$\begin{aligned}(x+\omega_b^a)+(y+\omega_b^a+z+\omega_b^a) &= (x+\omega_b^a+y+\omega_b^a)+z+\omega_b^a\\ &= (x+y+z)+\omega_b^a.\\ (x+\omega_b^a)+((-x)+\omega_b^a) &= (0+\omega_b^a)=\omega_b^a.\\ (x+\omega_b^a)\cdot((y+\omega_b^a)\cdot(z+\omega_b^a)) &= ((x+\omega_b^a)\cdot(y+\omega_b^a))\cdot(z+\omega_b^a)\\ &= (x\cdot y\cdot z)+\omega_b^a.\\ (x+\omega_b^a)\cdot(1+\omega_b^a) &= x+\omega_b^a=(1+\omega_b^a)\cdot(x+\omega_b^a).\\ (x+\omega_b^a)\cdot(y+\omega_b^a) &= (y+\omega_b^a)\cdot(x+\omega_b^a)=xy+\omega_b^a.\\ (x+\omega_b^a)\cdot(y+\omega_b^a) &= (y+\omega_b^a)\cdot(x+\omega_b^a)=xy+\omega_b^a.\end{aligned}$$

Hence, τ is a commutative ring with unity. □

6. **Conclusion**

In this paper, we have studied (a,b)-fuzzy subrings and (a,b)-fuzzy ideals of a ring. In the next studies, we will formulate the concept of (a,b)-intuitionistic fuzzy subrings and (a,b)-intuitionistic fuzzy ideals of a ring.

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Shriram Khanderao Nimbhorkar
Department of Mathematics,
Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431004, India
Email: `sknimbhorkar@gmail.com`
Jyoti Ashok Khubchandani
Department of Mathematics, Dr.Vitthalrao Vikhe Patil College of Engineering,
Ahmednagar 414001, India
Email: `khubchandani_jyoti@yahoo.com`