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FUZZY PAIRS IN FUZZY α-LATTICES

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ABSTRACT. In this paper, we introduce the notion of a fuzzy α-modular pair in a fuzzy α-lattice and obtain some results.

Key Words: Fuzzy α-lattice, fuzzy modular pair, fuzzy α-modular pair, α-comparable.
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1. INTRODUCTION

The concept of fuzzy ordering was defined by Zadeh [5] in 1971. Yuan and Wu [1] introduced the concept of a fuzzy sublattice. Ajmal and Thomas [8] defined a fuzzy lattice and a fuzzy sublattice as a fuzzy algebra in 1994. Chon [4] considered Zadeh's fuzzy order [6] and proposed a new notion of a fuzzy lattice and studied level sets of such structures. At the same time he also proved some results for distributive and modular fuzzy lattices. Mezzomo *et. al.* [3] changed the way to define the fuzzy supremum and the fuzzy infimum of a pair of elements by considering as a threshold fixed $\alpha \in [0, 1)$ instead of, as usual, zero.

The concept of a modular pair in a lattice is well investigated by Maeda and Maeda [2]. Recently, Wasadikar and Khubchandani [7] defined a fuzzy modular pair in a fuzzy lattice and obtained some properties of fuzzy modular pairs. In this paper, we introduce the notion of a fuzzy α-modular pair in a fuzzy α-lattice and prove some properties analogous to the classical theory.

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2. Preliminaries

In fuzzy sets, each element of a nonempty set X is mapped to $[0,1]$ by a membership function $\mu : X \to [0,1]$.
A mapping $A : X \times X \to [0,1]$ is called a fuzzy binary relation on X.

The following definition is from Zadeh [6]. A fuzzy binary relation A on X is called:

(i) fuzzy reflexive if $A(x,x) = 1$, for all $x \in X$;
(ii) fuzzy symmetric if $A(x,y) = A(y,x)$, for all $x, y \in X$;
(iii) fuzzy transitive if $A(x,z) \geq \sup_{y \in X} \min[A(x,y),\ A(y,z)]$;
(iv) fuzzy antisymmetric if $A(x,y) > 0$ and $A(y,x) > 0$ implies $x = y$.

Based on the above properties Zadeh [6] introduced the following concepts related to a fuzzy binary relation A on a set X:

(i) A is called a fuzzy equivalence relation on X if A is fuzzy reflexive, fuzzy symmetric and fuzzy transitive;
(ii) A is a fuzzy partial order relation if A is fuzzy reflexive, fuzzy antisymmetric and fuzzy transitive and the pair (X, A) is called a fuzzy partially ordered set or a fuzzy poset;
(iii) A is a fuzzy total order relation if it is a fuzzy partial order relation and $A(x,y) > 0$ or $A(y,x) > 0$, for all $x, y \in X$, and the fuzzy poset (X, A) is called of a fuzzy totally ordered set or a fuzzy chain.

Definition 2.1. [4, Definition 3.1] Let (X, A) be a fuzzy poset and let $Y \subseteq X$. An element $u \in X$ is said to be an upper bound for Y iff $A(y,u) > 0$, for all $y \in Y$. An upper bound u_0 for Y is the least upper bound (or supremum) of Y iff $A(u_0, u) > 0$, for every upper bound u for Y. We then write $u_0 = \sup Y = \vee Y$. If $Y = \{x, y\}$, then we write $\vee Y = x \vee y$.

Similarly, an element $v \in X$ is said to be a lower bound for Y iff $A(v, y) > 0$, for all $y \in Y$. A lower bound v_0 for Y is the greatest lower bound (or infimum) of Y iff $A(v, v_0) > 0$, for every lower bound v for Y. We then write $v_0 = \inf Y = \wedge Y$. If $Y = \{x, y\}$, then we write $\wedge Y = x \wedge y$.

3. Fuzzy α-lattices

Mezzomo and Bedregal [3] generalized the concept of a (fuzzy) upper bound as follows.

Definition 3.1. [3, Definition 3.1] Let (X, A) be a fuzzy poset. Let $Y \subseteq X$ and $\alpha \in [0, 1)$. An element $u \in X$ is said to be an α-upper bound for Y whenever $A(x, u) > \alpha$, for all $x \in Y$. An α-upper bound u_0 for Y is called a least α-upper bound (or α-Supremum) of Y iff $A(u_0, u) > \alpha$, for every α-upper bound u of Y.

Dually, an element $v \in X$ is said to be an α-lower bound for Y iff $A(v, x) > \alpha$, for all $y \in Y$. An α-lower bound v_0 for Y is called a greatest α-lower bound (or α-infimum) of Y iff $A(v, v_0) > \alpha$ for every α-lower bound v for Y.

We denote the least α-upper bound of the set {x, y} by $x \vee_\alpha y$ and the greatest α-lower bound of the set {x, y} by $x \wedge_\alpha y$.

Remark 3.2. [3, Remark 3.1] Since A is fuzzy antisymmetric, the least α-upper (greatest α-lower) bound, if it exists, is unique.

Proposition 3.3. [3, Proposition 3.1] *Let (X, A) be fuzzy poset, $\alpha \in [0, 1)$ and $x, y, z \in X$. If $A(x, y) > \alpha$ and $A(y, z) > \alpha$, then $A(x, z) > \alpha$.*

Definition 3.4. [3, Definition 3.2] A fuzzy poset (X, A) is a fuzzy α-lattice iff $x \vee_\alpha y$ and $x \wedge_\alpha y$ exists for all $x, y \in X$, for some $\alpha \in [0, 1)$.

Definition 3.5. [3, Definition 3.4] A fuzzy poset (X, A) is called fuzzy sup α-lattice, if each pair of element has α-supremum in X, denoted by $sup_\alpha\ X$.

Dually, it is called fuzzy inf α-lattice, if each pair of element has α-infimum in X, denoted by $inf_\alpha\ X$. A fuzzy semi α-lattice is a fuzzy poset which is a fuzzy sup α-lattice or a fuzzy inf α-lattice.

Definition 3.6. [3, Definition 3.5] Let (X, A) be a fuzzy poset and I be a fuzzy set on X. The α-supremum in I denoted by $sup_\alpha\ I$, is an element of X such that if $x \in X$ and $\mu_I(x) > \alpha$, then $A(x, sup_\alpha I) > \alpha$ and if $u \in X$ is such that $A(x, u) > \alpha$ whenever $\mu_I(x) > \alpha$, then $A(sup_\alpha I, u) > \alpha$.

Similarly, the α-infimum in I denoted by $inf_\alpha\ I$, is an element of X such that if $x \in X$ and $\mu_I(x) > \alpha$, then $A(inf_\alpha I, x) > \alpha$ and if $v \in X$ is such that $A(v, x) > \alpha$ whenever $\mu_I(x) > \alpha$, then $A(v, inf_\alpha I) > \alpha$.

Definition 3.7. [3, Definition 3.6] A fuzzy inf α-lattice is called inf complete if all of its nonempty fuzzy sets have α-infimum.

Similarly, a fuzzy sup α-lattice is called sup-complete if all of its nonempty fuzzy set admit α-supremum. A fuzzy α-lattice is complete whenever it is, simultaneously, inf-complete and sup-complete.

Proposition 3.8. [3, Proposition 3.2] *Let (X, A) be a complete fuzzy sup α-lattice (inf α-lattice) and I be a fuzzy set on X. Then, $sup_\alpha\ I$ ($inf_\alpha\ I$) exists and it is unique.*

Proposition 3.9. [3, Proposition 3.3] *Let $\mathcal{L} = (X, A)$ be a fuzzy sup α-lattice, then there exist an element $\top$ in X, such that $A(x, \top) > \alpha$ for all $x \in X$.*

Proposition 3.10. [3, Proposition 3.4] *Let $\mathcal{L} = (X, A)$ be a fuzzy inf α-lattice, then there exist an element $\perp$ in X, such that $A(\perp, x) > \alpha$ for all $x \in X$.*

Definition 3.11. [3, Definition 3.6] A fuzzy lattice (X, A) is bounded if there exists $\top$ and $\perp$ in X such that for any $x \in X$, $A(\perp, x) > \alpha$ and $A(x, \top) > \alpha$.

Corollary 3.12. [3, Corollary 3.1] *Every fuzzy lattice is a fuzzy α-lattice.*

We illustrate the concepts of an α-upper bound and α-lower bound with an example.

Example 3.13. Consider the set $X = \{x, y, z, w\}$, let α=0.2 and let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation defined as follows:
$A(x, x) = A(y, y) = A(z, z) = A(w, w) = 1.0$,
$A(w, z) = 0.2$, $A(w, y) = 0.5$, $A(w, x) = 0.9$,
$A(z, w) = 0.0$, $A(z, y) = 0.3$, $A(z, x) = 0.6$,
$A(y, w) = 0.0$, $A(y, z) = 0.0$, $A(y, x) = 0.4$,
$A(x, w) = 0.0$, $A(x, z) = 0.0$, $A(x, y) = 0.0$.
Then A is a fuzzy total order relation.
Let $Y = \{w, z\}$. Then x, y and z are the α-upper bounds of Y. Since $A(z, w) = 0.0$ and $A(w, z) = 0.2 \geq \alpha$, it follows that the α-supremum of Y is z and the α-infimum is w.
The fuzzy α-join and fuzzy α-meet tables are as follows:

$\vee_\alpha$	x	y	z	w
x	x	x	x	x
y	x	y	y	y
z	x	y	z	z
w	x	y	z	w

$\wedge_\alpha$	x	y	z	w
x	x	y	z	w
y	y	y	z	w
z	z	z	z	w
w	w	w	w	w

We note that (X, A) is a fuzzy lattice as well as a fuzzy α-lattice for $\alpha = 0.2$.

In Figure 1, we show the related tabular and graphical representations for the fuzzy relation A.

A	w	z	y	x
w	1.0	0.2	0.5	0.9
z	0.0	1.0	0.3	0.6
y	0.0	0.0	1.0	0.4
x	0.0	0.0	0.0	1.0

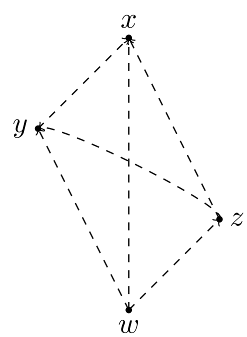

Figure 1

The following example shows that a subset of a fuzzy poset may not have a greatest α-lower bound (least α-upper bound).

Example 3.14. Let $X = \{x_1, y_1, z_1, w_1\}$.
Let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation defined as follows:
$A(x_1, x_1) = A(y_1, y_1) = A(z_1, z_1) = A(w_1, w_1) = 1.0$,
$A(x_1, y_1) = 0.20$, $A(x_1, z_1) = 0.30$, $A(x_1, w_1) = 0.90$,
$A(y_1, x_1) = 0.0$, $A(y_1, z_1) = 0.0$, $A(y_1, w_1) = 0.50$,
$A(z_1, x_1) = 0.0$, $A(z_1, y_1) = 0.0$, $A(z_1, w_1) = 0.70$,
$A(w_1, x_1) = 0.0$, $A(w_1, y_1) = 0.0$, $A(w_1, z_1) = 0.0$.
Then A is a fuzzy partial order relation.
The fuzzy α-join and fuzzy α-meet tables are as follows:

$\vee_\alpha$	x_1	y_1	z_1	w_1
x_1	x_1	y_1	z_1	w_1
y_1	y_1	y_1	w_1	w_1
z_1	z_1	w_1	z_1	w_1
w_1	w_1	w_1	w_1	w_1

$\wedge_\alpha$	x_1	y_1	z_1	w_1
x_1	x_1	x_1	x_1	x_1
y_1	x_1	y_1	x_1	y_1
z_1	x_1	x_1	z_1	z_1
w_1	x_1	y_1	z_1	w_1

We note that (X, A) is a fuzzy lattice.

In Figure 2, we show the related tabular and graphical representation for the fuzzy relation A.

A	x_1	y_1	z_1	w_1
x_1	1.0	0.20	0.30	0.90
y_1	0.0	1.0	0.0	0.50
z_1	0.0	0.0	1.0	0.70
w_1	0.0	0.0	0.0	1.0

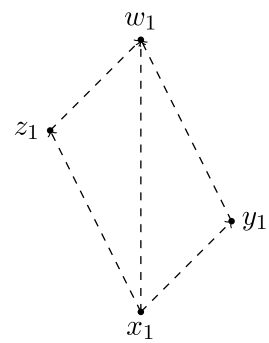

Figure 2

In Figure 3, we show the related tabular and graphical representations for the fuzzy relation A for $\alpha > 0.30$.
Here $x_1 \vee_\alpha w_1 = w_1$, $x_1 \wedge_\alpha w_1 = x_1$,
$y_1 \vee_\alpha w_1 = w_1$, $y_1 \wedge_\alpha w_1 = y_1$,
$z_1 \vee_\alpha w_1 = w_1$, $z_1 \wedge_\alpha w_1 = z_1$,
$y_1 \vee_\alpha z_1 = w_1$, $y_1 \vee_\alpha x_1 = w_1$, $z_1 \vee_\alpha x_1 = w_1$.
But $y_1 \wedge_\alpha z_1$, $y_1 \wedge_\alpha x_1$, $z_1 \wedge_\alpha x_1$ does not exist.

A	x_1	y_1	z_1	w_1
x_1	1.0	0.0	0.0	0.90
y_1	0.0	1.0	0.0	0.50
z_1	0.0	0.0	1.0	0.70
w_1	0.0	0.0	0.0	1.0

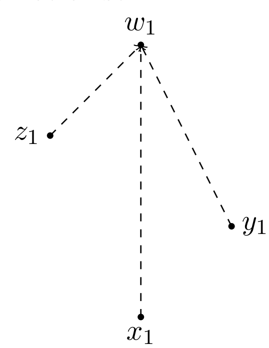

Figure 3

Remark 3.15. We note that Example 3.13 is an example of a fuzzy α-lattice for $\alpha = 0.2$ whereas Example 3.14, is not a fuzzy α-lattice for $\alpha > 0.30$.

Proposition 3.16. [3, Proposition 3.7] *Let (X, A) be a fuzzy α-lattice, $\alpha \in [0, 1)$ and let $x, y, z \in X$. The following statements hold:*
(i) $A(x, x\vee_\alpha y) > \alpha$, $A(y, x\vee_\alpha y) > \alpha$, $A(x\wedge_\alpha y, x) > \alpha$, $A(x\wedge_\alpha y, y) > \alpha$;

(ii) $A(x,z) > \alpha$ *and* $A(y,z) > \alpha$ *implies* $A(x \vee_\alpha y, z) > \alpha$;
(iii) $A(z,x) > \alpha$ *and* $A(z,y) > \alpha$ *implies* $A(z, x \wedge_\alpha y) > \alpha$;
(iv) $A(x,y) > \alpha$ *iff* $x \vee_\alpha y = y$;
(v) $A(x,y) > \alpha$ *iff* $x \wedge_\alpha y = x$;
(vi) If $A(y,z) > \alpha$, *then* $A(x \wedge_\alpha y, x \wedge_\alpha z) > \alpha$ *and* $A(x \vee_\alpha y, x \vee_\alpha z) > \alpha$;
(vii) If $A(x \vee_\alpha y, z) > \alpha$, *then* $A(x,z) > \alpha$ *and* $A(y,z) > \alpha$;
(viii) If $A(x, y \wedge_\alpha z) > \alpha$, *then* $A(x,y) > \alpha$ *and* $A(x,z) > \alpha$.

Proposition 3.17. [3, Proposition 3.8] *Let* (X,A) *be a fuzzy* α*-lattice and let* $x,\ y,\ z \in X$. *Then*
(i) $x \vee_\alpha x = x$ *and* $x \wedge_\alpha x = x$;
(ii) $x \vee_\alpha y = y \vee_\alpha x$ *and* $x \wedge_\alpha y = y \wedge_\alpha x$;
(iii) $(x \vee_\alpha y) \vee_\alpha z = x \vee_\alpha (y \vee_\alpha z)$ *and* $(x \wedge_\alpha y) \wedge_\alpha z = x \wedge_\alpha (y \wedge_\alpha z)$;
(iv) $(x \vee_\alpha y) \wedge_\alpha x = x$ *and* $(x \wedge_\alpha y) \vee_\alpha x = x$.

Lemma 3.18. *Let* (X,A) *be a fuzzy* α*-lattice and* $x, y, x', y' \in X$. *If* $A(x',x) > \alpha$ *and* $A(y',y) > \alpha$, *then* $A(x' \wedge_\alpha y', x \wedge_\alpha y) > \alpha$ *and* $A(x' \vee_\alpha y', x \vee_\alpha y) > \alpha$.

Proof. As $A(x',x) > \alpha$ so, by (vi) of Proposition 3.16,
we have that $A(x' \wedge_\alpha y', x \wedge_\alpha y') > \alpha$. (I)
Also, $A(y',y) > \alpha$ so, by (vi) of Proposition 3.16,
we have $A(x \wedge_\alpha y', x \wedge_\alpha y) > \alpha$. (II)
From (I) and (II) by fuzzy transitivity of A we have
$A(x' \wedge_\alpha y', x \wedge_\alpha y) > \alpha$.
Similarly, we can show that $A(x' \vee_\alpha y', x \vee_\alpha y) > \alpha$. □

Definition 3.19. [3, Definition 3.8] Let (X,A) be a fuzzy α-lattice. (X,A) is fuzzy distributive iff $x \wedge_\alpha (y \vee_\alpha z) = (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z)$ and $(x \vee_\alpha y) \wedge_\alpha (x \vee_\alpha z) = x \vee_\alpha (y \wedge_\alpha z)$.

Note that (X,A) is fuzzy distributive iff $A(x \wedge_\alpha (y \vee_\alpha z), (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z)) > \alpha$ and $A((x \vee_\alpha y) \wedge_\alpha (x \vee_\alpha z), x \vee_\alpha (y \wedge_\alpha z)) > \alpha$.
We now define fuzzy modularity in a fuzzy α-lattice.

Proposition 3.20. *(Modular inequality) Let* (X,A) *be a fuzzy* α*-lattice and let* $x, y, z \in X$. *Then* $A(x,z) > \alpha$ *implies* $A(x \vee_\alpha (y \wedge_\alpha z), (x \vee_\alpha y) \wedge_\alpha z) > \alpha$.

Proof. As $A(x, x \vee_\alpha y) > \alpha$ and $A(x,z) > \alpha$ by (iii) of Proposition 3.16,
we have $A(x, (x \vee_\alpha y) \wedge_\alpha z) > \alpha$. (I)

Since $A(y \wedge_\alpha z, y) > \alpha$ and $A(y, x \vee_\alpha y) > \alpha$ by fuzzy transitivity of A we have $A(y \wedge_\alpha z, x \vee_\alpha y) > \alpha$.
Using (iii) of Proposition 3.16, we have
$A(y \wedge_\alpha z, (x \vee_\alpha y) \wedge_\alpha z) > \alpha$. (II)
Thus by (I) and (II) and by (ii) of Proposition 3.16, we have
$A(x \vee_\alpha (y \wedge_\alpha z), (x \vee_\alpha y) \wedge_\alpha z) > \alpha$. □

Definition 3.21. Let (X, A) be a fuzzy α-lattice. (X, A) is fuzzy α-modular iff $A(x, z) > \alpha$ implies $x \vee_\alpha (y \wedge_\alpha z) = (x \vee_\alpha y) \wedge_\alpha z$ for all $x, y, z \in X$.

By the modular inequality, a fuzzy α-lattice (X, A) is fuzzy α-modular iff $A(x, z) > \alpha$ implies $A(x \vee_\alpha y) \wedge_\alpha z, x \vee_\alpha (y \wedge_\alpha z)) > \alpha$ for $x, y, z \in X$.

Proposition 3.22. *Let (X, A) be a fuzzy α-lattice. (X, A) be a fuzzy distributive lattice, then (X, A) is fuzzy α-modular lattice.*

Proof. Let $x, y, z \in X$. Suppose $A(x, z) > \alpha$.
Since (X, A) is fuzzy distributive so, we have
$(x \vee_\alpha y) \wedge_\alpha z = (x \wedge_\alpha z) \vee_\alpha (y \wedge_\alpha z)$. Thus,
$A((x \vee_\alpha y) \wedge_\alpha z, x \vee_\alpha (y \wedge_\alpha z)) = A((x \wedge_\alpha z) \vee_\alpha (y \wedge_\alpha z), x \vee_\alpha (y \wedge_\alpha z))$.(I)
As $A(x, z) > \alpha$ by (v) of Proposition 3.16, we have $x \wedge_\alpha z = x$.
So, (I) reduces to
$A((x \vee_\alpha y) \wedge_\alpha z, x \vee_\alpha (y \wedge_\alpha z)) = A(x \vee_\alpha (y \wedge_\alpha z), x \vee_\alpha (y \wedge_\alpha z)) > \alpha$.
Hence $(x \vee_\alpha y) \wedge_\alpha z = x \vee_\alpha (y \wedge_\alpha z)$.
Thus, (X, A) is fuzzy α-modular lattice. □

4. Fuzzy α-modular pairs in a fuzzy α-lattice

In this section, we define a fuzzy α-modular pair in a fuzzy α-lattice and we prove some propositions.

We recall the definition of a fuzzy modular pair in a fuzzy lattice from [7].

Definition 4.1. Let X be a nonempty set and $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$. Let $x, y \in X$. We say that (x, y) is a fuzzy meet-modular pair and we write $(x, y)_F M_m$ if whenever $A(z, y) > 0$, then
$(z \vee_F x) \wedge_F y = z \vee_F (x \wedge_F y)$.

We say that (x, y) is a fuzzy join-modular pair and we write $(x, y)_F M_j$ if whenever $A(y, z) > 0$, then $(z \wedge_F x) \vee_F y = z \wedge_F (x \vee_F y)$.
We write $(x, y)_F \overline{M_j}$ or $(x, y)_F \overline{M_m}$ when the pair (x, y) is not a fuzzy join-modular or fuzzy meet-modular pair respectively.

Definition 4.2. Let (X, A) be a fuzzy α-lattice. We say that (x, y) is a fuzzy α-modular pair and we write $(x, y)FM_\alpha$, if whenever $A(z, y) > \alpha$ for some $z \in X$, $\alpha \in [0, 1)$, then $(z \vee_\alpha x) \wedge_\alpha y = z \vee_\alpha (x \wedge_\alpha y)$.

We say that (x, y) is a fuzzy dual α-modular pair and we write $(x, y)FM^*_\alpha$, if whenever $A(y, z) > \alpha$ for some $z \in X$, then $(z \wedge_\alpha x) \vee_\alpha y = z \wedge_\alpha (x \vee_\alpha y)$.

We write $(x, y)\overline{FM_\alpha}$ when the pair (x, y) is not a fuzzy α-modular pair.

Example 4.3. Let $X = \{v, w, x, y, z\}$ and let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation defined as follows:

$A(v, v) = A(w, w) = A(x, x) = A(y, y) = A(z, z) = 1.0,$

$A(v, w) = 0.40,\ A(v, x) = 0.50,\ A(v, y) = 0.80,\ A(v, z) = 0.94,$

$A(w, v) = 0.0,\ A(w, x) = 0.20,\ A(w, y) = 0.60,\ A(w, z) = 0.90,$

$A(x, v) = 0.0,\ A(x, w) = 0.0,\ A(x, y) = 0.30,\ A(x, z) = 0.70,$

$A(y, v) = 0.0,\ A(y, w) = 0.0,\ A(y, x) = 0.0,\ A(y, z) = 0.40,$

$A(z, v) = 0.0,\ A(z, w) = 0.0,\ A(z, x) = 0.0,\ A(z, y) = 0.0.$

Then A is a fuzzy partial order relation.

The fuzzy α-join and α-fuzzy meet tables are as follows:

$\vee_\alpha$	v	w	x	y	z
v	v	w	x	y	z
w	w	w	x	y	z
x	x	x	x	y	z
y	y	y	y	y	z
z	z	z	z	z	z

$\wedge_\alpha$	v	w	x	y	z
v	v	v	v	v	v
w	v	w	w	w	w
x	v	w	x	x	x
y	v	w	x	y	y
z	v	w	x	y	z

We note that (X, A) is a fuzzy lattice.

Here for $A(v, x) = 0.50 > 0$, $(y, x)_F M_m$ holds in a fuzzy lattice (X, A) as $(v \vee_F y) \wedge_F x = y \wedge_F x = x = v \vee_F x = v \vee_F (y \wedge_F x)$.

For $A(w, y) = 0.60 > 0$, $(x, y)_F M_m$ holds in a fuzzy lattice (X, A) as $(w \vee_F x) \wedge_F y = x \wedge_F y = x = w \vee_F x = w \vee_F (x \wedge_F y)$.

In Figure 4, we show the related tabular and graphical representations for the fuzzy relation A.

A	v	w	x	y	z
v	1.0	0.40	0.50	0.80	0.94
w	0.0	1.0	0.20	0.60	0.90
x	0.0	0.0	1.0	0.30	0.70
y	0.0	0.0	0.0	1.0	0.40
z	0.0	0.0	0.0	0.0	1.0

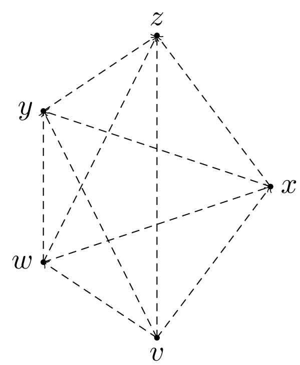

Figure 4

Now using Example 4.3 we construct an example which shows that a pair may be a fuzzy meet modular pair in a fuzzy lattice but may not be a fuzzy α-modular pair in a fuzzy α-lattice.

Example 4.4. We use Example 4.3 to construct a fuzzy α-lattice for $\alpha \geq 0.40$.

We have $A(v,v) = A(w,w) = A(x,x) = A(y,y) = A(z,z) = 1.0$,

$A(v,w) = 0.40,\ A(v,x) = 0.50,\ A(v,y) = 0.80,\ A(v,z) = 0.94$,

$A(w,v) = 0.0,\ A(w,x) = 0.0,\ A(w,y) = 0.60,\ A(w,z) = 0.90$,

$A(x,v) = 0.0,\ A(x,w) = 0.0,\ A(x,y) = 0.0,\ A(x,z) = 0.70$,

$A(y,v) = 0.0,\ A(y,w) = 0.0,\ A(y,x) = 0.0,\ A(y,z) = 0.40$,

$A(z,v) = 0.0,\ A(z,w) = 0.0,\ A(z,x) = 0.0,\ A(z,y) = 0.0$.

The fuzzy α-join and fuzzy α-meet tables are as follows:

$\vee_\alpha$	v	w	x	y	z
v	v	w	x	y	z
w	w	w	z	y	z
x	x	z	x	z	z
y	y	y	z	y	z
z	z	z	z	z	z

$\wedge_\alpha$	v	w	x	y	z
v	v	v	v	v	v
w	v	w	v	w	w
x	v	v	x	v	x
y	v	w	v	y	y
z	v	w	x	y	z

We note that for $A(v,x) = 0.50 > \alpha$, $(y,x)FM_\alpha$ holds as $(v \vee_\alpha y) \wedge_\alpha x = y \wedge_\alpha x = v = v \vee_\alpha v = v \vee_\alpha (y \wedge_\alpha x)$.

We note that for $A(w,y) = 0.60 > \alpha$, $(x,y)FM_\alpha$ does not hold as $(w \vee_\alpha x) \wedge_\alpha y = z \wedge_\alpha y = y$ and $w \vee_\alpha (x \wedge_\alpha y) = w \vee_\alpha v = w \neq y$.

Note that $(x,y)_F M_m$ holds but $(x,y)FM_\alpha$ does not hold for $\alpha \geq 0.40$.

In Figure 5, we show the related tabular and graphical representations for the fuzzy relation A.

A	v	w	x	y	z
v	1.0	0.40	0.50	0.80	0.94
w	0.0	1.0	0.0	0.60	0.90
x	0.0	0.0	1.0	0.0	0.70
y	0.0	0.0	0.0	1.0	0.40
z	0.0	0.0	0.0	0.0	1.0

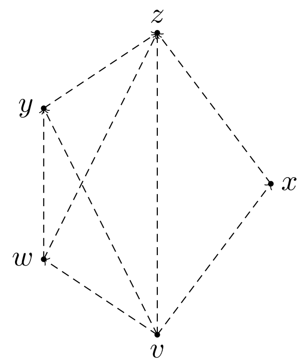

Figure 5

Remark 4.5. Let (X, A) be a fuzzy poset and $x, y \in X$. We say that x and y are α-comparable, if $A(x, y) > \alpha$ or $A(y, x) > \alpha$ for some $\alpha \in [0, 1)$.

Proposition 4.6. *Let (X, A) be a fuzzy α-lattice. If x and y are α-comparable, then $(y, x)FM_\alpha$ for some $\alpha \in [0, 1)$.*

Proof. Since x and y are α-comparable, then $A(x, y) > \alpha$ or $A(y, x) > \alpha$.
Case (1): Let $A(x, y) > \alpha$. Suppose that $A(z, x) > \alpha$ for some $z \in X$.
Then by fuzzy transitivity of A we have $A(z, y) > \alpha$, that is, $z \vee_\alpha y = y$.
As $A(z, x) > \alpha$ and $A(z, y) > \alpha$ so, by (iii) of Proposition 3.16, we get $A(z, x \wedge_\alpha y) > \alpha$.
Hence $z \vee_\alpha (y \wedge_\alpha x) = y \wedge_\alpha x = (z \vee_\alpha y) \wedge_\alpha x$.
Therefore $(y, x)FM_\alpha$ holds.

Case (2): Let $A(y, x) > \alpha$. Suppose that $A(z, x) > \alpha$.
Since $A(z, x) > \alpha$ and $A(y, x) > \alpha$ by (ii) of Proposition 3.16, we have $A(z \vee_\alpha y, x) > \alpha$ such that $z \vee_\alpha (y \wedge_\alpha x) = z \vee_\alpha y = (z \vee_\alpha y) \wedge_\alpha x$.
Hence $(y, x)FM_\alpha$ holds. □

Corollary 4.7. *Let (X, A) be a fuzzy α-lattice. Then $(x \wedge_\alpha y, x)FM_\alpha$, $(x \wedge_\alpha y, y)FM_\alpha$, $(x, x \vee_\alpha y)FM_\alpha$, $(y, x \vee_\alpha y)FM_\alpha$ and $(x \wedge_\alpha y, x \vee_\alpha y)FM_\alpha$.*

Proposition 4.8. *Let (X, A) be a fuzzy α-lattice. Suppose that $(x, y)FM_\alpha$ holds. Let $z \in X$. If $A(x \wedge_\alpha y, z) > \alpha$ and $A(z, y) > \alpha$, then $(z \vee_\alpha x) \wedge_\alpha y = z$.*

Proof. Let $(x, y)FM_\alpha$ hold and let $z \in X$.
Suppose that $A(x \wedge_\alpha y, z) > \alpha$, so by (iv) of Proposition 3.16, we have $(x \wedge_\alpha y) \vee_\alpha z = z$.
Since $(x, y)FM_\alpha$ holds, if $A(z, y) > \alpha$, then
$(z \vee_\alpha x) \wedge_\alpha y = z \vee_\alpha (x \wedge_\alpha y)$.
Hence $(z \vee_\alpha x) \wedge_\alpha y = z$. □

Proposition 4.9. *Let (X, A) be a fuzzy α-lattice. Let $x, y \in X$ be such that $(x, y)FM_\alpha$. If $A(x \wedge_\alpha y, x') > \alpha$, $A(x', x) > \alpha$, $A(x \wedge_\alpha y, y') > \alpha$ and $A(y', y) > \alpha$, then $(x', y')FM_\alpha$.*

Proof. To prove this we use Proposition 4.8.
Let $(x, y)FM_\alpha$ hold. Suppose that $A(x \wedge_\alpha y, y') > \alpha$ and $A(y', y) > \alpha$.
Since $A(x \wedge_\alpha y, x') > \alpha$ and $A(x \wedge_\alpha y, y') > \alpha$,
by (iii) of Proposition 3.16, we have $A(x \wedge_\alpha y, x' \wedge_\alpha y') > \alpha$. (I)
As $A(x', x) > \alpha$ and $A(y', y) > \alpha$ so by Lemma 3.18, we get
$A(x' \wedge_\alpha y', x \wedge_\alpha y) > \alpha$. (II)
From (I) and (II) by fuzzy antisymmetry of A we get
$x' \wedge_\alpha y' = x \wedge_\alpha y$. (III)
Now, let $z \in X$ be such that $A(z, y') > \alpha$.
As $A(z, y') > \alpha$ and $A(y', y) > \alpha$ by fuzzy antisymmetry of A we have $A(z, y) > \alpha$.
As $(x, y)FM_\alpha$ holds, we have $(z \vee_\alpha x) \wedge_\alpha y = z \vee_\alpha (x \wedge_\alpha y)$.
Thus, by (III) we obtain $(z \vee_\alpha x) \wedge_\alpha y = z \vee_\alpha (x' \wedge_\alpha y')$. (IV)
Since $A(x', x) > \alpha$ by (iii) of Proposition 3.16, we have
$A(z \vee_\alpha x', z \vee_\alpha x) > \alpha$.
As $A(y', y) > \alpha$ and $A(z \vee_\alpha x', z \vee_\alpha x) > \alpha$ again using Lemma 3.18, we have $A((z \vee_\alpha x') \wedge_\alpha y', (z \vee_\alpha x) \wedge_\alpha y) > \alpha$.
By using (IV) we get $A((z \vee_\alpha x') \wedge_\alpha y', z \vee_\alpha (x' \wedge_\alpha y')) > \alpha$.
Thus, $(x', y')FM_\alpha$ holds. □

5. Conclusion

In this paper, we have presented a novel approach to modularity in fuzzy α-lattices.

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