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# ON EDGE PRODUCT HYPERGRAPHS

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Abstract. In this paper we introduced the notion of an edge product hypergraph. A hypergraph $\mathcal{H}$ is said to be an edge product hypergraph if edges of hypergraph can be labeled with distinct positive integers such that the product of all the labels of edges incident to a vertex is again an edge label of $\mathcal{H}$ and if the product of any collection of edges is a label of an edge in $\mathcal{H}$ then, they are incident to a vertex. Here we have proved some important results by which one can verify that given hypergraph is a unit edge product hypergraph or not. We also found some results on domination number and inverse domination number of edge product hypergraph and its complement.

**Key Words:** edge product hypergraph, unit edge product hypergraph, domination number.
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## 1. Introduction

In graph theory, graph labeling plays an important role in many applications such as astronomy, radar, x-ray crystallography, circuit design, data base management, communication network addressing etc. The concept of graph labeling was presented by Alexander Rosa in his paper [6] in 1967. There are many different types of graph labeling such as vertex labeling, edge labeling etc. In graph labeling, we assign integers to the vertices or edges or both, subject to certain conditions. If the domain is the set of vertices (or edges) then the labeling is called the vertex (edge) labeling. If the labels are assigned to both vertices as well

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as edges of a graph then the labeling is called total labeling. Traditionally, the set of labels which we give to vertices or edges is a subset of integers. For more details about graph labeling reader may refer to Gallian [4]. In 1990 Harary [5] introduced the notion of sum graphs of a graph $G$. A graph $G(V, E)$ is said to be a sum graph if there exists a bijection labeling $f$ from the vertex set $V$ to a set $S$ of positive integers such that $xy \in E$ if and only if $f(x) + f(y) \in S$. The product analogue of sum graphs was first introduced by Thavamani [7] in 2011. He introduced edge product graph and edge product number of a graph. A graph $G$ is said to be an edge product graph if the edges of $G$ can be labeled with distinct positive integers such that the product of all the labels of the edges incident on a vertex is again an edge label of $G$ and if the product of any collection of edges is a label of an edge in $G$ then they are incident on a vertex.

Hypergraph theory has been introduced in the 1960's. It has many applications in different sciences. Hypergraph is a very useful tool to understand problems in a wide variety of scientific field. It also models many practical problems in many different sciences. Labeled hypergraphs are one of the most widely appropriate and general way to represent data on stateful machine. Hypergraph is a generalization of a graph in which any subset of a given set may be an edge rather than two element subsets. A hypergraph $\mathcal{H}$ is a pair $(V, E)$ where $V$ is a set of elements called vertices or nodes and $E$ is a set of non-empty subsets of $X$ called hyperedges or edges. It means that $E$ is a subset of $P(X) \setminus \phi$ where $P(X)$ is the power set of $X$. A hypergraph whose edges are assigned with weights are called weighted hypergraphs. The weighted hypergraphs gained much attention in computer vision for the purpose of representing geometrical information.

In the present paper, we introduced the notions of edge product hypergraph, unit edge product hypergraph and studied their properties on the line of [5] and [7]. We also proved some important results by which one can verify that the given hypergraph is an unit edge product hypergraph or not. Finally we obtain the dominating set and domination number of edge product hypergraph and its complement.

## 2. Preliminaries

We begin with recalling some basic definitions from [1]-[3] required for our purpose.

**Definition 2.1.** A hypergraph $\mathcal{H}$ is a pair $\mathcal{H}(V, E)$ where $V$ is a finite nonempty set and $E$ is a collection of subsets of $V$. The elements of $V$ are called vertices and the elements of $E$ are called edges or hyperedges. And $\cup_{e_i \in E} e_i = V$ and $e_i \neq \phi$ are required, for all $e_i \in E$. The number of vertices in $\mathcal{H}$ is called the order of the hypergraph and is denoted by $|V|$. The number of edges in $\mathcal{H}$ is called the size of $\mathcal{H}$ and is denoted by $|E|$. A hypergraph of order $n$ and size $m$ is called a $(n, m)$ hypergraph. The number $|e_i|$ is called the degree (cardinality) of the edges $e_i$. The rank of a hypergraph $\mathcal{H}$ is $r(\mathcal{H}) = max_{e_i \in E}|e_i|$.

**Definition 2.2.** For any vertex $v$ in a hypergraph $\mathcal{H}(V, E)$, the set $N[v] = \{u \in V : u \text{ is adjacent to } v\} \cup \{v\}$ is called the closed neighborhood of $v$ in $\mathcal{H}$ and each vertex in the set $N[v] - \{v\}$ is called neighbor of $v$. The open neighborhood of the vertex $v$ is the set $N[v] \setminus \{v\}$. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

**Definition 2.3.** A simple hypergraph (or sperner family) is a hypergraph $\mathcal{H}(V, E)$ where $E = \{e_1, e_2, \cdots, e_m\}$ such that $e_i \subset e_j$ implies $i = j$.

**Definition 2.4.** For any hypergraph $\mathcal{H}(V, E)$ two vertices $v$ and $u$ are said to be adjacent if there exists an edge $e \in E$ that contains both $v$ and $u$ and non adjacent otherwise.

**Definition 2.5.** For any hypergraph $\mathcal{H}(V, E)$ two edges are said to be adjacent if their intersection is nonempty. If a vertex $v_i \in V$ belongs to an edge $e_j \in E$ then we say that they are incident to each other.

**Definition 2.6.** An edge in a hypergraph $\mathcal{H}$ is called a pure hyperedge if it contains at least three vertices; otherwise it is called ordinary, and $\mathcal{H}$ is called a pure hypergraph if each edge of $\mathcal{H}$ is a pure hyperedge.

**Definition 2.7.** The vertex degree of a vertex $v$ is the number of vertices adjacent to the vertex $v$ in $\mathcal{H}$. It is denoted by $d(v)$.
The maximum (minimum) vertex degree of a hypergraph is denoted by $\Delta(\mathcal{H})(\delta(\mathcal{H}))$.

**Definition 2.8.** The edge degree of a vertex $v$ is the number of edges containing the vertex $v$. It is denoted by $d_E(v)$.

The maximum (minimum) edge degree of a hypergraph is denoted by $\Delta_E(\mathcal{H})(\delta_E(\mathcal{H}))$. A vertex of a hypergraph which is incident to no edge is called an isolated vertex. The edge degree (or vertex degree) of an isolated vertex is trivially 0. An edge of cardinality one is called a singleton (loop), a vertex of edge degree one is called a pendant vertex.

**Definition 2.9.** The hypergraph $\mathcal{H}(V, E)$ is called connected if for any pair of its vertices, there is a path connecting them. If $\mathcal{H}$ is not connected then it consists of two or more connected components, each of which is a connected hypergraph.

**Definition 2.10.** A hypergraph is said to be of rank $k$ if each of its edge contains at most $k$ vertices.

**Definition 2.11.** The complement of $\bar{\mathcal{H}}$ of a hypergraph $\mathcal{H}(V, E)$ is defined as $\bar{\mathcal{H}}(V, \bar{E})$ where $\bar{E} = \{\bar{e} | e \in E\}$ with $\bar{e} \in \bar{E}, \bar{e} = \{v \notin e | e \in E\}$.

**Definition 2.12.** For a hypergraph $\mathcal{H}(V, E)$, a set $D \subseteq V$ is called a dominating set of $\mathcal{H}$ if for every $v \in V \setminus D$ there exists $u \in D$ such that $u$ and $v$ are adjacent in $\mathcal{H}$, that is there exists $e \in E$ such that $u, v \in e$.

**Definition 2.13.** A dominating set $D$ of a hypergraph $\mathcal{H}$ is called a minimal dominating set, if no proper subset of $D$ is a dominating set of $\mathcal{H}$. The minimum cardinality of a minimal dominating set in a hypergraph $\mathcal{H}$ is called the domination number of $\mathcal{H}$ and is denoted by $\gamma(\mathcal{H})$.

**Definition 2.14.** Let $D \in D^0(\mathcal{H})$, the set of all minimum dominating sets (of cardinality $\gamma(\mathcal{H})$). An inverse dominating set with respect to $D$ is any dominating set $D'$ of $\mathcal{H}$ such that $D' \subseteq V \setminus D$. The inverse domination number of $\mathcal{H}$ is defined as

$$\gamma^{-1}(\mathcal{H}) = min\{|D'| \ \mid \ D \in D^0(\mathcal{H}), D' \text{is an inverse dominating set with respect to D}\}$$

In this paper, we consider a simple hypergraph $(n, m)$ without isolated vertices and of size $m > 1$.

## 3. Edge Product Hypergraph

In this section the notion of an edge function is given and using this edge function an edge product hypergraph is defined. Later the notion of a unit edge product hypergraph is introduced. These definitions are verified with examples and some important results are obtained. Lastly the results based on the domination number and inverse domination number of unit edge product hypergraph and its complement are obtained.

**Definition 3.1.** Let $\mathcal{H}(V, E)$ be a simple and connected hypergraph. Let $V(\mathcal{H})$ be the vertex set of $\mathcal{H}$ and $E(\mathcal{H})$ be the edge set of $\mathcal{H}$. Let $P$ be a set of positive integers such that $|E| = |P|$. Then any bijection $f : E \to P$ is called an edge function of the hypergraph $\mathcal{H}$.

**Definition 3.2.** The function

$$F(v) = \prod\{f(e)| \text{ edge } e \text{ is incident to the vertex } v\}$$

on $V(\mathcal{H})$ is called an edge product function of the edge function $f$.

**Definition 3.3.** The hypergraph $\mathcal{H}(V, E)$ is said to be an edge product hypergraph if there exists an edge function $f : E \to P$ such that the edge function $f$ and the corresponding edge product function $F$ of $f$ on $V(\mathcal{H})$ have the following two conditions:

(1) $F(v) \in P$, for every $v \in V$.
(2) If $f(e_1)\times f(e_2)\times\ldots\times f(e_p) \in P$, for some edges $e_1, e_2, \ldots, e_p \in E$ then the edges $e_1, e_2, \ldots, e_p$ are all incident to a vertex $v \in V$.

*Example* 3.4. Let $\mathcal{H}(V, E)$ be a hypergraph, where $V = \{v_1, v_2, \ldots, v_{20}\}$ and $E = \{e_1, e_2, \ldots, e_7\}$. In which the edges of $\mathcal{H}$ are defined as follows:
$e_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$,
$e_2 = \{v_1, v_2, v_3, v_{10}\}$,
$e_3 = \{v_1, v_2, v_3, v_{11}, v_{12}\}$,
$e_4 = \{v_4, v_5, v_{13}\}$,
$e_5 = \{v_4, v_5, v_{14}, v_{15}, v_{16}\}$,
$e_6 = \{v_4, v_5, v_{17}, v_{18}\}$, and
$e_7 = \{v_{19}, v_{20}\}$.

Now define the edge function $f : E \to P$ by $f(e_1) = 11$, $f(e_2) = 30$, $f(e_3) = 4$, $f(e_4) = 3$, $f(e_5) = 20$, $f(e_6) = 2$, $f(e_7) = 1320$.

The edge product function $F$ of $f$ is defined by, $F(v_1) = 1320$, $F(v_2) = 1320$, $F(v_3) = 1320$, $F(v_4) = 1320$, $F(v_5) = 1320$, $F(v_6) = 11$, $F(v_7) = 11$, $F(v_8) = 11$, $F(v_9) = 11$, $F(v_{10}) = 30$, $F(v_{11}) = 4$, $F(v_{12}) = 4$, $F(v_{13}) = 3$, $F(v_{14}) = 20$, $F(v_{15}) = 20$, $F(v_{16}) = 20$, $F(v_{17}) = 2$, $F(v_{18}) = 2$, $F(v_{19}) = 1320$, $F(v_{20}) = 1320$.

Hence the given hypergraph is a edge product hypergraph.

**Definition 3.5.** For an edge product hypergraph $\mathcal{H}(V, E)$ there exists an edge function $f : E \to P$ such that an element $1 \in P$ then the hypergraph $\mathcal{H}$ is called a unit edge product hypergraph.

**Theorem 3.6.** *Let $\mathcal{H}$ be a unit edge product hypergraph with an edge $e^* \in E$ and $f(e^*) = 1$. Then $e^*$ must be adjacent to all the edges of $\mathcal{H}$.*

*Proof.* Let $\mathcal{H}(V, E)$ be a unit edge product hypergraph with $f(e^*) = 1$. Let $e_j$ be any edge in $\mathcal{H}$ such that $f(e_j) \in P$. Suppose that $e_j$ is not adjacent to $e^*$. It means that there is no common vertex in $e_j$ and $e^* \Rightarrow e_j \cap e^* = \emptyset$. Now, $f(e_j) = f(e_j) \cdot 1 \Rightarrow f(e_j) = f(e_j) \cdot f(e^*) \in P$.

Therefore edges $e_j$ and $e^*$ are incident to a vertex $v \in V$ that is $v \in e_j \cap e^*$. This is a contradiction to our assumption and hence $e^*$ must be adjacent to all the edges of $\mathcal{H}$. □

*Example* 3.7. Let $\mathcal{H}(V, E)$ be a hypergraph, where $V = \{v_1, v_2, \ldots, v_{24}\}$ and $E = \{e_1, e_2, \ldots, e_6\}$. In which the edges of $\mathcal{H}$ are defined as follows:
$e_1 = \{v_1, v_2, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{23}\}$,
$e_2 = \{v_1, v_2, v_3, v_4, v_5\}$,
$e_3 = \{v_1, v_2, v_6, v_7, v_8, v_9, v_{10}\}$,
$e_4 = \{v_{14}, v_{15}, v_{16}, v_{17}, v_{18}\}$,
$e_5 = \{v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{22}\}$, and
$e_6 = \{v_{23}, v_{24}\}$.

Define the edge function $f : E \to P$ by $f(e_1) = 1$, $f(e_2) = 2^4$, $f(e_3) = 2^7$, $f(e_4) = 2^5$, $f(e_5) = 2^6$, $f(e_6) = 2^{11}$.

The edge product function $F$ of $f$ is defined by, $F(v_1) = F(v_2) = 2^{11}$, $F(v_3) = F(v_4) = F(v_5) = 2^4$, $F(v_6) = F(v_7) = F(v_8) = F(v_9) = F(v_{10}) = 2^7$, $F(v_{14}) = (v_{15}) = F(v_{16}) = 2^{11}$, $F(v_{17}) = F(v_{18}) = 2^5$, $F(v_{11}) = F(v_{12}) = F(v_{13}) = 1$, $F(v_{19}) = F(v_{20}) = F(v_{21}) = F(v_{22}) = 2^6$, $F(v_{23}) = 2^{11}$, $F(v_{24}) = 2^{11}$.

Hence the given hypergraph $\mathcal{H}(V, E)$ is a unit edge product hypergraph.

*Example* 3.8. Let $\mathcal{H}(V, E)$ be a hypergrapgh, where $V = \{v_1, v_2, \ldots, v_{22}\}$ and $E = \{e_1, e_2, \ldots e_5\}$. In which the edges of $\mathcal{H}$ are defined as follows:
$e_1 = \{v_1, v_2, v_3, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$,
$e_2 = \{v_1, v_2, v_3, v_4\}$,
$e_3 = \{v_1, v_2, v_3, v_5, v_6\}$,
$e_4 = \{v_1, v_2, v_3, v_7, v_8, v_9\}$, and
$e_5 = \{v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}\}$.

Define the edge function $f : E \to P$ by $f(e_1) = 1$, $f(e_2) = 2$, $f(e_3) = 3$, $f(e_4) = 4$, $f(e_5) = 24$,

The edge product function $F$ of $f$ is defined by, $F(v_1) = F(v_2) = F(v_3) = 24$, $F(v_4) = 2, F(v_5) = 3 = F(v_6)$, $F(v_7) = F(v_8) = F(v_9) = 4$, $F(v_{10}) = (v_{11}) = 1$, $F(v_{12}) = F(v_{13}) = F(v_{14}) = F(v_{15}) = F(v_{16}) = F(v_{17}) = 24$, $F(v_{18}) = F(v_{19}) = F(v_{20}) = F(v_{21}) = F(v_{22}) = 24$.

Hence the given hypergraph $\mathcal{H}(V, E)$ is a unit edge product hypergraph.

**Theorem 3.9.** *Let $\mathcal{H}(V, E)$ be a hypergraph with an edge $e^* \in E$ such that $f(e^*) = 1$. Let for any other edge (excluding $e^*$), there is exactly* *one distinct vertex* $x_i \in e_i \cap e^*$, *for* $i = 1, 2, \ldots, m-1$ *and* $e_i \cap e_j = \phi$, *for* $i \neq j$ , $i, j = 1, 2, \ldots, m-1$. *Then* $\mathcal{H}$ *is a unit edge product hypergraph.*

*Proof.* Let $\mathcal{H}(V, E)$ be a hypergraph satisfying the hypothesis. Let $e_1, e_2, \ldots, e_{m-1}$ be other edges in $\mathcal{H}$ and $e_i \cap e_j = \phi$ for $i \neq j$, $i, j = 1, 2, \ldots, m-1$. Now for each edge $e_i$ we have only one distinct vertex $x_i \in e_i \cap e^*$ that is $x_1, x_2, \ldots, x_{m-1}$ are the vertices in $e^*$. Let $t_1, t_2, \ldots, t_q$ be the other vertices in $e^*$ such that $t_1, t_2, \ldots, t_q \notin e_i$ for $i = 1, 2, \ldots, m-1$ which implies $t_1, t_2, \ldots, t_q$ are pendant vertices in $\mathcal{H}$.

Now let $e_i$ be the edge with vertices $y_1^i, y_2^i, \cdots, y_{r_i}^i$ in $\mathcal{H}$, for $i = 1, 2, \ldots, m-1$, where $r_1, r_2, \ldots, r_{m-1}$ are the non-negative integers representing the number of elements in $e_1, e_2, \ldots, e_{m-1}$ respectively.

Thus,

$$V(\mathcal{H}) = \{x_1, x_2, \ldots, x_{m-1}, t_1, t_2, \ldots, t_q, y_1^1, y_2^1, \ldots, y_{r_1}^1, y_1^2, y_2^2, \ldots, y_{r_2}^2, \ldots, y_1^{m-1}, y_2^{m-1}, \ldots, y_{r_{m-1}}^{m-1}\}$$

and

$$E(\mathcal{H}) = \{e^*, e_1, e_2, \ldots, e_{m-1}\},$$

where $e^* = \{x_1, x_2, \ldots, x_{m-1}, t_1, t_2, \ldots, t_q\}$ and $e_i = \{y_1^i, y_2^i, \ldots, y_{r_i}^i\}$ for $i = 1, 2, \ldots, m-1$ is the vertex set and edge set of the given hypergraph respectively.

The set of all elements of $P = \{1, p_1, p_2, \ldots, p_{m-1}\}$ where $p_i$ denote the $i^{th}$ prime number, when we enumerate prime numbers in the increasing order. The edge function $f : E \to P$ defined by $f(e^*) = 1, f(e_i) = p_i$ for $i = 1, 2, \ldots m-1$.

The edge product function $F$ of $f$ is defined by, $F(x_i) = p_i$ for $1 \leq i \leq m-1$.

$F(t_j) = 1$ for $1 \leq j \leq q$

$F(y_1^1) = F(y_2^1) = F(y_3^1) = \cdots = F(y_{r_1}^1) = p_1$

$F(y_1^2) = F(y_2^2) = F(y_3^2) = \cdots = F(y_{r_2}^2) = p_2$

$\vdots$

$F(y_1^{m-1}) = F(y_2^{m-1}) = F(y_3^{m-1}) = \cdots = F(y_{r_{m-1}}^{m-1}) = p_{m-1}$

Here for every vertex $v \in V(\mathcal{H})$, we have $F(v) \in P$ and if the product of a collection of more than one element of $P$ is in $P$ then the collection consists of exactly two elements 1 and $p_i$. For 1 and $p_i$, we have edges $e^*$ and $e_i$ incident to a vertex $x_i \in V(\mathcal{H})$. Hence the given hypergraph is a unit edge product hypergraph. □

*Example* 3.10. Let $\mathcal{H}(V, E)$ be a hypergraph. Where $V = \{v_1, v_2, \ldots, v_{21}\}$ and $E = \{e_1, e_2, \ldots, e_7\}$. In which the edges of $\mathcal{H}$ are defined as follows: $e_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$,

$e_2 = \{v_1, v_{10}\}$,
$e_3 = \{v_2, v_{11}\}$,
$e_4 = \{v_3, v_{12}, v_{13}\}$,
$e_5 = \{v_4, v_{14}, v_{15}\}$,
$e_6 = \{v_5, v_{16}, v_{17}, v_{18}\}$, and
$e_7 = \{v_6, v_{19}, v_{20}, v_{21}\}$.

Define edge function $f : E \to P$ by $f(e_1) = 1$, $f(e_2) = 2$, $f(e_3) = 3$, $f(e_4) = 5$, $f(e_5) = 7$, $f(e_6) = 11$, $f(e_7) = 13$.

The edge product function $F$ of $f$ is defined by $F(v_1) = F(v_{10}) = 2$, $F(v_2) = F(v_{11}) = 3$, $F(v_3) = F(v_{12}) = F(v_{13}) = 5$, $F(v_4) = F(v_{14}) = F(v_{15}) = 7$, $F(v_5) = F(v_{16}) = (v_{17}) = F(v_{18}) = 11$, $F(v_6) = F(v_{19}) = F(v_{20}) = F(v_{21}) = 13$, $F(v_7) = F(v_8) = F(v_9) = 1$.

Hence the given hypergraph $\mathcal{H}(V, E)$ is a unit edge product hypergraph.

**Theorem 3.11.** *Let $\mathcal{H}(V, E)$ be a unit edge product hypergraph and $e^* \in E$ such that $f(e^*) = 1$. Then $\mathcal{H}$ contains at least one edge which is adjacent to only $e^*$.*

*Proof.* Let $\mathcal{H}$ be a unit edge product hypergraph with $e^* \in E$ such that $f(e^*) = 1$. Let $f$ be an edge function and $F$ be an edge product function of $f$. Let us suppose that $p$ be the largest element in the set of positive integers $P$. Then for bijection $f : E \to P$ there exists an edge $e \in E$ such that $f(e) = p$. Consider there is no edge in $\mathcal{H}$ which is adjacent to only $e^*$. This implies an edge $e$ is adjacent to some other edge $e_j \in E$ in $\mathcal{H}$. Hence we have a vertex $v \in e \cap e_j$ or $v \in e \cap e_j \cap e^*$. We also obtain $F(v) = f(e) \cdot f(e_j) > p$ or $F(v) = f(e) \cdot f(e_j) \cdot f(e^*) > p$, which is a contradiction. Thus $\mathcal{H}$ contains at least one edge which is adjacent to only $e^*$. $\square$

**Theorem 3.12.** *Let $\mathcal{H}$ be a unit edge product hypergraph and $e^* \in E$ such that $f(e^*) = 1$. Let $v_1, v_2, \ldots, v_r$ be the pendant vertices in $e^*$. Then $\gamma(\mathcal{H}) \leq |e^*| - r$.*

*Proof.* Let $\mathcal{H}$ be a unit edge product hypergraph with $e^* \in E$ and $f(e^*) = 1$ and $v_1, v_2, \ldots, v_r$ be the pendant vertices in $e^*$. By Theorem 3.6 we have $x_i \in (e_i \cap e^*)$. Hence $x_1, x_2, \ldots, x_{m-1}$ be non pendant vertices in $e^*$ (which may or may not be distinct). Now each $x_i$ dominates rest of all vertices of $e_i$, for $i = 1, 2, \ldots, m-1$. Therefore $\{x_1, x_2, \ldots, x_{m-1}\}$ is a dominating set of $\mathcal{H}$ with cardinality $|e^*| - r$. Hence $\gamma(\mathcal{H}) \leq |e^*| - r$. $\square$

*Remark* 3.13. The bound given in the Theorem 3.12 is sharp:
For, consider the hypergraph with the vertex set $V = \{v_1, v_2, \dots, v_{19}\}$ and the edge set $E = \{e_1, e_2, \dots, e_5\}$. In which the edges of $\mathcal{H}$ are defined as follows: $e_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$,
$e_2 = \{v_1, v_9\}$,
$e_3 = \{v_2, v_{10}, v_{11}\}$,
$e_4 = \{v_3, v_{12}, v_{13}, v_{14}\}$, and
$e_5 = \{v_4, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}\}$.

We define the edge function $f : E \to P$ by $f(e_1) = 1$, $f(e_2) = 2$, $f(e_3) = 3$, $f(e_4) = 5$, $f(e_5) = 7$, where $P = \{1, 2, 3, 5, 7\}$.

The edge product function $F$ of $f$ is defined by, $F(v_1) = F(v_9) = 2$, $F(v_2) = F(v_{10}) = F(v_{11}) = 3$, $F(v_3) = F(v_{12}) = F(v_{13}) = F(v_{14}) = 5$, $F(v_4) = F(v_{15}) = F(v_{16}) = (v_{17}) = F(v_{18}) = 7$, $F(v_6) = F(v_7) = F(v_5) = F(v_8) = 1$.

Thus, $\mathcal{H}(V, E)$ is a unit edge product hypergraph. Here $e_1 \in E$ and $f(e_1) = 1$, $|e_1| = 8$. The vertices $\{v_5, v_6, v_7, v_8\}$ are pendant vertices in $e_1$. clearly the set $D = \{v_1, v_2, v_3, v_4\}$ is a minimum dominating set of a given hypergraph and $\gamma(\mathcal{H}) = 4$ that is $\gamma(\mathcal{H}) = |e_1| - r$.

In order to avoid the trivial anomalies, whenever we talk about $\bar{\mathcal{H}}$, we restrict ourselves to those hypergraphs which satisfies the condition that, every vertex $v$ of $\mathcal{H}$ is incident with some edge $e$ of cardinality, $2 \leq |e| \leq |v| - 2$, and avoiding $v$ and $d_E(v) < |E|$ and $|V| \geq 4$.

**Lemma 3.14.** *Let $\mathcal{H}(V, E)$ be a unit edge product hypergraph and $e^* \in E$ such that $f(e^*) = 1$ and $e^*$ contains two pendant vertices. Then $\gamma(\bar{\mathcal{H}}) = \gamma^{-1}(\bar{\mathcal{H}}) = 1$.*

*Proof.* Let $\mathcal{H}$ be a unit edge product hypergraph with $e^* \in E$ and $f(e^*) = 1$. Let $u$ and $v$ be the pendant vertices in $e^*$. Then by Theorem 3.11, there exists an edge $e_k$ which is adjacent to only $e^*$. For the vertex $w \in X|_{e_k \cup \{u\}}$ we have $\bar{e}_k = X|_{e_k} \in \bar{E}$ such that $u, w$ are in $\bar{e}_k$. Hence the vertex $u$ dominates the vertex $w$ in $\bar{\mathcal{H}}$. Now let $x \in e_k$. Then there exists $e_j \in E$ such that $x \notin e_j$. Since $u$ and $x$ are not in $e_j$, the edge $\bar{e}_j = X|_{e_j} \in \bar{E}$ contains the vertices $u$ and $x$ both. Therefore the vertex $x$ is dominated by $u$. Hence $\{u\}$ is a dominating set of $\bar{\mathcal{H}}$ and $\{v\}$ is the inverse dominating set of $\bar{\mathcal{H}}$. □

## 4. Non unit Edge Product Hypergraphs

In this section all the edge product hypergraphs are non unit edge product hypergraph unless otherwise stated. This section deals with disconnectedness, adjacency, edge degree, dominating set and domination number of edge product hypergraph. Finally we proved that for a edge product hypergraph $\gamma(\bar{\mathcal{H}}) = \gamma^{-1}(\bar{H}) = 1$.

**Theorem 4.1.** *Edge product hypergraph is always disconnected.*

*Proof.* Let $\mathcal{H}(V, E)$ be an edge product hypergraph. Let $f : E \to P$ be an edge function and $F$ be an edge product function of $f$. Let us suppose that $p$ be the largest element in the set of positive integers $P$. Then for the bijection $f : E \to P$, there exists an edge $e \in E$ such that $f(e) = p$. Suppose the edge $e$ is adjacent to any other edge $e_j$ in $\mathcal{H}$ then we have $x \in (e \cap e_j)$ and $F(x) = f(e)f(e_j) > f(e) = p$, which is a contradiction. Hence the edge $e$ is not adjacent to any other edge in $\mathcal{H}$. Thus $\mathcal{H}$ is a disconnected hypergraph. □

*Note* 4.2. To make the hypergraph a edge product hypergraph one can add any edge of cardinality less than or equal to $n$. To avoid inconvenience we add a subset of $V$ of cardinality two.

**Corollary 4.3.** *If $\mathcal{H}(V, E)$ is a edge product hypergraph then $K_2$ is the component of $\mathcal{H}$.*

**Theorem 4.4.** *Let $\mathcal{H}(V, E)$ be an edge product hypergraph. Let $u$ be a non pendant vertex and $e_j \in E$ such that $F(u) = f(e_j)$. Let $e_1, e_2, \ldots, e_p$ be the collection of edges incident to a vertex $u$. Then if an edge $e_j \in E$ is adjacent to any edge (excluding $e_1, e_2, \ldots, e_p$) in $\mathcal{H}$ then that edge must contains a vertex $v$ such that $d_E(v) \geq 3$.*

*Proof.* Let $\mathcal{H}$ be an edge product hypergraph. Let $u$ be a non pendant vertex and $e_j \in E$ such that $F(u) = f(e_j)$. Let $u$ be incident to the edges $e_1, e_2, \ldots, e_p$ and $p \geq 2$. Therefore $F(u) = f(e_1) \times f(e_2) \times \ldots \times f(e_p) \in P$. Let edges $e_j$ and $e_m$ are adjacent and $e_m \neq e_i$, for $i = 1, 2, \ldots, p$. Then we have a vertex $x \in e_j \cap e_m$ and $F(x) = f(e_j) \times f(e_m) \in P$ which implies $F(x) = f(e_1) \times f(e_2) \times \ldots \times f(e_p) \times f(e_m) \in P$. Hence the edges $e_1, e_2, \ldots, e_p, e_m$ are incident to a vertex $v \in V(\mathcal{H})$ and $d_E(v) \geq 3$. Hence the proof. □

**Theorem 4.5.** *Let $\mathcal{H}(V, E)$ be an edge product hypergraph. Let $u$ be a non pendant vertex such that edges $e_1, e_2, \ldots, e_p$ are incident to it and $e_j \in E$ such that $F(u) = f(e_j)$. If edge $e_j$ is adjacent to any edge $e_m$ (excluding $e_1, e_2, \ldots, e_p$) in $\mathcal{H}$. Then $e_m$ must be adjacent to the edges $e_1, e_2, \ldots, e_p$ in $\mathcal{H}$.*

**Theorem 4.6.** *Let $\mathcal{H}(V, E)$ be an edge product hypergraph. Let $e_1, e_2, \ldots, e_p$ be the edges incident to a vertex $u$ and $l_1, l_2, \ldots, l_q$ be the edges incident to a vertex $v$. If there exists a proper sub collection $e_1, e_2, \ldots, e_r$ of $e_1, e_2, \ldots, e_p$ and $l_1, l_2, \ldots, l_s$ of $l_1, l_2, \ldots, l_q$ such that $f(e_1) \times f(e_2) \times \ldots \times f(e_r) = f(l_1) \times f(l_2) \times \ldots \times f(l_s)$. Then there exists vertices $w_1$ and $w_2$ such that $d_E(w_1) = p-r+s$ and $d_E(w_2) = q-s+p$ in $\mathcal{H}$.*

*Proof.* Let $\mathcal{H}(V, E)$ be an edge product hypergraph satisfying the hypothesis. Let $e_1, e_2, \ldots, e_p$ be the edges incident to a vertex $u$ and $e_1, e_2, \ldots, e_r$ is the sub collection of $e_1, e_2, \ldots, e_p$ and $r < p$. Now

$$F(u) = f(e_1) \times f(e_2) \times \ldots \times f(e_r) \times f(e_{r+1}) \times f(e_{r+2}) \times \ldots \times f(e_p) \in P$$

implies

$$F(u) = f(l_1) \times f(l_2) \times \ldots \times f(l_s) \times f(e_{r+1}) \times f(e_{r+2}) \times \ldots \times f(e_p) \in P.$$

Therefore the edges $l_1, l_2, \ldots, l_s, e_{r+1}, e_{r+2}, \ldots e_p$ are all incident to a vertex say $w_1 \in V$ and the edge degree of $w_1$ is $p - r + s$. Similarly,

$$F(v) = f(l_1) \times f(l_2) \times \ldots \times f(l_s) \times f(l_{s+1}) \times f(l_{s+2}) \times \ldots \times f(l_q) \in P$$

implies

$$F(v) = f(e_1) \times f(e_2) \times \ldots \times f(e_r) \times f(l_{s+1}) \times f(l_{s+2}) \times \ldots \times f(l_q) \in P$$

and $s < q$. Hence the edges $e_1, e_2, \ldots, e_r, l_{s+1}, l_{s+2}, \ldots, l_q$ all are incident to a vertex, say $w_2 \in V$ and edge degree of $w_2$ is $q - s + r$. □

**Theorem 4.7.** *If $\mathcal{H}(V, E)$ is a edge product hypergraph. Then any singleton subset of $\mathcal{H}$ can not be a dominating set of $\mathcal{H}$.*

*Proof.* Let $\mathcal{H}(V, E)$ be an edge product hypergraph. Suppose that $D = \{x\}$ is a dominating set of $\mathcal{H}$. By Theorem 4.1 $\mathcal{H}$ is disconnected which implies $\mathcal{H}$ has at least two components say $\mathcal{H}_1$ and $\mathcal{H}_2$. Then the vertex $x$ is in $\mathcal{H}_1$ or in $H_2$. If the vertex $x$ is in $\mathcal{H}_1$ then for every vertex $v \in \mathcal{H}_2$, we have no vertex in $D$, which is adjacent to $v$ a contradiction to our hypothesis that $D$ is a dominating set. Hence any singleton subset of $\mathcal{H}$ can not be a dominating set of $\mathcal{H}$. □

**Corollary 4.8.** *If $\mathcal{H}(V, E)$ is a edge product hypergraph then $\gamma(\mathcal{H}) \geq 2$.*

*Proof.* let $\mathcal{H}(V, E)$ be an edge product hypergraph. Therefore by Theorem 4.7 $\gamma(\mathcal{H}) \geq 2$. □

**Lemma 4.9.** *Let $\mathcal{H}(V, E)$ be an edge product hypergraph. Then $\gamma(\bar{\mathcal{H}}) = \gamma^{-1}(\bar{H}) = 1$.*

*Proof.* Let $\mathcal{H}$ be an edge product hypergraph. Let $e_k = \{u, v\}$ be a $K_2$ component in $\mathcal{H}$. Let $w \in X|_{\{u,v\}}$ then there exists $e \in E$ such that $w \notin e$. Since $e_k$ is a $K_2$ component of $\mathcal{H}$, $\bar{e} = X|_e$ contains vertices $w, u, v$. Hence $u$ and $v$ are dominated by $w$. Now for any vertex $x \in X|_{\{u,v\}}$, we have $\bar{e}_k = X|_{e_k}$ contains both $x$ and $w$. Hence $w$ dominates $x$. Therefore $\{w\}$ is a dominating set of $\bar{\mathcal{H}}$. Similarly $\{x\}$ forms an inverse domination of $\bar{\mathcal{H}}$ and $x \in X|_{\{u,v\}}$. Thus $\gamma(\bar{\mathcal{H}}) = \gamma^{-1}(\bar{\mathcal{H}}) = 1$. □

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