Journal of Hyperstructures 10 *(1) (2020), 13-21.*
ISSN: 2322-1666 print/2251-8436 online

I-VAGUE IDEALS IN NEAR-RINGS

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ABSTRACT. In this paper the authers study the concepts of I-Vague sub near-rings and I-Vague ideals in near-rings. Some properties are illustrated corresponding to I-Vague sub near-rings and I-Vague ideals in near-rings with an example.

Key Words: *I*- Vague sub near-ring, *I*- Vague ideals.
2010 Mathematics Subject Classification: 03E72.

1. Introduction

In Mathematics, Fuzzy sets, subsets and some properties are introduced by L. A. Zadeh [2], W. Liu [13] have extended the concepts of ideals to Fuzzy sets and some authors have extended that work further. Concepts of fuzzy ideals are used by different authors [4,6,7,8] for further studies in vague sets. W. L. Gau and D. J. Buehrer [12] introduced two membership functions in vague sets, one is truth membership and another is false membership function. R. Biswas [5] introduced Vague groups & T. Eswarlal [9] extended those concepts to Vague field and Vague vector space. Then Seung Dong Kim & Hee Sik Kim [7] studied fuzzy sub near-ring and fuzzy ideals of near-ring. T. Zelalem [10] introduced I-vague sets and I-Vague relations, and again he [11] studied further on I-Vague groups. Then L. Bhasker [3] extended that part of fuzzy ideals in near-ring to Vague ideal of a near-ring.

Received: 27 November 2020, Accepted: 09 May 2021. Communicated by Kamal Bahmanpour;
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2. Preliminaries

Definition 2.1. [1] A non-empty set R with two binary operations "+" and "." satisfying the following axioms:
(1) $(R, +)$ is a group,
(2) $(R, .)$ is a semigroup,
(3) $x.(y + z) = x.y + x.z$ for all $x, y, z \in R$.
Precisely speaking, it is a left near-ring because it satisfies the left distributive law. We will use "near-ring", instead of "left near-ring". We denote xy instead of $x.y$. Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for some $x, y \in R$. Let R and S be near-rings. A map $f : R \to S$ is called a (near-ring) homomorphism if $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$ for any $x, y \in R$. An ideal I of a near-ring R is a subset of R such that
(4)$(I, +)$ is a normal subgroup of $(R, +)$,
(5) $RI \subseteq I$,
(6) $(r + i)s - rs \in I$ for any $r, s \in R$
Note that I is a left ideal of R if I satisfies (4) and (5), and I is a right ideal of R if I satisfies (4) and (6).
Throughout this paper let $I = (I, +, -, \vee, \wedge, 0, 1)$ be a dually residuated lattice ordered semigroup satisfying $1 - (1 - a) = a$ for all $a \in I$.

Definition 2.2. [9] An I-vague set A on a non-empty set X is a pair (t_A, f_A) where $t_A : X \to I$ and $f_A : X \to I$ with $t_A(x) \leq 1 - f_A(x)$ for all $x \in X$.

Definition 2.3. [9] The interval $[t_A(x), 1 - f_A(x)]$ is called the I-vague value of $x \in X$ and is denoted by $V_A(x)$.

Definition 2.4. [9] Let $A = (t_A, f_A)$ be an I-vague set of a non-empty set X. For $\alpha, \beta \in I$ and $\alpha \leq \beta$ the (α, β) cut of the I-vague set A denoted by $A_{(\alpha,\beta)}$ is a crisp subset of the set X is given by

$$A_{(\alpha,\beta)} = \{x \in X : V_A(x) \geq [\alpha, \beta]\}.$$

Definition 2.5. [9] Let G be a group. An I-vague set of a group G is called an I-vague group G if,
(i)$V_A(xy) \geq iinf\{V_A(x), V_A(y)\}$ for all $x, y \in G$, and
(ii)$V_A(x^{-1}) \geq V_A(x)$ for all $x \in G$.

Definition 2.6. [2] Let A be a vegaue set of a near-ring R. Then A is called vague sub near-ring of R if for all $x, y \in R$, it satisfies
(i) $V_A(x + y) \geq min\{V_A(x), V_A(y)\}$

(ii) $V_A(-x) = V_A(x)$
(iii) $V_A(xy) \geq min\{V_A(x), V_A(y)\}$.

Definition 2.7. [2] Let A be a vague set of near-ring R, then A is said to be a Vague ideal of R if for all $x, y, z, i \in R$, it satisfies
(i) $V_A(x + y) \geq min\{V_A(x), V_A(y)\}$
(ii) $V_A(-x) = V_A(x)$
(iii) $V_A(z + x - z) \geq V_A(x)$
(iv) $V_A(xy) \geq V_A(x)$
(v) $V_A(x(x(y + i) - xy)) \geq V_A(i)$ or $V_A(xz - xy) \geq V_A(z - y)$.
A is said to be a vague right ideal if it satisfies (i), (ii), (iii) and (iv). Similarly, A is said to be a vague left ideal if it satisfies (i), (ii), (iii) and (v).

3. I-Vague ideals of near-rings

Definition 3.1 Let A be an I-vague set of near-ring R. Then A is called I-vague sub near-ring of R if it satisfies the following conditions for all $x, y \in R$,
(i) $V_A(x + y) \geq iinf\{V_A(x), V_A(y)\}$
(ii) $V_A(-x) = V_A(x)$
(iii) $V_A(xy) \geq iinf\{V_A(x), V_A(y)\}$.

Definition 3.2 Let A be an I-vague set of a near-ring R, then A is said to be an I-vague ideal of R if and only if for all $x, y, z, i \in R$, it satisfies
(i) $V_A(x + y) \geq iinf\{V_A(x), V_A(y)\}$
(ii) $V_A(-x) = V_A(x)$
(iii) $V_A(z + x - z) \geq V_A(x)$
(iv) $V_A(xy) \geq V_A(x)$
(v) $V_A[x(x(y + i) - xy)] \geq V_A(i)$ or $V_A(xz - xy) \geq V_A(z - y)$.
A is said to be I-vague right idealof R if it satisfies (i), (ii), (iii) and (iv). Similarly, A is said to be an I-vague left ideal of R if it satisfies (i), (ii), (iii) and (v).
Note : (i) In above definition the conditions (i) and (ii) together can be written as $V_A(x - y) \geq iinf\{V_A(x), V_A(y)\}$.
(ii)If A is a vague ideal of R, then $V_A(x+y) = V_A(y+x)$ for all $x, y \in R$.
(iii) If A is an I-vague ideal of R then $V_A(0) \geq V_A(x)$ for all $x \in R$.

Example 3.3 Let $R_1 = Z_4 = \{0, 1, 2, 3\}$ be a near-ring under the addition and the multiplication of residue classes modulo-4. An I-vague set $A = (t_A, f_A)$ of R_1 defined as $t_A : R_1 \to I$ and $f_A : R_1 \to I$ such that

$t_A(x) = \begin{cases} 0.5 & x = 0,1 \\ 0.5 & x = 2,3 \end{cases}$ and $f_A(x) = \begin{cases} 0.5 & x = 0,1 \\ 0.5 & x = 2,3 \end{cases}$
clearly A is an I-vague ideal of R_1.

Remark 3.4 Let A be an I-vague ideal of R, then the condition $V_A(xz-xy) \geq V_A(z-y)$ is equivalent to the condition $V_A((x(y+i)-xy) \geq V_A(i)$.
Proof. Let

$$V_A(xz - xy) \geq V_A(z - y)$$

Put $z = y + i$

$$\begin{aligned} V_A((x(y + i) - xy) &\geq V_A(y + i - y)). \\ &= V_A(i). \end{aligned}$$

Conversely, let

$$V_A((x(y + i) - xy) \geq V_A(i).$$

$$\begin{aligned} xz - xy &= x(y - y + z) - xy \\ &= x(y + i) - xy. \qquad (\because i = -y + z.) \end{aligned}$$

$$\begin{aligned} V_A(xz - xy) &= V_A(x(y + i) - xy) \\ &\geq V_A(i) \\ &= V_A(-y + z) \\ &= V_A(z - y). \end{aligned}$$

Hence the proof is done.

Lemma 3.5 Let R be a near-ring and A be an I-vague set of R satisfies the condition $V_A(x - y) \geq iinf\{V_A(x), V_A(y)\}$ Then the followings are hold (i) $V_A(0) \geq V_A(x)$ (ii) $V_A(-x) \geq V_A(x)$.
Proof. (i)

$$\begin{aligned} V_A(0) &= V_A(x - x) \\ &\geq iinf\{V_A(x), V_A(-x)\} \\ &= iinf\{V_A(x), V_A(x)\} \\ &= V_A(x). \end{aligned}$$

(ii)

$$\begin{aligned} V_A(-x) &= V_A(0-x) \\ &\geq iinf\{V_A(0), V_A(-x)\} \\ &\geq iinf\{V_A(x), V_A(-x)\} \\ &= iinf\{V_A(x), V_A(x)\} \\ &= V_A(x). \end{aligned}$$

Hence the proof is done.

Lemma 3.6 Let A be an I-vague ideal of near-ring R. If $V_A(x-y) = V_A(0)$ then $V_A(x) = V_A(y)$.
Proof. Let A be an I-vague ideal of near-ring R.
Suppose that $V_A(x-x) = V_A(0)$ for all $x, y \in R$.
Now,

$$\begin{aligned} V_A(x) &= V_A(x-y+y) \\ &\geq iinf\{V_A(x-y), V_A(y)\} \\ &= iinf\{V_A(0), V_A(x)\} \qquad (\because V_A(x-y) = V_A(0)) \\ &\geq iinf\{V_A(y), V_A(y)\} \qquad (\because V_A(0) \geq V_A(x), \forall x \in R) \\ &= V_A(y). \end{aligned}$$

Conversely,

$$\begin{aligned} V_A(y) &= V_A(y-x+x) \\ &\geq iinf\{V_A(y-x), V_A(y)\} \\ &= iinf\{V_A(x-y), V_A(y)\} \qquad (\because V_A(y-x) = V_A(x-y)) \\ &= iinf\{V_A(0), V_A(x)\} \qquad (\because V_A(x-y) = V_A(0)) \\ &\geq iinf\{V_A(x), V_A(x)\} \qquad (\because V_A(0) \geq V_A(x), \forall x \in R) \\ &= V_A(x). \end{aligned}$$

so we get, $V_A(x) = V_A(y)$

Definition 3.7 Let A be an I-vague ideal of near-ring R and g be a function defined on R. Then the I-vague set B in $g(R)$ defined by,

$$V_B(y) = \sup_{x \in g^{-1}(y)} V_A(x) \qquad \forall y \in g(R)$$

is called the image of A under g.
Similarly, if B is an I- vague set in $g(R)$ then the I-vague set $A = B \circ g$

in R (i.e. the I-vague set defined as $V_A(x) = V_B[g(x)] \quad \forall x \in R$).

Theorem 3.8 A near-ring homomorphic pre-image of an I-vague left(right) ideal is an I-vague left (right) ideal.
Proof. Let $\psi : R \to S$ be a near-ring homomorphism and B be an I-vague left ideal of S where A be the pre-image of B under ψ in R.
Let us show that A is an I-vague ideal in R.
Now, $\forall x, y \in R$

$$\begin{aligned} V_A(x-y) &= V_B[\psi(x-y)] \\ &= V_B[\psi(x) - \psi(y)] \\ &\geq iinf\{V_B(\psi(x), V_B(\psi(y))\} \\ &= iinf\{V_A(x), V_B(y)\}. \end{aligned}$$

$$\begin{aligned} V_B(xy) &= V_B[\psi(xy)] \\ &= V_B[\psi(x)\psi(y)] \\ &\geq V_B[\psi(y)] \\ &= V_A(y). \end{aligned}$$

$$\begin{aligned} V_A(y+x-y) &= V_B[\psi(y+x-y)] \\ &= V_B\{[psi(y) + \psi(x) - \psi(y)]\} \\ &\geq V_B[\psi(x)] \\ &= V_A(x). \end{aligned}$$

It shows A is an I-vague left ideal of R.
Suppose B is an I-vague right ideal of S then $\forall x, y, i \in R$,

$$\begin{aligned} V_A[(x+i)y - xy] &= V_B[\psi((x+i)y - xy)] \\ &= V_B[(\psi(x) + \psi(i))\psi(y) - \psi(x)\psi(y)] \\ &= V_B[\psi(i)\psi(y)] \\ &\geq V_B[\psi(i)] \\ &= V_A(i). \end{aligned}$$

Hence A is an I-vague right ideal of R.

Definition 3.9 We say that an I- vague set A in near-ring R has the sup property if, for any subset T of R there exist $t_0 \in T$ such that

$$V_A(t_0) = \sup_{t \in T} V_A(t).$$

Theorem 3.10 A near-ring homomorphic image of an I-vague left (right) ideal having sup property is an I-vague left (right) ideal.
Proof. Let $\psi : R \to S$ be a near-ring homorphism, and A be an I-vague left ideal of R with the sup property. Let B be the image of A in S under ψ.
For $x, y \in R$ we get $\psi(x), \psi(y) \in \psi(R)$.
Let $x_0 \in \psi^{-1}(\psi(x)), y_0 \in \psi^{-1}(\psi(y))$ such that

$$V_A(x_0) = \sup_{t \in \psi^{-1}(\psi(x))} V_A(t) \qquad V_A(y_0) = \sup_{t \in \psi^{-1}(\psi y x))} V_A(t).$$

respectively. Then

$$\begin{aligned} V_B[\psi(x) - \psi(y)] &= \sup_{t \in \psi^{-1}(\psi(x) - \psi(y))} V_A(t) \\ &\geq V_A(x_0 - y_0) \\ &\geq iinf\{V_A(x_0), V_A(y_0)\} \\ &= iinf\{\sup_{t \in \psi^{-1}(\psi(x))} V_A(t), \sup_{t \in \psi^{-1}(\psi(y))} V_A(t)\} \\ &= iinf\{V_B(\psi(x)), V_B(\psi(y))\}. \end{aligned}$$

and,

$$\begin{aligned} V_B(\psi(x)\psi(y)) &= \sup_{t \in \psi^{-1}(\psi(x)\psi(y))} V_A(t) \\ &\geq V_A(x_0 y_0) \\ &\geq iinf\{V(y_0)\} \\ &= iinf\{\sup_{t \in \psi^{-1}(\psi(y))} V_A(t)\} \\ &= iinf\{V_B(\psi(y))\}. \end{aligned}$$

and,

$$\begin{aligned}
V_B(\psi(y+x-y)) &= \sup_{t\in\psi^{-1}(\psi(y+x-y))} V_A(t) \\
&= \sup_{t\in\psi^{-1}(\psi(y)+\psi(x)-\psi(y))} V_A(t) \\
&\geq V_A(y_0+x_0-y_0) \\
&= V_A(x_0) \\
&= \sup_{t\in\psi^{-1}(\psi(x))} V_A(t) \\
&= V_B(\psi(x)).
\end{aligned}$$

It implies that B is an I-vague left ideal of $\psi(R)$.
Now, let A be an I-vague right ideal of R.
Let $\psi(i) \in \psi(R)$ and $i_0 \in \psi^{-1}(\psi(i))$ such that

$$V_A(i_0) = \sup_{t\in\psi^{-1}(\psi(i))} V_A(t)$$

then,

$$\begin{aligned}
V_B(\psi((x+i)y-xy)) &= V_B((\psi(x)+\psi(i))\psi(y)-\psi(x)\psi(y)) \\
&= \sup_{t\in\psi^{-1}((\psi(x)+\psi(i))\psi(y)-\psi(x)\psi(y))} V_A(t) \\
&\geq V_A[(x_0+i_0)y_0-x_0y_0] \\
&\geq V_A(i_0) \\
&= \sup_{t\in\psi^{-1}(\psi(i))} V_A(t) \\
&= V_B[\ \psi(i)]
\end{aligned}$$

It implies B is an I-vague right ideal of $\psi(R)$.

4. Conclusions

In this paper, the concepts of an I-vague sub near-ring and I-vague ideals of near-ring are discussed. Also properties related to I-vague ideals of near-ring are proved. Here we have observed what happens with the homomorphic image and pre-image of I- vague ideals with the help of some previous concepts.

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