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ON THE MATRIX OF RANK ONE OVER A UFD

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Abstract. In this paper we characterize all matrices of rank one over a unique factorization domain (UFD). Also we find the R-module generated by the rows and the R-module generated by the columns of a matrix of rank one and assert some properties of them.

Key Words: unique factorization domain, rank of a matrix, irreducible element.
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1. Introduction

Let R denotes a commutative ring with identity and A be an $m \times n$ matrix over R. Each of the m rows of A can be regarded as an element of R^n and each of the n columns of A can be regarded as an element of R^m. The i-th row of A will be denoted by $Row_i(A)$ and the j-th column of A will be denoted by $Col_j(A)$. Thus if $A = (a_{ij})_{m\times n}$, then $Row_i(A) = (a_{i1}, ..., a_{in})$ and $Col_j(A) = (a_{1j}, ..., a_{mj})^t$. The R-submodule of R^n generated by $Row_1(A), ..., Row_m(A)$ is denoted by $< A >_r$ and the R-submodule of R^m generated by $Col_1(A), ..., Col_n(A)$ is denoted by $< A >_c$. The set of all $m \times n$ matrices with entries from R will be denoted by $\mathbb{M}_{m\times n}(R)$. For each $t = 1, ..., r = min\{m, n\}$, $I_t(A)$ will denote the ideal in R generated by all $t \times t$ minors of A. Thus we have the following ascending chain of ideals in R:

$$I_r(A) \subseteq I_{r-1}(A) \subseteq ... \subseteq I_2(A) \subseteq I_1(A) \subseteq R.$$

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It will be notationally convenient to extend the definition of $I_t(A)$ to all values of $t \in \mathbb{Z}$ as follows: $I_t(A) = 0$, if $t > min\{m, n\}$ and $I_t(A) = R$, if $t \leq 0$. Then we have $I_t(A) \subseteq I_{t-1}(A)$, for all $t \in \mathbb{Z}$.
The rank of A, denoted by $\text{rk}(A)$, is the following integer: $\text{rk}(A)$=$\max\{t \mid Ann_R(I_t(A)) = 0\}$ ([1]). Suppose F is a field and $A \in \mathbb{M}_{m\times n}(F)$. In most elementary textbooks in linear algebra, the classical rank of A, denoted by $\text{rank}_F(A)$ is defined to be the maximum number of linearly independent rows (or columns) of A. It is well known that $\text{rank}_F(A)$ is the largest integer t such that A contains a $t \times t$ submatrix whose determinant is nonzero. (See [2, Chapter 3, Theorem 3.22]). Since F is a field, $Ann_F(I_t(A)) = 0$ if and only if $I_t(A) \neq 0$. Thus $\text{rk}(A)$ is the largest integer t such that A contains a $t \times t$ submatrix whose determinant is nonzero. In other words, $\text{rk}(A)$ =$\text{rank}_F(A)$.
We can carry this discussion one step further. Suppose that R is an integral domain with quotient field F. Let $A \in \mathbb{M}_{m\times n}(R)$. Since $R \subseteq F$, $\mathbb{M}_{m\times n}(R) \subseteq \mathbb{M}_{m\times n}(F)$, and we can view A as a matrix in $\mathbb{M}_{m\times n}(F)$. Since R is an integral domain, $\text{Ann}_R(I_t(A)) = 0$ if and only if $I_t(A) \neq 0$. Thus, $\text{rk}(A)$=$\max\{t \mid A \text{ has a nonzero } t \times t \text{ } minor\}$. Now this number $\max\{t \mid A \text{ has a nonzero } t \times t \text{ } minor\}$ is the same whether we view A as a matrix in $\mathbb{M}_{m\times n}(R)$ or $\mathbb{M}_{m\times n}(F)$. Hence, $\text{rk}(A)$ is just the classical rank of A when A is viewed as a matrix in $\mathbb{M}_{m\times n}(F)$. So $\text{rk}(A)$ =$\text{rank}_F(A)$, in this case.

2. Matrix of rank one

Let R be a commutative ring. Elements a, b of R are said to be associates if $a \mid b$ and $b \mid a$. A nonunit and nonzero element $p \in R$ is called an irreducible element, If $p = ab$ implies that either a or b is a unit element of R. Recall that an integral domain R is a unique factorization domain (UFD) provided every nonzero nonunit element of R can be written $a = p_1...p_n$, with $p_1, ..., p_n$ irreducible and if $a = q_1...q_m$ (q_i irreducible) then $n = m$ and for some permutation σ of $\{1, ..., n\}$, p_i and q_i are associates for every i. Note that in a unique factorization domain (UFD), a greatest common divisor (GCD) of any collection of elements always exists. Also, for every a, b, c in a UFD, if $a \mid bc$ and a, b are relatively prime (i.e. $\text{GCD}(a, b) = 1$), then $a \mid c$.
In the next Theorem we characterize all $m \times n$ matrices of rank one over a unique factorization domain.

Theorem 2.1. *Let R be a UFD and $0 \neq A = (a_{ij}) \in \mathbb{M}_{m\times n}(R)$ be a matrix of rank one. Let $x_j = GCD(a_{1j}, ..., a_{mj})$, $1 \leq j \leq n$. If l-th column of A is nonzero, then*

$$A = \begin{pmatrix} \frac{a_{1l}}{x_l}x_1 & ... & a_{1l} & ... & \frac{a_{1l}}{x_l}x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{a_{ml}}{x_l}x_1 & ... & a_{ml} & ... & \frac{a_{ml}}{x_l}x_n \end{pmatrix}.$$

Proof. Let $(a_{ij}) \in \mathbb{M}_{m\times n}(R)$ be a matrix of rank one and l-th column of A be nonzero. We consider two cases.
Case 1: Suppose that $GCD(a_{1l}, \ldots, a_{ml}) = 1$. Assume that $a_{i_1 l}, ..., a_{i_t l} \neq 0$, where $1 \leq i_1 < ... < i_t \leq m$ and $a_{il} = 0$, for all $i \neq i_k$, $1 \leq k \leq t$. Put $d_k = GCD(a_{i_k l}, a_{(i_k+1)l})$, $1 \leq k \leq t$ and, for the moment, fix j, $1 \leq j \neq l \leq n$. Since $\text{rk}(A) = 1$, hence for $i = 1, \ldots, t$ we have $a_{i_k l}a_{(i_k+1)j} = a_{i_k j}a_{(i_k+1)l}$. Thus $\frac{a_{i_k l}}{d_k}a_{(i_k+1)j} = a_{i_k j}\frac{a_{(i_k+1)l}}{d_k}$ and so $\frac{a_{i_k l}}{d_k} \mid a_{i_k j}$ which implies that there exists $r_{kj} \in R$ such that $a_{i_k j} = \frac{a_{i_k l}}{d_k}r_{kj}$ and so $a_{(i_k+1)j} = \frac{a_{(i_k+1)l}}{d_k}r_{kj}, 1 \leq k \leq t$. Therefore $a_{i_k j} = \frac{a_{i_k l}}{d_k}r_{kj} = \frac{a_{i_k l}}{d_{k-1}}r_{(k-1)j}$, $2 \leq k \leq t$ and $a_{i_1 j} = \frac{a_{1l}}{d_1}r_{1j}$. Hence, $r_{kj}d_{k-1} = r_{(k-1)j}d_k$. Now, we show by induction that $\frac{d_k}{GCD(d_1,\ldots,d_k)} \mid r_{kj}$. For $k = 2$, since $r_{2j}d_1 = r_{1j}d_2$, hence $\frac{d_2}{GCD(d_1,d_2)} \mid r_{2j}$. Assume that $d_k s_k = r_{kj}GCD(d_1, \ldots, d_k)$, for some $s_k \in R$. We have $r_{(k+1)j}d_k s_k = r_{kj}d_{k+1}s_k$. Thus

$$r_{(k+1)j}GCD(d_1, \ldots, d_k) = d_{k+1}s_k. \tag{2.1}$$

On the other hand, from $r_{(k+1)j}d_k = r_{kj}d_{k+1}$ we obtain $\frac{d_{k+1}}{GCD(d_k,d_{k+1})} \mid r_{(k+1)j}$ and so there exists $s'_k \in R$ such that

$$r_{(k+1)j}GCD(d_k, d_{k+1}) = d_{k+1}s'_k. \tag{2.2}$$

Combining (2.1) and (2.2) we have

$$\begin{aligned} d_{k+1}s_k GCD(d_k, d_{k+1}) &= r_{(k+1)j}GCD(d_1, \ldots, d_k)GCD(d_k, d_{k+1}) \\ &= d_{k+1}s'_k GCD(d_1, \ldots, d_k). \end{aligned}$$

Thus $s_k GCD(d_k, d_{k+1}) = s'_k GCD(d_1, \ldots, d_k)$ and so $\frac{GCD(d_k,d_{k+1})}{GCD(d_1,\ldots,d_{k+1})} \mid s'_k$. Now, by (2.2), we have $\frac{d_{k+1}GCD(d_k,d_{k+1})}{GCD(d_1,\ldots,d_{k+1})} \mid r_{(k+1)j}GCD(d_k, d_{k+1})$ and hence
$\frac{d_{k+1}}{GCD(d_1,\ldots,d_{k+1})} \mid r_{(k+1)j}$ which completes the induction. Therefore $d_t =$

$\frac{d_t}{GCD(d_1,\ldots,d_t)} \mid r_{tj}$. Since $r_{tj}d_{t-1} = r_{(t-1)j}d_t$, hence $d_{t-1} \mid r_{(t-1)j}$. Continuing this process, we have $d_k \mid r_{kj}$, $1 \le k \le t$. As a consequence, $a_{i_kj} = \frac{a_{i_kl}}{d_k} r_{kj} = a_{i_kl}\frac{r_{kj}}{d_k}$, $1 \le k \le t$. Also for $i \ne i_k, 1 \le k \le t$, we have $a_{i1} = 0$. Since $\text{rk}(A) = 1$, hence $a_{i2}a_{11} = a_{12}a_{i1} = 0$. So $a_{i2} = 0$. Thus for all $1 \le i \le m$, we have $a_{ij} = \dfrac{a_{il}}{x_l} x_j$.

Case 2: Suppose that $GCD(a_{1l},\ldots,a_{ml}) = x_l$ is not a unit element of R. For the moment, fix j, $1 \le j \ne l \le n$. By the same argument and notation as in case 1, we have $\frac{d_t}{GCD(d_1,\ldots,d_t)} \mid r_{tj}$. Thus $\frac{d_t}{x_l} \mid r_{tj}$ and therefore there exists $r'_{tj} \in R$ such that $r_{tj} = \frac{d_t}{x_l} r'_{tj}$. Thus $a_{tj} = \frac{a_{tl}}{x_l} r'_{tj}$. On the other hand, $r_{(t-1)j}d_t = r_{tj}d_{t-1}$. Hence $r_{(t-1)j}d_t = \frac{d_t}{x_l} r'_{tj} d_{t-1}$ and so $r_{(t-1)j} = r'_{tj}\frac{d_{t-1}}{x_l}$. Therefore $a_{(t-1)j} = \frac{a_{(t-1)1}}{d_{(t-1)j}} r_{(t-1)j} = \frac{a_{(t-1)l}}{x_l} r'_{tj}$. Continuing this process we obtain $r_{kj} = r'_{tj}\frac{d_k}{x_l}$ and so $a_{i_kj} = \frac{a_{i_kl}}{x_l} r'_{tj}$, $1 \le k \le t, 1 \le j \ne l \le n$. Thus in fact, $r'_{tj} = GCD(a_{1j},\ldots,a_{mj}) = x_j$. Hence

$$A = \begin{pmatrix} \dfrac{a_{1l}}{x_l}x_1 & \ldots & a_{1l} & \ldots & \dfrac{a_{1l}}{x_l}x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{a_{ml}}{x_l}x_1 & \ldots & a_{ml} & \ldots & \dfrac{a_{ml}}{x_l}x_n \end{pmatrix}.$$

□

Corollary 2.2. *Let R be UFD and $A = (a_{ij}) \in \mathbb{M}_{m\times n}(R)$ be a matrix of rank one with nonzero column l. Let $x_j = GCD(a_{1j},\ldots,a_{mj})$, $1 \le j \le n$, I be the ideal of R generated by $x_1,\ldots,x_n$ and J be the ideal of R generated by $\dfrac{a_{1l}}{x_l},\ldots,\dfrac{a_{ml}}{x_l}$. Then*

(1) $< A >_c = I < (\dfrac{a_{1l}}{x_l},\ldots,\dfrac{a_{ml}}{x_l})^t >$;

(2) $< A >_r = J < (x_1,\ldots,x_n) >$.

Proof. By Theorem 2.1, it is obvious. □

Let $\mu(M)$ denotes the minimal number of generators of M. It is well known that if A is a matrix over a field F, then $\text{rk}(A) = s$ if and only if the dimension of column space of A (equal to the dimension of row space of A) is s. Let R be a principal ideal domain (PID) and $A \in \mathbb{M}_{m\times n}(R)$. Then $< A >_c$ is a submodule of R^m. Since R is a PID and R^m is a free R-module, then $< A >_c$ is a free R- module. In fact $< A >_c$ is free of rank s if and only if $\mathrm{rk}(A) = s$. (See Proposition 2.3, from [3, Proposition 7-2-11]).

Proposition 2.3. *If A is an $n \times m$ matrix of rank $r > 0$ over a principal ideal domain R, then A is equivalent to a matrix of the form $\begin{pmatrix} L_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, where L_r is an $r \times r$ diagonal matrix with nonzero diagonal entries $d_1, ..., d_r$ such that $d_1 \mid ... \mid d_r$. The ideals $(d_1), ..., (d_r)$ in R are uniquely determined by the equivalence class of A.*

Thus if R is either a field or a PID and A is a matrix over R, then $\mathrm{rk}(A) = s$ if and only if $\mu(< A >_r) = \mu(< A >_c) = s$.
Now, let R be an integral domain with quotient field F and $A = (a_{ij}) \in \mathbb{M}_{m\times n}(R)$ be a matrix with $\mu(< A >_r) = 1$ (or $\mu(< A >_c) = 1$). Since $R \subseteq F$, $\mathbb{M}_{m\times n}(R) \subseteq \mathbb{M}_{m\times n}(F)$, and we can view A as a matrix in $\mathbb{M}_{m\times n}(F)$. Thus the dimension of row space (or the dimension of row space) of A is 1. So $\mathrm{rank}_F(A) = \mathrm{rk}(A) = 1$. Hence we have the following Proposition.

Proposition 2.4. *Let R be an integral domain and $A \in \mathbb{M}_{m\times n}(R)$. Let $\mu(< A >_r) = 1$ or $\mu(< A >_c) = 1$. Then $rk(A) = 1$.*

One of the interesting question is " If A is a matrix of rank 1 over a UFD, whether $\mu(< A >_c)$ or $\mu(< A >_r)$ is 1?." Here we give some example which shows that it is not true in general (Example 2.6). Further we use the following Lemma.

Lemma 2.5. *Let (R, P) be a local integral domain and I be a finitely generated ideal of R. If $I < (y_1, ..., y_n) >$ is a nonzero cyclic R-module, then I is a principal ideal of R.*

Proof. Let $I =< a_1, ..., a_m >$ and $I < (y_1, ..., y_n) >=< (b_1, ..., b_n) >$, for some $b_i \in R$, $1 \le i \le n$. Then $a_i(y_1, ..., y_n) = s_i(b_1, ..., b_n)$, for some $s_i \in R$, $1 \le i \le n$. Hence for all $1 \le i \le m$ and $1 \le j \le n$ we have

$$a_i y_j = s_i b_j. \tag{2.3}$$

On the other hand, $(b_1, ..., b_n) = \sum_{i=1}^{m} r_i a_i(y_1, ..., y_n)$, for some $r_i \in R$. Thus $b_j = \sum_{i=1}^{m} r_i a_i y_j$, $1 \le j \le n$. So by (2.3), $b_j = \sum_{i=1}^{m} r_i a_i y_j = \sum_{i=1}^{m} r_i s_i b_j$. Since R is an integral domain and $(b_1, ..., b_n) \neq 0$, hence $\sum_{i=1}^{m} r_i s_i = 1$. Thus there exists some $1 \le k \le m$, such that $s_k \notin P$. So s_k is a unit element of R. Thus $(b_1, ..., b_n) = s_k^{-1} a_k(y_1, ..., y_n)$. Let i, $1 \le i \le m$ be arbitrary and fixed. By (2.3), we have $a_i y_j = s_i b_j =$

$s_is_k^{-1}a_ky_j$. Since $(y_1, ..., y_n) \neq 0$, then $a_i \in < a_k >$. Therefore I is a principal ideal. □

Example 2.6. Let (R, P) be a local UFD and $p, q \in R$ be two irreducible elements of R which are not associates and $a \in R$. Let $A = \begin{pmatrix} p & pa \\ q & qa \end{pmatrix}$. Thus $< A >_c = < (p, q)^t >$, whence $< A >_r = < p, q >< (1, a) >$ is not cyclic. Because if $< A >_r$ be a cyclic R-module, then by Lemma 2.5, $< p, q >$ is a principal ideal. Let $< p, q > = < x >$, for some element $x \in R$. Thus there exist some $r, s \in R$ such that $p = rx$ and $q = sx$. Since p, q are two irreducible elements of R, hence r, s are unit elements or x is unit. If x is a unit element, then $< p, q > = R$, a contradiction, because $p, q \in P$. Therefore r, s are unit elements of R. So $q = sx = sr^{-1}p$ and $p = rx = rs^{-1}q$. This means that p, q are associates, a contradiction. Thus $< A >_r = < p, q >< (1, a) >$ is not cyclic. Similarly we have $A = \begin{pmatrix} p & q \\ pa & qa \end{pmatrix}$ is a matrix of rank one such that $< A >_r = < (p, q) >$ is a cyclic R-module but $< A >_c = < (p, q)^t >$ is not a cyclic module.

Now, we show that if $< A >_c$ ($< A >_r$) is a cyclic module, then $< A >_r$ ($< A >_c$) is always in the form of above.

Proposition 2.7. *Let (R, P) be a local UFD and $A = (a_{ij}) \in \mathbb{M}_{m\times n}(R)$ be a matrix of rank one with nonzero column l. Then*

(1) *If $< A >_c$ is a nonzero cyclic R-module, then $< A >_r = < a_{1k}, ..., a_{mk} >< (r_1, ..., 1, ..., r_n) >$, for some $1 \leq k \leq n$ and $r_i \in R$ (1 is in k-th place).*
(2) *If $< A >_r$ is a nonzero cyclic R-module, then $< A >_c = < x_1, ..., x_n >< (s_1, ..., 1, ..., s_m)^t >$, for some $s_i \in R$ (1 is in l-th place).*

Proof. Let $< A >_c$ be a cyclic R-module. By Corollary 2.2 and Lemma 2.5, $< x_1, ..., x_n >$ is a principal ideal. Since R is a local ring, hence there exists some nonzero element x_k, $1 \leq k \leq n$ such that $< x_1, ..., x_n > = < x_k >$. Thus $x_i = r_ix_k$, $1 \leq i \leq n$. Since $< A >_c$ is nonzero and $< x_1, ..., x_n > = < x_k >$, hence k-th column of A is nonzero, so by Corollary 2.2, we have $< A >_r = < \frac{a_{1k}}{x_k}, ..., \frac{a_{mk}}{x_k} >< (x_1, ..., x_n) > = x_k < \frac{a_{1k}}{x_k}, ..., \frac{a_{mk}}{x_k} >< (x_1, ..., x_n) > = < a_{1k}, ..., a_{mk} >< (r_1, ..., 1, ..., r_n) >$.
Now, let $< A >_r$ be a cyclic R-module. So by Corollary 2.2 and

Lemma 2.5, $<\frac{a_{1l}}{x_l},...,\frac{a_{ml}}{x_l}>$ is a principal ideal. Since R is a local ring, hence there exists some nonzero element a_{kl}, $1 \leq k \leq m$ such that $<\frac{a_{1l}}{x_l},...,\frac{a_{ml}}{x_l}>=<\frac{a_{kl}}{x_l}>$. Thus $\frac{a_{il}}{x_l} = s_i\frac{a_{kl}}{x_l}$, $1 \leq i \leq m$. Therefore $a_{il} = s_i a_{kl}$. So $x_l = GCD(a_{1l},...,a_{ml}) = a_{kl}$. Hence by Corollary 2.2, $<A>_c=<x_1,...,x_n><(\frac{a_{1l}}{x_l},...,\frac{a_{ml}}{x_l})^t>=<x_1,...,x_n><(s_1,...,1,...,s_m)^t>$.

□

Proposition 2.8. *Let R be a UFD and $A \in \mathbb{M}_{n\times n}(R)$ be a matrix of rank one. Then $A^k = (\operatorname{tr} A)^{k-1}A$, for every $k \in \mathbb{N}$.*

Proof. Let A be a nonzero matrix of rank one, then by Theorem 2.1, there exists some $1 \leq l \leq n$ such that

$$A = \begin{pmatrix} \frac{a_{1l}}{x_l}x_1 & \dots & a_{1l} & \dots & \frac{a_{1l}}{x_l}x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{a_{nl}}{x_l}x_1 & \dots & a_{nl} & \dots & \frac{a_{nl}}{x_l}x_n \end{pmatrix}.$$

We have

$$A^2 = \begin{pmatrix} \sum_{i=1}^n \frac{a_{1l}}{x_l}x_i\frac{a_{il}}{x_l}x_1 & \dots & \sum_{i=1}^n \frac{a_{1l}}{x_l}x_i a_{il} & \dots & \sum_{i=1}^n \frac{a_{1l}}{x_l}x_i\frac{a_{il}}{x_l}x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^n \frac{a_{nl}}{x_l}x_i\frac{a_{il}}{x_l}x_1 & \dots & \sum_{i=1}^n \frac{a_{nl}}{x_l}x_i a_{il} & \dots & \sum_{i=1}^n \frac{a_{nl}}{x_l}x_i\frac{a_{il}}{x_l}x_n \end{pmatrix} =$$

$$\begin{pmatrix} \sum_{i=1}^n \frac{a_{il}}{x_l}x_i\frac{a_{1l}}{x_l}x_1 & \dots & \sum_{i=1}^n \frac{a_{il}}{x_l}x_i a_{1l} & \dots & \sum_{i=1}^n \frac{a_{il}}{x_l}x_i\frac{a_{nl}}{x_l}x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^n \frac{a_{il}}{x_l}x_i\frac{a_{nl}}{x_l}x_1 & \dots & \sum_{i=1}^n \frac{a_{il}}{x_l}x_i a_{nl} & \dots & \sum_{i=1}^n \frac{a_{il}}{x_l}x_i\frac{a_{nl}}{x_l}x_n \end{pmatrix} = (\operatorname{tr} A)A.$$

Hence $A^k = (\operatorname{tr} A)^{k-1}A$, for every $k \in \mathbb{N}$. □

Corollary 2.9. *Let R be a UFD and $A \in \mathbb{M}_{n\times n}(R)$ be a matrix of rank one. Then $\operatorname{tr}(A^k) = (\operatorname{tr} A)^k$.*

Proof. By Proposition 2.8, $A^k = (\operatorname{tr} A)^{k-1}A$. Thus $\operatorname{tr}(A^k) = (\operatorname{tr} A)^k = (\operatorname{tr} A)^{k-1}\operatorname{tr} A = (\operatorname{tr} A)^k$.

Corollary 2.10. *Let R be a UFD and $0 \neq A \in \mathbb{M}_{n\times n}(R)$ be a matrix of rank one. Then*

(1) *A is nilpotent if and only if $\operatorname{tr} A = 0$.*

(2) *A is idempotent if and only if* $\operatorname{tr} A = 1$.

Proof. By Proposition 2.8, $A^k = (\operatorname{tr} A)^{k-1} A$, for every $k \in \mathbb{N}$. Thus, since R is an integral domain, hence $A^k = (\operatorname{tr} A)^{k-1} A = 0$ if and only if $\operatorname{tr} A = 0$ and $A^2 = A$ if and only if $\operatorname{tr} A = 1$. □

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