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DIFFERENTIAL TRANSFORMATION METHOD FOR SOLVING HYBRID FUZZY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, Differential Transformation Method (DTM) was studied for solving Hybrid Fuzzy Differential Equations (HFDEs). The proposed method was also illustrated by some examples and the error comparison was made using Runge-Kutta method of order 4 (RK4).

Key Words: Hybrid systems, Fuzzy differential equations, Seikkala derivative; Differential Transformation method.
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1. Introduction

Hybrid systems are developed to model, design and validate interactive systems of computer programs and continuous systems; i.e. control systems that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are called hybrid fuzzy differential systems. HFDEs are thought of an important research branch of fuzzy differential equations. Stability properties and analytical results of HFDEs can be found in [1]-[3].

The concept of fuzzy derivative was first introduced by S. L. Chang, L. A. Zadeh in [4]. It was followed up by D. Dubois, H. Prade in [5],

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who defined and used the extension principle. Numerical methods of fuzzy differential equation have been studied by numerous authors such as [6]-[8]. Furthermore, there are some numerical techniques for solving hybrid fuzzy differential equations [9]-[12].

All of the above-mentioned methods have introduced discrete solutions; but DTM [13]-[18] is used for finding analytical approximate solutions of hybrid fuzzy differential equations. In this paper, DTM was applied for solving hybrid fuzzy differential equations, based on the Seikkala's derivative.

The rest of this paper is organized as follows. In Section 2, some basic definitions are listed for fuzzy valued functions and fuzzy differential equations. In Section 3, hybrid fuzzy differential systems are reviewed. In Section 4, the application of DTM is extended to construct approximate solutions for hybrid fuzzy differential equations. Numerical experiments are provided in Section 5 and conclusion is made in Section 6.

2. Preliminaries

Denote by E^1 the set of all functions $u : \mathbb{R} \to [0,1]$ such that (i) u is normal; that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$, (ii) u is fuzzy convex; i.e. for $x,\ y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$, (iii) u is upper semi continuous, and (iv) $[u]^0 \equiv$ the closure of $\{x \in \mathbb{R} : u(x) > 0\}$ is compact.
For $0 < \alpha \leq 1$, $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$ is defined. For later purposes, $\hat{0} \in E^1$ is defined as $\hat{0}(x) = 1$ if $x = 0$ and $\hat{0}(x) = 0$ if $x \neq 0$.
Then, the Seikkala derivative [19] of $x : I \to E^1$ is reviewed where $I \subset \mathbb{R}$ is an interval. If $[x(t)]^\alpha = [\underline{x}^\alpha(t), \overline{x}^\alpha(t)]$ for all $t \in I$ and $\alpha \in [0,1]$, then $[x'(t)]^\alpha = [(\underline{x}^\alpha)'(t), (\overline{x}^\alpha)'(t)]$ if $[x'(t)]^\alpha \in E^1$. Also, consider the initial value problem (IVP)

$$\begin{cases} x'(t) = f(t, x(t)), \\ \quad x(0) = x_0, \end{cases} \tag{2.1}$$

where $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous. (2.1) should be interpreted using the Seikkala derivative and $x_0 \in E^1$.
Let $[x_0]^\alpha = [\underline{x}_0{}^\alpha, \overline{x}_0{}^\alpha]$ and $[x(t)]^\alpha = [\underline{x}^\alpha(t), \overline{x}^\alpha(t)]$. Using the-Zadeh extension principle, $f : [0,\infty) \times E^1 \to E^1$, is obtained where
$[f(t,x)]^\alpha = [\min\{f(t,u) : u \in [\underline{x}^\alpha(t), \overline{x}^\alpha(t)]\}, \max\{f(t,u) : u \in [\underline{x}^\alpha(t), \overline{x}^\alpha(t)]\}]$.
Then, $x : [0,\infty) \to E^1$ is a solution of (2.1) using the Seikkala derivative and

$x_0 \in E^1$ if

$$(\underline{x}^\alpha)'(t) = \min\{f(t,u) : u \in [\underline{x}^\alpha(t), \overline{x}^\alpha(t)]\},\ \underline{x}^\alpha(0) = \underline{x}_0^\alpha,$$

$$(\overline{x}^\alpha)'(t) = \max\{f(t,u) : u \in [\underline{x}^\alpha(t), \overline{x}^\alpha(t)]\},\ \overline{x}^\alpha(0) = \overline{x}_0^\alpha,$$

for all $t \in [0,\infty)$ and $\alpha \in [0,1]$. Finally, consider an $f : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is continuous and IVP

$$\left\{ \begin{array}{c} x'(t) = f(t,x(t),k), \\ x(0) = x_0. \end{array} \right. \tag{2.2}$$

As in [20] to interpret (2.2) using the Seikkala derivative and $x_0, k \in E^1$, by the Zadeh extension principle, $f : [0,\infty) \times E^1 \times E^1 \to E^1$ is used where

$$\begin{aligned} [f(t,x,k)]^\alpha = [&\min\{f(t,u,u_k) : u \in [\underline{x}^\alpha(t), \overline{x}^\alpha(t)], u_k \in [\underline{k}^\alpha, \overline{k}^\alpha]\}, \\ &\max\{f(t,u,u_k) : u \in [\underline{x}^\alpha(t), \overline{x}^\alpha(t)], u_k \in [\underline{k}^\alpha, \overline{k}^\alpha]\}]. \end{aligned}$$

where $k^\alpha = [\underline{k}^\alpha, \overline{k}^\alpha]$. Then $x : [0,\infty) \to E^1$ is a solution of (2.2) using the Seikkala derivative and $x_0, k \in E^1$ if

$$(\underline{x}^\alpha)'(t) = \min\{f(t,u,u_k) : u \in [\underline{x}^\alpha(t), \overline{x}^\alpha(t)], u_k \in [\underline{k}^\alpha, \overline{k}^\alpha]\},\ \underline{x}^\alpha(0) = \underline{x}_0^\alpha,$$

$$(\overline{x}^\alpha)'(t) = \max\{f(t,u,u_k) : u \in [\underline{x}^\alpha(t), \overline{x}^\alpha(t)], u_k \in [\underline{k}^\alpha, \overline{k}^\alpha]\},\ \overline{x}^\alpha(0) = \overline{x}_0^\alpha,$$

for all $t \in [0,\infty)$ and $\alpha \in [0,1]$. (see[20], p.45)

3. Hybrid fuzzy differential system

Consider the hybrid fuzzy differential system

$$\left\{ \begin{array}{c} x'(t) = f(t,x(t),\lambda_k(x_k)),\ t \in [t_k, t_{k+1}], \\ x(t_k) = x_k, \end{array} \right. \tag{3.1}$$

where $'$ denotes Seikkala differentiation and

$0 \le t_0 < t_1 < \cdots < t_k < \cdots, t_k \to \infty, f \in C[\mathbb{R}^+ \times E^1 \times E^1, E^1], \lambda_k \in [E^1, E^1]$.

To be specific the system would look like

$$x'(t) = \left\{ \begin{array}{c} x_0'(t) = f(t,x_0(t),\lambda_0(x_0)),\ x_0(t_0) = x_0,\ t_0 \le t \le t_1, \\ x_1'(t) = f(t,x_1(t),\lambda_1(x_1)),\ x_1(t_1) = x_1,\ t_1 \le t \le t_2, \\ \vdots \\ x_k'(t) = f(t,x_k(t),\lambda_k(x_k)),\ x_k(t_k) = x_k,\ t_k \le t \le t_{k+1}, \\ \vdots \end{array} \right.$$

Assuming that the existence and uniqueness of solution of (3.1) hold for each $[t_k, t_{k+1}]$, by the solution of (3.1) the following function is generated:

$$x(t) = x(t, t_0, x_0) = \begin{cases} x_0(t) \; t_0 \le t \le t_1, \\ x_1(t) \; t_1 \le t \le t_2, \\ \vdots \\ x_k(t) \; t_k \le t \le t_{k+1}, \\ \vdots \end{cases}$$

(3.1) may be replaced with an equivalent system

$$\begin{cases} \underline{x}'(t) = \underline{f}(t, x, \lambda_k(x_k)) = h_k(t, \underline{x}, \overline{x}), \; \underline{x}(t_k) = \underline{x}_k, \\ \overline{x}'(t) = \overline{f}(t, x, \lambda_k(x_k)) = g_k(t, \underline{x}, \overline{x}), \; \overline{x}(t_k) = \overline{x}_k, \end{cases}$$

which possesses a unique solution $(\underline{x}, \overline{x})$ which is a fuzzy function. That is for each t, the pair $[\underline{x}(t;r), \overline{x}(t;r)]$ is a fuzzy number, where $\underline{x}(t;r), \overline{x}(t;r)$ are respectively solutions of the parametric form given by

$$\begin{cases} \underline{x}'(t;r) = h_k[t, \underline{x}(t;r), \overline{x}(t;r)], \; \underline{x}(t_k;r) = \underline{x}_k(r), \\ \overline{x}'(t;r) = g_k[t, \underline{x}(t;r), \overline{x}(t;r)], \; \overline{x}(t_k;r) = \overline{x}_k(r), \end{cases} \tag{3.2}$$

for $r \in [0, 1]$.

4. Differential transformation method

In this section, for a hybrid fuzzy differential equation (3.1), differential transformation method is developed via the application of differential transformation method for fuzzy differential equation in [21] when f and λ_k in (3.1) can be obtained via the Zadeh extension principle from $f \in C[\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ and $\lambda_k \in C[\mathbb{R}, \mathbb{R}]$.

Basic definitions and fundamental operations of the differential transform are introduced in [13, 14]. Differential transform of the function $u(x)$ is in the following form

$$U(k) = \frac{1}{k!} \left[\frac{d^k u(x)}{dx^k} \right]_{(x_0)}, \tag{4.1}$$

where $u(x)$ is original function and $U(k)$ is transformed function. The inverse differential transform of $U(k)$ is defined as

$$u(x) = \sum_{k=0}^{\infty} U(k)(x - x_0)^k. \tag{4.2}$$

When (x_0) are taken as (0), the function $u(x)$, (4.2) is expressed as follows:

$$u(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k u(x)}{dx^k} \right]_{(0)} x^k. \tag{4.3}$$

TABLE 1. The original function and transformed function

$u(x)$	$U(k)$
$u(x) = f(x) \pm g(x)$	$U(k) = F(k) \pm G(k)$
$u(x) = \lambda f(x),\ \lambda \in \mathbb{R}$	$U(k) = \lambda F(k)$
$u(x) = x^m$	$U(k) = \delta(k-m) = \begin{cases} 1, & k = m \\ 0, & \text{otherwise} \end{cases}$
$u(x) = d^r f(x)/dx^r,\ r \in \mathbb{N}$	$U(k) = (k+1)\cdots(k+r)F(k+r)$
$u(x) = f(x)g(x)$	$U(k) = \sum_{r=0}^{k} F(r)G(k-r)$

Eq. (4.3) implies that the concept of differential transform is derived from Taylor series expansion.

In this paper, the lower case letters represent original function and upper case letters stand for the transformed function (T-function). The fundamental mathematical operations performed by differential transform method can be readily obtained, as listed in Table 1.

The differential transform of fuzzy function $x(t,r) = \big(\underline{x}(t,r), \overline{x}(t,r)\big)$ can be defined as follows:

$$\underline{X}(k,r) = \frac{1}{k!}\left[\frac{d^k \underline{x}(t,r)}{dt^k}\right]_{(t_0)}, \tag{4.4}$$

and

$$\overline{X}(k,r) = \frac{1}{k!}\left[\frac{d^k \overline{x}(t,r)}{dt^k}\right]_{(t_0)}, \tag{4.5}$$

The inverse differential transform of $\underline{X}(k,r)$ and $\overline{X}(k,r)$ is defined respectively as

$$\underline{x}(t,r) = \sum_{k=0}^{\infty} \underline{X}(k,r)(t-t_0)^k. \tag{4.6}$$

and

$$\overline{x}(t,r) = \sum_{k=0}^{\infty} \overline{X}(k,r)(t-t_0)^k. \tag{4.7}$$

When (t_0) are taken as (0), the functions $\underline{x}(t,r)$ and $\overline{x}(t,r)$ of (4.4) and (4.5), are expressed as follows:

$$\underline{X}(k,r) = \frac{1}{k!}\left[\frac{d^k \underline{x}(t,r)}{dt^k}\right]_{(0)}, \tag{4.8}$$

and

$$\overline{X}(k,r) = \frac{1}{k!}\left[\frac{d^k\overline{x}(t,r)}{dt^k}\right]_{(0)}, \tag{4.9}$$

Eqs. (4.8) and (4.9) imply that the concept of fuzzy differential transform is derived from fuzzy Taylor series expansion.

5. DTM with fixed grid size

The objective of this section is to find the solution of (3.2) at the equally spaced grid points $t_0, t_1, t_2, \cdots, t_N$, where $t_i = a + ih$, for each $i = 0, 1, \cdots, N$ and $h = \frac{(b-a)}{N}$.
The domain of interest $[a, b]$ is divided to N sub-domains and the approximation functions in each sub-domain are $x_i(t, r), i = 0, 1, 2, \cdots, N-1$, respectively. Taking differential transformation of (3.2), transformed equation describes the relationship between spectrum of $x(t, r)$ as

$$(k+1)\underline{X}(K+1,r) = H(t,\underline{X}(t,r),\overline{X}(t,r)), \tag{5.1}$$

$$(k+1)\overline{X}(K+1,r) = G(t,\underline{X}(t,r),\overline{X}(t,r)) \tag{5.2}$$

where $H(.)$ denotes transformed function of $h_k(t,\underline{x}(t,r),\overline{x}(t,r))$, and $G(.)$ denotes transformed function of $g_k(t,\underline{x}(t,r),\overline{x}(t,r))$. From the initial condition, the following can be obtained:

$$\underline{X}(0,r) = \underline{x}_0(r),\ \overline{X}(0,r) = \overline{x}_0(r).$$

In the first sub-domain, $\underline{x}(t,r), \overline{x}(t,r)$ can be described by $\underline{x}_0(t,r)$ and $\overline{x}_0(t,r)$, respectively. They can be represented in terms of their nth-order Taylor polynomials with respect to a, that is

$$\underline{x}_0(t,r) = \underline{X}_0(0,r) + \underline{X}_0(1,r)(t-a) + \underline{X}_0(2,r)(t-a)^2 + \cdots + \underline{X}_0(n,r)(t-a)^n,$$
$$\overline{x}_0(t,r) = \overline{X}_0(0,r) + \overline{X}_0(1,r)(t-a) + \overline{X}_0(2,r)(t-a)^2 + \cdots + \overline{X}_0(n,r)(t-a)^n,$$

where the subscript 0 denotes that the Taylor polynomial is expanded to $t_0 = a$. Once the Taylor polynomial is obtained $x(t_1, r)$ can be evaluated as

$$\begin{aligned}
\underline{x}(t_1,r) &= \underline{X}_0(0,r) + \underline{X}_0(1,r)(t_1-a) + \cdots + \underline{X}_0(n,r)(t_1-a)^n \\
&= \underline{X}_0(0,r) + \underline{X}_0(1,r)h + \cdots + \underline{X}_0(n,r)h^n \\
&= \sum_{j=0}^{n} \underline{X}_0(j,r)h^j, \\
\overline{x}(t_1,r) &= \overline{X}_0(0,r) + \overline{X}_0(1,r)(t_1-a) + \cdots + \overline{X}_0(n,r)(t_1-a)^n \\
&= \overline{X}_0(0,r) + \overline{X}_0(1,r)h + \cdots + \overline{X}_0(n,r)h^n \\
&= \sum_{j=0}^{n} \overline{X}_0(j,r)h^j.
\end{aligned}$$

The final value, $x_0(t_1, r)$ of the first sub-domain is the initial value of the second sub-domain, i.e. $x_1(t_1, r) = X_1(0, r) = x_0(t_1, r)$. In a similar manner $x(t_2, r)$ can be represented as

$$\begin{aligned}
\underline{x}(t_2, r) \approx \underline{x}_1(t_2, r) &= \underline{X}_1(0, r) + \underline{X}_1(1, r)h + \cdots + \underline{X}_1(n, r)h^n \\
&= \sum_{j=0}^{n} \underline{X}_1(j, r)h^j, \\
\overline{x}(t_2, r) \approx \overline{x}_1(t_2, r) &= \overline{X}_1(0, r) + \overline{X}_1(1, r)h + \cdots + \overline{X}_1(n, r)h^n \\
&= \sum_{j=0}^{n} \overline{X}_1(j, r)h^j.
\end{aligned}$$

Hence, the solution on the grid points (t_{i+1}) can be obtained as follows:

$$\begin{aligned}
\underline{x}(t_{i+1}, r) \approx \underline{x}_i(t_{i+1}, r) &= \sum_{j=0}^{n} \underline{X}_i(j, r)h^j, \\
\overline{x}(t_{i+1}, r) \approx \overline{x}_i(t_{i+1}, r) &= \sum_{j=0}^{n} \overline{X}_i(j, r)h^j.
\end{aligned}$$

Remark 5.1. Convergence of DTM for hybrid fuzzy differential equations should be mentioned as well. Since the fuzzy differential transform has been derived from fuzzy Taylor series expansion, convergence of DTM can be proven similar to that given in [7].

6. Examples

To present a clear overview of this study and illustrate the above-discussed technique, the following examples are considered.
The bound of errors for these examples is used as follows:

$$\text{bound of error} = \max\{\text{error of } \underline{x}(t, r),\ \text{error of } \overline{x}(t, r)\}$$

Example 1. Consider the following hybrid fuzzy differential equation

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)),\ t \in [t_k, t_{k+1}], t_k = k, \quad k = 0, 1, 2, \cdots \\ x(0, r) = [0.75 + 0.25r, 1.125 - 0.125r],\ 0 \le r \le 1, \end{cases} \tag{6.1}$$

where

$$m(t) = \begin{cases} 2(t(mod\ 1)), & \text{if } t(mod\ 1) \le 0.5, \\ 2(1 - t(mod\ 1)), & \text{if } t(mod\ 1) > 0.5, \end{cases}$$

and

$$\lambda_k(\mu) = \begin{cases} 0, & \text{if } k = 0, \\ \mu, & \text{if } k \in \{1, 2, \cdots\}. \end{cases}$$

The hybrid fuzzy IVP(6.1) is equivalent to the following system of fuzzy IVPs:

$$\begin{cases} x_0'(t) = x_0(t), t \in [0,1] \\ x_0(0,r) = [0.75 + 0.25r, 1.125 - 0.125r] \qquad , \ 0 \le r \le 1, \\ x_i'(t) = x_i(t) + m(t)x_i(t_i), \ t \in [t_i, t_{i+1}], \ x_i(t_i) = x_{i-1}(t_i), \quad i = 1, 2, \cdots . \end{cases}$$

In (6.1), $x(t) + m(t)\lambda_k(x(t_k))$ is a continuous function of t, x, and $\lambda_k(x(t_k))$. Therefore, using Example 6.1 of Kaleva [22], for each $k = 0, 1, 2, \cdots$. The fuzzy IVP

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), \ t \in [t_k, t_{k+1}], t_k = k, \\ x(t_k) = x_{tk}, \end{cases}$$

has a unique solution on $[t_k, t_{k+1}]$.
For $t \in [0,1]$ the following is given

$$\begin{cases} x_0'(t) = x_0(t), \ t \in [0,1] \\ x_0(0,r) = [0.75 + 0.25r, 1.125 - 0.125r], \ 0 \le r \le 1, \end{cases}$$

or

$$\begin{cases} \underline{x}_0'(t,r) = \underline{x}_0(t,r), \ \underline{x}_0(0,r) = (0.75 + 0.25r), \\ \overline{x}_0'(t,r) = \overline{x}_0(t,r), \ \overline{x}_0(0,r) = (1.125 - 0.125r). \end{cases} \tag{6.2}$$

Taking fuzzy differential transform of (6.2), results in

$$\begin{cases} (k+1)\underline{X}_0(k+1,r) = \underline{X}_0(k,r), \\ (k+1)\overline{X}_0(k+1,r) = \overline{X}_0(k,r). \end{cases} \tag{6.3}$$

From the initial conditions, the following can be written

$$\underline{X}_0(0,r) = (0.75 + 0.25r), \ \overline{X}_0(0,r) = (1.125 - 0.125r). \tag{6.4}$$

Substituting Eqs. (6.4) in (6.3), lead to all spectra that can be found as

$$\begin{aligned} \underline{x}_0(t,r) = {} & (0.75 + 0.25r) + (0.75 + 0.25r)t + (0.75 + 0.25r)t^2 \\ & + (0.75 + 0.25r)t^3 + \cdots . \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} \overline{x}_0(t,r) = {} & (1.125 - 0.125r) + (1.125 - 0.125r)t + (1.125 - 0.125r)t^2 \\ & + (1.125 - 0.125r)t^3 + \cdots . \end{aligned} \tag{6.6}$$

For $t \in [1,2]$:

$$\begin{cases} x'(t,r) = x(t,r) + m(t)\lambda_k(x(t_k,r)), \ t \in [1,2], \\ x(t_k,r) = x_{t_k}. \end{cases} \tag{6.7}$$

Let $N = 10$ and $h = 0.1$. The differential equation of a system between t_i and t_{i+1} can be represented as

$$\begin{cases} x'(t^*,r) = x(t^*,r) + m(t^* + t_i)\lambda_k(x(t^*,r)), \ t \in [1,2], \\ x(t_i,r) = x_{t_i} \end{cases} \tag{6.8}$$

where $t^* = t - t_i$. Taking differential transformation of (6.8), it can be obtained that for $i = 1, 2, \cdots, 6$

$$\begin{cases} \underline{X}_i(k+1,r) = & [\underline{X}_i(k,r) + 2(\delta(k-1) + t_i\delta(k) - \delta(k))\underline{x}(1,r)]/(k+1), \\ \overline{X}_i(k+1,r) = & [\overline{X}_i(k,r) + 2(\delta(k-1) + t_i\delta(k) - \delta(k))\overline{x}(1,r)\ ]/(k+1), \end{cases}$$

and for $i = 7, \cdots, 10$

$$\begin{cases} \underline{X}_i(k+1,r) = & [\underline{X}_i(k,r) + 2(2\delta(k) - t_i\delta(k) - \delta(k-1))\underline{x}(1,r)]/(k+1), \\ \overline{X}_i(k+1,r) = & [\overline{X}_i(k,r) + 2(2\delta(k) - t_i\delta(k) - \delta(k-1))\overline{x}(1,r)\ ]/(k+1). \end{cases}$$

From the initial conditions:

$$\underline{X}_0(0,r) = \underline{x}_0(1,r),\ \overline{X}_0(0,r) = \overline{x}_0(1,r). \tag{6.9}$$

For $t \in [0,1]$, the exact solution of (6.1) satisfies

$$x(t;r) = \big((0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t\big).$$

For $t \in [1,1.5]$, the exact solution of (6.1) satisfies

$$x(t;r) = x(1;r)(3e^{t-1} - 2t).$$

Therefore,

$$x(1;r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e],$$
$$x(1.5;r) = x(1;r)(3\sqrt{e} - 3).$$

For $t \in [1.5,2]$, the exact solution of (6.1) satisfies

$$x(t;r) = x(1;r)(2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)). \tag{6.10}$$

Therefore,

$$x(2.0;r) = x(1;r)(2 + 3e - 4\sqrt{e}). \tag{6.11}$$

The exact and approximate solutions (DTM4 and RK4) and (DTM5 and RK4) are compared and plotted for $0 \le r \le 1$, at $t = 2$ in Figures 1 and 2, respectively.

The errors of approximate solutions are shown by DTM and RK4 in Table 2.

Example 2. Consider hybrid fuzzy differential equation

(6.12)
$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), \ t \in [t_k, t_{k+1}], t_k = k, \ k = 0, 1, 2, \cdots \\ x(0, r) = [0.75 + 0.25r, 1.125 - 0.125r], \ 0 \le r \le 1, \end{cases}$$

where $m(t) = |sin(\pi t)|$ and

$$\lambda_k(\mu) = \begin{cases} 0, & if \ k = 0, \\ \mu, & if \ k \in \{1, 2, \cdots\}. \end{cases}$$

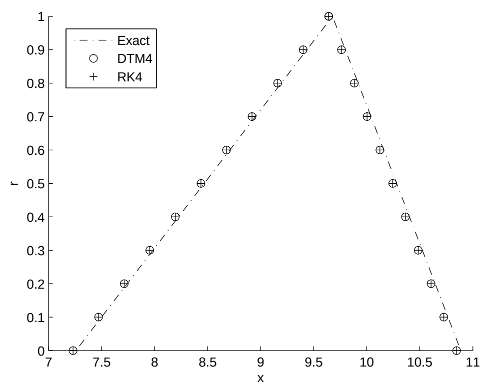

FIGURE 1. Exact and approximate solutions for $0 \le r \le 1$, at $t = 2$.

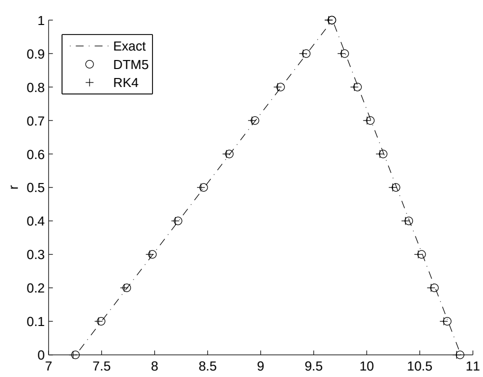

FIGURE 2. Exact and approximate solutions for $0 \le r \le 1$, at $t = 2$.

TABLE 2. Comparison of errors with Exp.1 for $0 \leq r \leq 1$, at $t = 2$.

r	DTM4	DTM5	DTM8	RK4
0.0	3.9854628e-2	6.4688403e-3	1.2249628e-5	3.9836828e-2
0.1	3.9411799e-2	6.3969643e-3	1.2113521e-5	3.9414765e-2
0.2	3.8968970e-2	6.3250883e-3	1.1977414e-5	3.8957103e-2
0.3	3.8526140e-2	6.2532123e-3	1.1841307e-5	3.8532936e-2
0.4	3.8083311e-2	6.1813363e-3	1.1705200e-5	3.8080745e-2
0.5	3.7640482e-2	6.1094603e-3	1.1569093e-5	3.7655315e-2
0.6	3.7197653e-2	6.0375843e-3	1.1432986e-5	3.7197653e-2
0.7	3.6754824e-2	5.9657083e-3	1.1296879e-5	3.6739991e-2
0.8	3.6311994e-2	5.8938323e-3	1.1160772e-5	3.6317928e-2
0.9	3.5869165e-2	5.8219563e-3	1.1024665e-5	3.5860265e-2
1.0	3.5426336e-2	5.7500803e-3	1.0888558e-5	3.5438203e-2

Since, $x(t) + m(t)\lambda_k(x(t_k))$ is a continuous function of t, x, and $\lambda_k(x(t_k))$ in (6.1). Therefore, the fuzzy IVP

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), \ t \in [t_k, t_{k+1}], t_k = k, \\ x(t_k) = x_{tk}, \end{cases} \tag{6.13}$$

has a unique solution on $[t_k, t_{k+1}]$ (see [22]). For $t \in [0, 1]$:

$$\begin{cases} x'(t) = x(t), \ t \in [0, 1], \\ x(0, r) = [0.75 + 0.25r, 1.125 - 0.125r], \ 0 \leq r \leq 1. \end{cases}$$

The approximate solution is given by

$$\underline{x}_0(t, r) = (0.75 + 0.25r) + (0.75 + 0.25r)t + (0.75 + 0.25r)t^2 + (0.75 + 0.25r)t^3 + \cdots .$$

and

$$\overline{x}_0(t, r) = (1.125 - 0.125r) + (1.125 - 0.125r)t + (1.125 - 0.125r)t^2 + (1.125 - 0.125r)t^3 + \cdots .$$

For $t \in [1, 2]$:

$$\begin{cases} x'(t, r) = x(t, r) + m(t)\lambda_k(x(t_k, r)), \ t \in [1, 2], \\ x(t_k, r) = x_{t_k}. \end{cases} \tag{6.14}$$

Let $N = 10$ and $h = 0.1$. The differential equation of a system between t_i and t_{i+1} can be represented as

$$\begin{cases} x'(t^*, r) = x(t^*, r) - sin(\pi(t^* + t_i))x(1, r), \ t \in [1, 2], \\ x(t_i, r) = x_{t_i} \end{cases} \tag{6.15}$$

where $t^* = t - t_i$. Taking differential transformation of (6.15), it can be obtained that, for $i = 1, 2, \cdots, 10$

$$\begin{aligned}(k+1)\underline{X}_i(k+1,r) &= \underline{X}_i(k,r) - \Big(cos(\pi t_i)\frac{\pi^k}{k!}sin(\frac{\pi k}{2})\\ &+ sin(\pi t_i)\frac{\pi^k}{k!}cos(\frac{\pi k}{2})\Big)\underline{x}_0(1,r)\Big), \end{aligned} \tag{6.16}$$

$$\begin{aligned}(k+1)\overline{X}_i(k+1,r) &= \overline{X}_i(k,r) - \Big(cos(\pi t_i)\frac{\pi^k}{k!}sin(\frac{\pi k}{2})\\ &+ sin(\pi t_i)\frac{\pi^k}{k!}cos(\frac{\pi k}{2})\Big)\overline{x}_0(1,r)\Big), \end{aligned} \tag{6.17}$$

From the initial conditions, the following can be written:

$$\underline{X}(0,r) = \underline{x}_0(1,r),\ \overline{X}(0,r) = \overline{x}_0(1,r). \tag{6.18}$$

For $t \in [1,2]$, the exact solution of (6.12) satisfies

$$x(t,r) = \big((0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t\big).$$

For $t \in [1,2]$, the exact solution of (6.12) satisfies

$$\underline{x}(t;r) = \underline{x}(1;r)\frac{\pi cos(\pi t) + sin(\pi t)}{\pi^2+1} + \frac{e^t}{e}\underline{x}(1;r)(1 + \frac{\pi}{\pi^2+1}),$$

$$\overline{x}(t;r) = \overline{x}(1;r)\frac{\pi cos(\pi t) + sin(\pi t)}{\pi^2+1} + \frac{e^t}{e}\overline{x}(1;r)(1 + \frac{\pi}{\pi^2+1}).$$

Therefore,

$$x(1;r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e],$$

and

$$x(2;r) = (\frac{\pi}{\pi^2+1} + e(1 + \frac{\pi}{\pi^2+1}))x(1;r).$$

The exact and approximate solutions (DTM4 and RK4) and (DTM5 and RK4) are compared and plotted for $0 \le r \le 1$, at $t = 2$ in Figure 3, 4, respectively. Errors of approximate solutions are demonstrated by DTM and RK4 in Table 3.

7. Conclusion

In this paper, the fuzzy differential transformation method was introduced for approximate solution of hybrid fuzzy differential equations and it was illustrated by some numerical examples. Useful comparison results were obtained to show that DTM was remarkably effective and simple.

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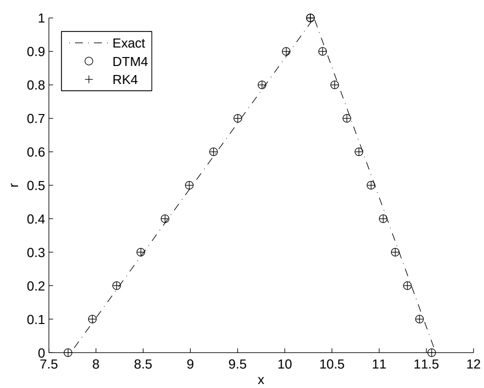

FIGURE 3. Exact and approximate solutions for $0 \leq r \leq 1$, at $t = 2$.

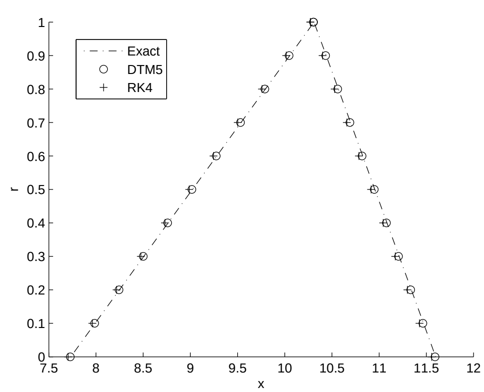

FIGURE 4. Exact and approximate solutions for $0 \leq r \leq 1$, at $t = 2$.

Table 3. Comparison of errors with Exp. 2 for $0 \leq r \leq 1$, at $t = 2$.

r	errorDTM4	errorDTM5	errorDTM8	errorRK4
0.0	4.2650069e-2	6.9010022e-3	1.3051973e-5	4.2436476e-2
0.1	4.2176180e-2	6.8243244e-3	1.2906951e-5	4.1986875e-2
0.2	4.1702290e-2	6.7476466e-3	1.2761929e-5	4.1499344e-2
0.3	4.1228400e-2	6.6709688e-3	1.2616907e-5	4.1047055e-2
0.4	4.0754511e-2	6.5942910e-3	1.2471885e-5	4.0566510e-2
0.5	4.0280621e-2	6.5176132e-3	1.2326863e-5	4.0112609e-2
0.6	3.9806731e-2	6.4409354e-3	1.2181842e-5	3.9625078e-2
0.7	3.9332842e-2	6.3642576e-3	1.2036820e-5	3.9137547e-2
0.8	3.8858952e-2	6.2875798e-3	1.1891798e-5	3.8687946e-2
0.9	3.8385062e-2	6.2109020e-3	1.1746776e-5	3.8200414e-2
1.0	3.7911173e-2	6.1342242e-3	1.1601754e-5	3.7750813e-2

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