Journal of Hyperstructures 6 *(1) (2017), 40-51.*
ISSN: 2322-1666 print/2251-8436 online

APPROXIMATE METHODS FOR SOLVING LOCAL FRACTIONAL INTEGRAL EQUATIONS

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Abstract. This paper presents new analytical approximate methods such as local fractional variational iteration method and local fractional decomposition method for a family of the linear and nonlinear integral equations of the second kind within local fractional derivative operators. Some examples are presented to illustrate the efficiency and accuracy of the proposed methods. The obtained results reveal that the proposed methods are very efficient and simple tools for solving local fractional integral equations.

Key Words: Local fractional integral equations, Local fractional variational iteration method, Local fractional decomposition method, Analytical approximate solutions.
2010 Mathematics Subject Classification: Primary: 26A33; Secondary: 34A12, 35R11.

1. Introduction

An integral equation is defined as an equation in which the unknown function to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation and partial differential equation can be transformed into problems of solving some approximate integral equations [1].

The theory of integral equations has close contacts with many different areas of mathematics. Foremost among these are differential equations

Received: 26 July 2016, Accepted: 04 November 2016. Communicated by Ali Taghavi;
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and operator theory. Many problems of mathematical physics can be stated in the form of integral equations [2].

The theory of local fractional integrals and derivatives was dealing with fractal functions, and was successfully applied in fractional Fokker-Planck equation, anomalous diffusion and relaxation equation in fractal space, fractal heat conduction equation, fractal-time dynamical systems, fractal elasticity, local fractional diffusion equation, local fractional Laplace equation, local fractional ordinary differential equations, local fractional partial differential equation, local fractional integral equations, fractional Brownian motion in local fractional derivatives sense, fractal signals, local fractional short time transforms and local fractional wavelet transform [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

Our aim in this paper is to investigate the applications of the local fractional variational iteration method and local fractional decomposition method for solving the integral equations in the sense of local fractional derivative operators. To illustrate the validity and advantages of the methods, we will apply it to the local fractional Fredholm integral equation and Volterra integro-differential equation as follows:

$$\psi(x) = \varphi(x) + \frac{1}{\Gamma(1+\alpha)} \int_a^b \Omega(x,\tau) F\left(\psi(\tau)\right) (d\tau)^\alpha, 0 < \alpha \leq 1 \tag{1.1}$$

$$\psi^{(k\alpha)}(x) = \varphi(x) + \frac{1}{\Gamma(1+\alpha)} \int_a^x \Omega(x,\tau)\psi(\tau)(d\tau)^\alpha, \psi^{(m\alpha)} = a_m, \tag{1.2}$$

where m=0,1,..., k-1, $\Omega(x,\tau)$ is the kernel of the local fractional integral equation, $\varphi(x)$ and $F(\psi)$ are known functions. The limits of integration a and b are constants and $\psi(x)$ is the unknown solution of integral equations. The paper has been organized as follows. In Section 2, we give the concept of local fractional calculus. In Section 3, we give analysis of the methods used. In Section 4, we consider several illustrative examples. Finally, in Section 5, we present our conclusions.

2. The Theory of Local Fractional Calculus

In this section we present some basic definitions and notations of the local fractional operators (see [21, 22, 23, 24, 25, 26, 27]).

Definition 2.1. The local fractional derivative of $\psi(x)$ of order α at $x = x_0$ is given by

$$\psi^{(\alpha)}(x_0) = \frac{d^\alpha}{dx^\alpha}\psi(x)|_{x=x_0} = \lim_{x\to x_0}\frac{\triangle^\alpha(\psi(x)-\psi(x_0))}{(x-x_0)^\alpha}, \tag{2.1}$$

where $\triangle^\alpha(\psi(x)-\psi(x_0) \cong \Gamma(\alpha+1)(\psi(x)-\psi(x_0))$.

The formulas of local fractional derivatives of special functions used in the paper are as follows:

$$D_x^{(\alpha)}a\psi(x) = aD_x^{(\alpha)}\psi(x) \tag{2.2}$$

$$\frac{d^\alpha}{dx^\alpha}\left(\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\right) = \frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)} \tag{2.3}$$

Definition 2.2. The local fractional integral of $\psi(x)$ in the interval $[a, b]$ is given by

$$\begin{aligned} {}_aI_b^{(\alpha)}\psi(x) &= \frac{1}{\Gamma(1+\alpha)}\int_a^b \psi(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\lim_{\triangle t\longrightarrow 0}\sum_{j=0}^{N-1}\psi(t_j)(\triangle t_j)^\alpha. \end{aligned} \tag{2.4}$$

where the partition of the interval $[a, b]$ is denoted as $(t_j, t_{j+1}), j = 0, ..., N-1$, $t_0 = a$ and $t_N = b$ with $\triangle t_j = t_{j+1} - t_j$ and $\triangle t = \max\{\triangle t_0, \triangle t_1, \dots\}$.

The formulas of local fractional integrals of special functions used in the paper are as follows:

$${}_0I_x^{(\alpha)}a\psi(t) = a{}_0I_x^{(\alpha)}\psi(t), \tag{2.5}$$

$${}_0I_x^{(\alpha)}\left(\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right) = \frac{x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \tag{2.6}$$

Definition 2.3. The Mittage Leffler function, sine function and cosine function are defined as

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}, 0 < \alpha \leq 1 \tag{2.7}$$

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}, 0 < \alpha \leq 1 \tag{2.8}$$

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k\alpha}}{\Gamma(1+2k\alpha)}, 0 < \alpha \leq 1. \tag{2.9}$$

3. Analysis of the Methods

In this section, two methods, namely the variational iteration method and Adomian decomposition method with local fractional derivative operators are analyzed and utilized for solving the local fractional integral equations of the second kind.

3.1 Local Fractional Variational Iteration Method

For solving (1.1) by local fractional variational iteration method, first we differentiate once from both sides of equation (1.1) with respect to x gives

$$\psi^{(\alpha)}(x) = \varphi^{(\alpha)}(x) + \frac{1}{\Gamma(1+\alpha)} \int_a^b \frac{\partial^\alpha \Omega(x,\tau)}{\partial x^\alpha} F\left(\psi(\tau)\right)(d\tau)^\alpha, \tag{3.1}$$

According to local fractional variational iteration method correction functional can be written in the following form [22, 23, 24]:

$$\begin{aligned}\psi_{n+1}(x) = \psi_n(x) +_0 I_x^{(\alpha)} \Bigg(\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} \Big[\psi_n^{(\alpha)}(\xi) - \varphi^{(\alpha)}(\xi) \\ - \frac{1}{\Gamma(1+\alpha)} \int_a^b \frac{\partial^\alpha \Omega(\xi,\tau)}{\partial \xi^\alpha} F\left(\psi_n^{\sim}(\tau)\right)(d\tau)^\alpha,\end{aligned} \tag{3.2}$$

where $\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}$ is a general fractal Lagranges multiplier. To make the above correction functional stationary with respect to ψ_n , we have :

$$\begin{aligned}\delta^\alpha \psi_{n+1}(x) = \quad & \delta^\alpha \psi_n(x) + \delta_0^\alpha I_x^{(\alpha)} \Bigg(\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} \Big[\psi_n^{(\alpha)}(\xi) - \varphi^{(\alpha)}(\xi) \\ & - \frac{1}{\Gamma(1+\alpha)} \int_a^b \frac{\partial^\alpha \Omega(\xi,\tau)}{\partial \xi^\alpha} F\left(\psi_n^{\sim}(\tau)\right)(d\tau)^\alpha, \\ = \quad & \delta^\alpha \psi_n(x) +_0 I_x^{(\alpha)} \left[\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} \delta^\alpha \left(\psi_n^{(\alpha)}(\xi) \right) \right] \\ = \quad & \delta^\alpha \psi_n(x) + \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \delta^\alpha \psi_n^{(\alpha)}(x) +_0 I_x^{(\alpha)} \left[\frac{\lambda^\alpha(\xi)^\alpha}{\Gamma(1+\alpha)} \delta^\alpha \left(\psi_n(\xi)\right] \right. .\end{aligned}$$

From the above relation for any $\delta^\alpha \psi_n$, we obtain

$$1 + \frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} \mid_{\xi=x} = 0, \left[\frac{\lambda^\alpha(\xi)^\alpha}{\Gamma(1+\alpha)} \right]^{(\alpha)} \mid_{\xi=x} = 0 \tag{3.3}$$

This in turn gives

$$\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} = -1. \tag{3.4}$$

Substituting the identified Lagrange multiplier (3.4) into (3.2), result in the following iterative formula:

$$\psi_{n+1}(x) = \psi_n(x) -_0 I_x^{(\alpha)} \left[\psi_n^{(\alpha)}(\xi) - \varphi^{(\alpha)}(\xi) - \frac{1}{\Gamma(1+\alpha)} \int_a^b \frac{\partial^\alpha \Omega(\xi,\tau)}{\partial \xi^\alpha} F\left(\psi_n(\tau)\right)(d\tau)^\alpha. \right. \tag{3.5}$$

Finally, we obtain the exact solution or an approximate solution of the equation (1.1) as follows:

$$\psi(x) = \lim_{n\to\infty} \psi_n(x). \tag{3.6}$$

3.2 Local Fractional Decomposition Method By integrating both sides of (1.2) leads to

$$\begin{aligned} & L_x^{(-k\alpha)}\left[\psi^{(k\alpha)}(x)\right] \\ &= L_x^{(-k\alpha)}\left[\varphi(x)\right] + L_x^{(-k\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^x \Omega(x,\tau)\psi(\tau)(d\tau)^\alpha\right] \end{aligned} \tag{3.7}$$

where $L_x^{(-k\alpha)}[\cdot] =_0 I_x^{(k\alpha)}[\cdot] = \overbrace{{}_0I_{x_0}^{(\alpha)} I_x^{(\alpha)} \ldots {}_0 I_x^{(\alpha)}}^{k-time}[\cdot]$.

Thus, we obtain

$$\begin{aligned} \psi(x) &= a_0 + \cdots + a_{k-1}\frac{x^{(k-1)\alpha}}{\Gamma\left(1+(k-1)\alpha\right)} + L_x^{(-k\alpha)}\left[\varphi(x)\right] \\ &\quad + L_x^{(-k\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^x \Omega(x,\tau)\psi(\tau)(d\tau)^\alpha\right], \end{aligned} \tag{3.8}$$

where the initial conditions $\psi(0), \psi^{(\alpha)}(0), \ldots, \psi^{((k-1)\alpha)}(0)$ are used.

We then use the decomposition series

$$\psi(x) = \sum_{n=0}^{\infty} \psi_n(x). \tag{3.9}$$

in both sides(3.8) to obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \psi_n(x) &= a_0 + \cdots + a_{k-1}\frac{x^{(k-1)\alpha}}{\Gamma\left(1+(k-1)\alpha\right)} + L_x^{(-k\alpha)}\left[\varphi(x)\right] \\ &\quad + L_x^{(-k\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^x \Omega(x,\tau)\left[\sum_{n=0}^{\infty}\psi_n(\tau)\right](d\tau)^\alpha\right], \end{aligned} \tag{3.10}$$

or equivalently

$$\begin{aligned}\psi_0(x)+\psi_1(x)+\cdots &= a_0+\cdots+a_{k-1}\frac{x^{(k-1)\alpha}}{\Gamma\left(1+(k-1)\alpha\right)}+L_x^{(-k\alpha)}\left[\varphi(x)\right]\\ &+L_x^{(-k\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^x \Omega(x,\tau)\psi_0(\tau)(d\tau)^\alpha\right]\\ &+L_x^{(-k\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^x \Omega(x,\tau)\psi_1(\tau)(d\tau)^\alpha\right]+\cdots.\end{aligned}\tag{3.11}$$

To determine the components $\psi_0(x), \psi_1(x), \cdots$ of the solution $\psi(x)$ we set the recurrence relations:

$$\begin{aligned}\psi_0(x) &= a_0+\cdots+a_{k-1}\frac{x^{(k-1)\alpha}}{\Gamma\left(1+(k-1)\alpha\right)}+L_x^{(-k\alpha)}\left[\varphi(x)\right],\\ \psi_{n+1}(x) &= L_x^{(-k\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^x \Omega(x,\tau)\psi_n(\tau)(d\tau)^\alpha\right].\end{aligned}\tag{3.12}$$

4. Applications to Solve Integral Equations Involving Local Fractional operators

To illustrate the ability and simplicity of the proposed methods, some illustrative examples are provided here.

I. Solving the linear and Nonlinear Fredholm Integral Equation with Local Fractional Variational Iteration Method

Example 4.1. Consider the following linear Fredholm integral equation involving local fractional derivative operator:

$$\psi(x)=E_\alpha(x^\alpha)-\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)}\frac{\tau^\alpha}{\Gamma(1+\alpha)}\psi(\tau)(d\tau)^\alpha,\tag{4.1}$$

with the exact solution $\psi(x)=E_\alpha(x^\alpha)$.

Differentiating both sides of (4.1) with respect to x yields

$$\psi^{(\alpha)}(x)=E_\alpha(x^\alpha)-1+\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{\tau^\alpha}{\Gamma(1+\alpha)}\psi(\tau)(d\tau)^\alpha.\tag{4.2}$$

The iterative formula can be expressed as the following:

$$\begin{aligned}\psi_{n+1}(x) &= \psi_n(x)-I_x^{(\alpha)}\left[\psi_n^{(\alpha)}(\xi)-E_\alpha(\xi^\alpha)+1\right.\\ &-\left.\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{r^\alpha}{\Gamma(1+\alpha)}\psi_n(r)(dr)^\alpha,\right.\end{aligned}\tag{4.3}$$

where we used $\dfrac{\lambda(\xi)^{\alpha}}{\Gamma(1+\alpha)} = -1$.

Notice that the initial condition $\psi(0) = 1$ is obtained by substituting $x = 0$ into (4.1). Therefore, we have

$$\begin{aligned}
\psi_0(x) &= 1, \\
\psi_1(x) &= E_\alpha(x^\alpha) - \frac{1}{2}\frac{x^\alpha}{\Gamma(1+\alpha)}, \\
\psi_2(x) &= E_\alpha(x^\alpha) - \frac{1}{6}\frac{x^\alpha}{\Gamma(1+\alpha)}, \\
\psi_3(x) &= E_\alpha(x^\alpha) - \frac{1}{18}\frac{x^\alpha}{\Gamma(1+\alpha)}, \\
&\vdots \\
\psi_n(x) &= E_\alpha(x^\alpha) - \frac{1}{2\times 3^{n-1}}\frac{x^\alpha}{\Gamma(1+\alpha)}, n \geq 1.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\psi(x) &= \lim_{n\to\infty} \psi_n(x) \\
&= \lim_{n\to\infty}\left[E_\alpha(x^\alpha) - \frac{1}{2\times 3^{n-1}}\frac{x^\alpha}{\Gamma(1+\alpha)}\right] \\
&= E_\alpha(x^\alpha), \qquad (4.4)
\end{aligned}$$

which is the exact solution.

Example 4.2. Consider the following nonlinear Fredholm integral equation involving local fractional derivative operators:

$$\begin{aligned}
\psi(x) &= \cos_\alpha(x^\alpha) - \frac{x^\alpha}{\Gamma(1+\alpha)} + \\
&\quad \frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)}\left[\psi^2(\tau) + \sin^2_\alpha(\tau^\alpha)\right](d\tau)^\alpha, \qquad (4.5)
\end{aligned}$$

with the exact solution $\psi(x) = \cos_\alpha(x^\alpha)$.

In the same procedure, the iterative formula can be expressed as the following:

$$\begin{aligned}
&\psi_{n+1}(x) = \psi_n(x) -_0 I_x^{(\alpha)}\Big[\psi_n^{(\alpha)}(\xi) + \sin_\alpha(\xi^\alpha) + 1 \\
&- \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left[\psi^2(\tau) + \sin^2_\alpha(\tau^\alpha)\right](d\tau)^\alpha, \qquad (4.6)
\end{aligned}$$

By using this iterative formula and taking $\psi_0(x) = \cos_\alpha(x^\alpha)$, we have:

$$\begin{aligned}
\psi_0(x) &= \cos_\alpha(x^\alpha), \\
\psi_1(x) &= \cos_\alpha(x^\alpha), \\
\psi_2(x) &= \cos_\alpha(x^\alpha), \\
&\vdots \\
\psi_n(x) &= \cos_\alpha(x^\alpha), n \geq 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\psi(x) &= \lim_{n\to\infty} \psi_n(x) \\
&= \lim_{n\to\infty} \cos_\alpha(x^\alpha) \\
&= \cos_\alpha(x^\alpha), \qquad (4.7)
\end{aligned}$$

which is the exact solution.

II. Solving the Volterra integro-differential Equation with Local Fractional Decomposition Method

Example 4.3. We consider the Volterra integro-differential equation involving local fractional derivative operator:

$$\psi^{(2\alpha}(x) = 1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{(x-\tau)^\alpha}{\Gamma(1+\alpha)} \psi(\tau)(d\tau)^\alpha, \qquad (4.8)$$

with initial conditions

$$\psi(0) = 1, \psi^{(\alpha)}(0) = 0.$$

Let the solution in the series form

$$\psi(x) = \sum_{n=0}^{\infty} \psi_n(x). \qquad (4.9)$$

Applying the integral operator $L_x^{(-2\alpha)}$ to both sides of (4.8), and using the given initial condition we obtain

$$\begin{aligned}
\psi(x) &= 1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \\
& \quad L_x^{(-2\alpha)} \left[\frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{(x-\tau)^\alpha}{\Gamma(1+\alpha)} \psi(\tau)(d\tau)^\alpha \right]. \qquad (4.10)
\end{aligned}$$

Then substituting (4.9) in(4.10), we have that

$$\sum_{n=0}^{\infty}\psi_n(x) = 1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+ L_x^{(-2\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-\tau)^{\alpha}}{\Gamma(1+\alpha)}\left(\sum_{n=0}^{\infty}\psi_n(x)\right)(d\tau)^{\alpha}\right]. \tag{4.11}$$

Making use of (4.11), we give the recurrence relations in the following form:

$$\psi_0(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)},$$

$$\psi_{n+1}(x)=L_x^{(-2\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-\tau)^{\alpha}}{\Gamma(1+\alpha)}\psi_n(\tau)(d\tau)^{\alpha}\right]. \tag{4.12}$$

From (4.12), we obtain

$$\begin{aligned}
\psi_0(x) &= 1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)},\\
\psi_1(x) &= \frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{7\alpha}}{\Gamma(1+7\alpha)},\\
\psi_2(x) &= \frac{x^{8\alpha}}{\Gamma(1+8\alpha)}+\frac{x^{9\alpha}}{\Gamma(1+9\alpha)}+\frac{x^{10\alpha}}{\Gamma(1+10\alpha)}+\frac{x^{11\alpha}}{\Gamma(1+11\alpha)},\\
&\vdots
\end{aligned}$$

This gives the solution in a series form

$$\begin{aligned}
\psi(x) &= 1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\cdots\\
&= \sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\\
&= E_{\alpha}(x^{\alpha}). \qquad (4.13)
\end{aligned}$$

Example 4.4. Consider the Volterra integro-differential equation involving local fractional derivative operator:

$$\psi^{(4\alpha}(x)=-1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}-\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-\tau)^{\alpha}}{\Gamma(1+\alpha)}\psi(\tau)(d\tau)^{\alpha}, \tag{4.14}$$

subject to the initial conditions given by

$$\psi(0)=-1,\psi^{(\alpha)}(0)=1,\psi^{(2\alpha)}(0)=1,\psi^{(3\alpha)}(0)=-1. \tag{4.15}$$

In view of (3.12) and (4.14), we arrive at the following iteration formula:

$$
\begin{aligned}
\psi_0(x) &= -1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)} \\
&\quad -\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}-\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}, \\
\psi_{n+1}(x) &= -L_x^{(-4\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(x-\tau)^{\alpha}}{\Gamma(1+\alpha)}\psi_n(\tau)(d\tau)^{\alpha}\right].
\end{aligned}
\tag{4.16}
$$

From (4.16), we obtain

$$
\begin{aligned}
\psi_0(x) &= -1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)} \\
&\quad -\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}-\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}, \\
\psi_1(x) &= \frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{7\alpha}}{\Gamma(1+7\alpha)}+\frac{x^{8\alpha}}{\Gamma(1+8\alpha)}+\cdots \\
&\;\;\vdots
\end{aligned}
$$

This gives the solution in a series form

$$
\begin{aligned}
\psi(x) &= \left[\frac{x^{\alpha}}{\Gamma(1+\alpha)}-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}\cdots\right] \\
&\quad -\left[1-\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}\cdots\right],
\end{aligned}
$$

and finally in its closed form gives

$$
\psi(x)=\sin_{\alpha}(x^{\alpha})-\cos_{\alpha}(x^{\alpha}). \tag{4.17}
$$

5. Conclusions

In this work, we have successfully provided new applications of the local fractional variational iteration method and local fractional decomposition method for solving several integral equations of the second kind with local fractional derivative operators. The analytical approximate solutions for local fractional integral equations were obtained and four examples were given, in order to illustrate the high efficiency and accuracy of the proposed techniques to solve the local fractional integral equations.

Acknowledgments

The author is very grateful to the referees for their valuable suggestions and opinions.

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