Journal of Hyperstructures 6 *(Spec. 13th AHA)* (*2017*), *38-55.*
ISSN: 2322-1666 print/2251-8436 online

THE $\mathbf{H_v}$-MATRIX REPRESENTATIONS

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ABSTRACT. The Theory of Representations of Hyperstructures was started in mid 80's but that time there was not any general definition of hyperfield. The H_v-structures, were introduced in 4^{th} AHA Congress 1990, and at the same time, the general definition of the hyperfield, was given. Since then the Theory of Representations is refereed mainly on H_v-groups by H_v-matrices. In H_v-structures the weak axioms replace the "equality" by the "non empty intersection". The characteristic property of H_v-structures, is that a partial order on H_v-structures on the same underline set, is defined. The weak properties increase extremely the number of hyperstructures defined in the same set. In representation theory the researchers have to treat well almost all the classical algebraic structures from semigroups to Lie-algebras. We present the problems, some new results and we give to researchers open problems in mathematics from hyperstructures.

Key Words:hope, H_v-structure, h/v-structure, H_v-field, h/v-field. H_v-matrix.
2010 Mathematics Subject Classification: Primary: 20N20; Secondary: 16Y99.

1. Introduction

Our object is the largest class of hyperstructures called H_v-structures introduced in 1990 [17], which satisfy the weak axioms where the non-empty intersection replaces the equality. Abbreviation: hyperoperation=hope.

Definition 1.1. *Algebraic hyperstructure* is a set H with a **hope** $\cdot :$ $H \times H \to P(H) - \{\varnothing\}$. Abbreviate *WASS* the *weak associativity*: $(xy)z \cap$

Received: 17 October 2017. Communicated by Ali Taghavi;
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$x(yz) \neq \varnothing, \forall x, y, z \in H$ and COW the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$. The hyperstructure $(H, \cdot)$ is called H_v-semigroup if it is WASS, it is called **$\mathbf{H_v}$-group** if it is reproductive H_v-semigroup, i.e. $xH = Hx = H, \forall x \in H$.

Motivation. The quotient of a group by an invariant subgroup, is a group. The quotient of a group by a subgroup is a hypergroup (F. Marty, 1934). The quotient of a group by a partition (or equivalently to any equivalence) is an H_v-group.

In an H_v-semigroup $(H, \cdot)$, *powers* are defined by *n-ary circle hope*: the union of hyperproducts, with all possible patterns of parentheses on them. $(H, \cdot)$ is *cyclic of period s*, if there is h, called *generator*, and a natural s, the minimum: $H = h^1 \cup \ldots \cup h^s$. If there is an h and s, the minimum: $H = h^s$, then $(H, \cdot)$, is called *single-power cyclic of period s.*

Definition 1.2. An $(R, +, \cdot)$ is called **$\mathbf{H_v}$-ring** if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is weak distributive with respect to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \varnothing, (x+y)z \cap (xz+yz) \neq \varnothing, \forall x, y, z \in R.$$

An H_v-ring is called *additive* if the addition is hope and the multiplication is ordinary operation and is called *multiplicative* if its product is hope and addition is operation.

Let $(R, +, \cdot)$ be an H_v-ring, $(M, +)$ be a COW H_v-group and there exists an external hope

$$\cdot : R \times M \to P(M) : (a, x) \to ax$$

such that, $\forall a, b \in R$ and $\forall x, y \in M$, we have

$$a(x+y) \cap (ax+ay) \neq \varnothing, (a+b)x \cap (ax+bx) \neq \varnothing, (ab)x \cap a(bx) \neq \varnothing,$$

then M is called an H_v-module over F. In the case of an H_v-field F, which is defined later, instead of an H_v-ring R, then the **$\mathbf{H_v}$-vector space** is defined.

For more definitions and applications on H_v-structures see books and papers as [1], [13], [14], [15], [11], [13], [15], [20], [21], [22], [23].

Definition 1.3. Let $(H, \cdot), (H, *)$ be H_v-semigroups on the same set H, the hope $(\cdot)$ is called **smaller** than the $(*)$, and $(*)$ **greater** than $(\cdot)$, iff there exists an automorphism

$$f \in Aut(H, *) \text{ such that } xy \subset f(x * y),\ \forall x, y \in H.$$

We write $\cdot \leq *$ and $(H, *)$ contains $(H, \cdot)$. If $(H, \cdot)$ is a structure, then $(H, *)$ is called H_b-*structure*.

Little Theorem. Greater hopes than WASS or COW, are also WASS or COW, respectively.

The Litle Theorem leads to a partial order, on posets, on H_v-structures [20], [22].

Let $(H, \cdot)$ be hypergroupoid. We *remove* $h \in H$, if we take the restriction of $(\cdot)$ in the set $H - \{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h by $\underline{h}$ and h does not appear. $\underline{h} \in H$ *merges* with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$.

The main tool in hyperstructures is the *fundamental relation*. M. Koscas 1970, [8], defined in hypergroups the relation β and its transitive closure β^*. β^* is also defined in H_v-groups and connect hyperstructures with the classical structures. T. Vougiouklis [15], [17], [18], [20] introduced the γ^* and ϵ^* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively, and he named all these relations, *fundamental*.

Definition 1.4. The fundamental relations β^*, γ^* and ϵ^*, are defined, in H_v-groups, H_v-rings and H_v-vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively.

On the motivation remark: Let $(G, \cdot)$ be group and R be any partition in G, then $(G/R, \cdot)$ is H_v-group, thus the quotient$(G/R, \cdot)/\beta^*$ is a group, the *fundamental* one.

Theorem 1.5. *Let $(H, \cdot)$ be H_v-group and U the set of all finite products of elements of H. We define β in H by $x\beta y$ iff $\{x, y\} \subset u, u \in U$. Then β^* is the transitive closure of β.*

Denote $[x]$ the fundamental class of the element $x \in H$. Therefore $\beta^*(x) = [x]$.

For proof, see [20]. Analogous theorems are for H_v-rings, H_v-vector spaces and so on. An element is called single if its fundamental class is singleton so, $[x] = \{x\}$.

General structures are defined using the fundamental ones. There was no general definition of a hyperfield but from 1990, there exists [17], [20]:

Definition 1.6. An H_v-ring $(R,+,\cdot)$ is called $\mathbf{H_v}$**-field** if R/γ^* is a field.

Definition 1.7. [26],[32],[33] Let $(L,+)$ be H_v-vector space on $(F,+,\cdot)$, $\phi : F \to F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of F/γ^*. Let ω_L be the core of the canonical map $\phi' : L \to L/\epsilon^*$ and denote 0 the zero of L/ϵ^*. Consider the *bracket (commutator) hope*:

$$[,] : L \times L \to P(L) : (x,y) \to [x,y]$$

then **L** is an H_v**-Lie algebra** over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1,y] + \lambda_2[x_2,y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x,y_1] + \lambda_2[x,y_2]) \neq \varnothing,$
$\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

(L2) $[x,x] \cap \omega_L \neq \varnothing, \quad \forall x \in L$

(L3) $([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) \cap \omega_L \neq \varnothing, \forall x,y,z \in L$

A well known class of hopes is defined on classical ones are [11], [13], [20]:

Definition 1.8. Let $(G,\cdot)$ be groupoid, then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called P-*hopes*: $\forall x,y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G,\underline{P})$,$(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called P-*hyperstructures*. The most usual case is if $(G,\cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G,\underline{P})$ is a semihypergroup.

A **generalization of P-hopes** used in Santilli's isotheory, is [6], [36] : Let $(G,\cdot)$ be abelian group, $P \subset G$ with $\#P < 1$. We define the hope $\times_p$ as follows:

$$x \times_p y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ and } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G,\times_p)$ is abelian H_v-group.

Definition 1.9. [16] An H_v-structure is called **very thin** if all hopes are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in H there exists a hope $(\cdot)$ and a pair $(a,b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products, are singletons.

From the properties of the very thin hopes the *Attach Construction* is obtained [24]: Let $(H,\cdot)$ be an H_v-semigroup and $v \notin H$. We extend the $(\cdot)$ into $\underline{H} = H \cup \{v\}$ by:

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

The $(\underline{H},\cdot)$ is an H_v-group, where $(\underline{H},\cdot)/\beta^* \cong \mathbf{Z}_2$ and v is a single.

A class of H_v-structures is the following [29], [30]:

Definition 1.10. Let $(G,\cdot)$ be groupoid (resp. hypergroupoid) and $f : G \to G$ be a map. We define a hope (∂), called *theta-hope*, we write *∂-hope*, on G as follows

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \quad \forall x,y \in G.$$

$$(\text{resp. } x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x,y \in G)$$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is COW, then ∂ is COW.

If $(G,\cdot)$ is a groupoid (or hypergroupoid) and $f : G \to P(G) - \{\varnothing\}$ be any multivalued map. We define the (∂), on G as follows $x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x,y \in G$

Motivation for the theta-hope is the map *derivative* where only the product of functions is used. Basic property: if $(G,\cdot)$ is semigroup then $\forall f$, the (∂) is WASS. **Example** (a) In integers $(\mathbf{Z},+,\cdot)$ fix $n \neq 0$, a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x, \forall x \in \mathbf{Z} - \{0\}$. Then $(\mathbf{Z},\partial_+,\partial.)$, where $\partial_+ and \partial.$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with

$$(\mathbf{Z},\partial_+,\partial.)/\gamma^* \cong \mathbf{Z}_n.$$

(b) In $(\mathbf{Z},+,\cdot)$ with $n \neq 0$, take f such that $f(n) = 0$ and $f(x) = x, \forall x \in \mathbf{Z}-\{n\}$. Then $(\mathbf{Z},\partial_+,\partial.)$ is an H_v-ring, moreover, $(\mathbf{Z},\partial_+,\partial.)/\gamma^* \cong \mathbf{Z}_n$.

Special case of the above is for n=p, prime, then $(\mathbf{Z},\partial_+,\partial.)$ is an H_v-field.

The uniting elements method was introduced by Corsini-Vougiouklis [2] in 1989. This leads, through hyperstructures, to structures satisfying additional properties.

The uniting elements method is the following: Let $\mathbf{G}$ be algebraic structure and d, a property which is not valid. Suppose that d is described by a set of equations; then, take the partition in $\mathbf{G}$ for which it is put together, in the same class, every pair of elements that causes the non-validity of the property d. The quotient by this partition G/d is an H_v-structure. Then, quotient out of the H_v-structure G/d by the fundamental relation β^*, is a stricter structure $(G/d)\beta^*$ for which d is valid, is obtained.

It is important if more properties are desired, then we have the following [20]:

Theorem 1.11. *Let $(R, +, \cdot)$ be a ring,*
and $F = \{f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n}\}$ be a system of equations on R consisting of two subsystems
$F_m = \{f_1, \ldots, f_m\}$ and $F_n = \{f_{m+1}, \ldots, f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the ring $R_m = (R/\sigma_m)/\gamma^*$. Then

$$(R/\sigma)/\gamma^* \cong (R_m/\sigma_n)/\gamma^*.$$

Combining the uniting elements procedure with the enlarging theory or the ∂-theory, we can obtain analogous results [30], [34], [37], [38].

Theorem 1.12. *In $(\mathbf{Z}_n, +, \cdot)$, with $n = ms$ we enlarge the multiplication only in the product of the special elements $0 \cdot m$ by setting $0 \otimes m = \{0, m\}$ and the rest results remain the same. Then*

$$(\mathbf{Z}_n, +, \otimes)/\gamma^* \cong (\mathbf{Z}_m, +, \cdot).$$

Remark that we can enlarge other products as well, for example $2 \cdot m$ by setting $2 \otimes m = \{2, m+2\}$, then the result remains the same. In this case 0 and 1 remain scalars.

Corollary. In the ring $(\mathbf{Z}_n, +, \cdot)$, with $n = ps$ where p is prime, we enlarge only the product $0 \cdot p$ by $0 \otimes p = \{0, p\}$ and the rest remain the same. Then $(\mathbf{Z}_n, +, \cdot)$ is very thin H_v-field.

2. h/v-groups, small H_v-fields

In the Definition 1.6, was introduced a new class of which is the following [26], [28]:

Definition 2.1. The H_v-semigroup $(H, \cdot)$ is called **h/v-group** if h/β^* is a group.

The class of h/v-groups is more general than the H_v-groups since in h/v-groups the reproductivity is not valid. The *reproductivity of classes* is valid instead, i.e. if H is partitioned into equivalence classes, then x[y]= [xy]= [x]y, $\forall x, y \in H$, since the quotient is reproductive. Similarly, *h/v-rings, h/v-fields, h/v-vector spaces* etc, are defined.

Remark 2.2. From definition of the H_v-field, we remark that the reproduction axiom in the product, is not assumed, the same is also valid for the definition of the h/v-field. Thus, an H_v-field is an h/v-field where the reproduction axiom for the sum is also valid.

The reproduction axiom in classical group theory is equivalent to the existence of the unit element and the existence of an inverse element for any given element. From the definition of the h/v-group, since a generalization of the reproductivity is valid, we have to extend the above two axioms on the equivalent classes.

Definition 2.3. Let $(H, \cdot)$ be an H_v-semigroup, and [x] the fundamental, or equivalent classes, of the element $x \in H$. We call **unit class** the class [e] if we have

$$([e] \cdot [x]) \cap [x] \neq \varnothing \text{ and } ([x] \cdot [e]) \cap [x] \neq \varnothing, \forall x \in H,$$

and for each element $x \in H$, we call **inverse class** of [x], the class $[x]^{-1}$, if we have

$$([x] \cdot [x]^{-1}) \cap [e] \neq \varnothing \text{ and } ([x]^{-1} \cdot [x]) \cap [e] \neq \varnothing.$$

The 'enlarged' hyperstructures were examined in the sense that a new element appears in one result. In enlargement or reduction, most useful are those H_v-structures or h/v-structures with the same fundamental structure [24], [34], [38].

construction 2.4. (a) Let $(H, \cdot)$ be an H_v-semigroup and $v \notin H$. We extend $(\cdot)$ into $\underline{H} = H \cup \{v\}$ as follows:

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

The $(\underline{H}, \cdot)$ is an h/v-group, called **attach**, where $(\underline{H}, \cdot)/\beta^* \cong \mathbf{Z}_2$ and v is single element. The core of $(\underline{H}, \cdot)$ is H. The scalars and units of $(H, \cdot)$ are scalars and units in $(\underline{H}, \cdot)$, as well. If $(H, \cdot)$ is COW (resp. commutative) then $(\underline{H}, \cdot)$ is COW (resp. commutative).

(b) Let $(H,\cdot)$ be H_v-semigroup and $\{v_1,\dots,v_n\}\cap H=\varnothing$, is an ordered set, where if $v_i<v_j$, when $i<j$. Extend $(\cdot)$ in $\underline{H}_n = H\cup\{v_1,\dots,v_n\}$ as follows:

$$x\cdot v_i = v_i\cdot x = v_i, v_i\cdot v_j = v_j\cdot v_i = v_j, \forall i<j \text{ and}$$

$$v_i\cdot v_i = H\cup\{v_1,\dots,v_{i-1}\}, \forall x\in H, i\in\{1,\dots,n\}.$$

Then $(\underline{H}_n,\cdot)$ is h/v-group, called **attach elements**, where $(\underline{H}_n,\cdot)/\beta^*\cong \mathbf{Z}_2$ and v_n is single.

(c) Let $(H,\cdot)$ be H_v-semigroup, $v\notin H$, and $(\underline{H},\cdot)$ be its attached h/v-group. Take an element $0\notin H$ and define in $\underline{H}_o = H\cup\{v,0\}$ two hopes:
hypersum $(+)$: $0+0=x+v=v+x=0$, $0+v=v+0=x+y=v$, $0+x=x+0=v+v=H$, $\forall x,y\in H$
hyperproduct $(\cdot)$: remains the same as in $\underline{H}$ moreover $0\cdot 0 = v\cdot x = x\cdot 0 = 0, \forall x\in \underline{H}$
Then $(\underline{H}_o,+,\cdot)$ is h/v-field with $(\underline{H}_o,+,\cdot)/\gamma^*\cong \mathbf{Z}_3$. $(+)$ is associative, $(\cdot)$ is WASS and weak distributive with respect to $(+)$. 0 is zero absorbing and single but not scalar in $(+)$. $(\underline{H}_o,+,\cdot)$ is called the **attached h/v-field** of the H_v-semigroup $(H,\cdot)$.

We present some results on the reproductivity. Let U be the set of all finite products of elements of a hypergroupoid $(H,\cdot)$. Consider the relation defined as follows:

$$xLy \text{ iff there exists } u\in U \text{ such that } ux\cap uy\neq\varnothing.$$

Then the transitive closure L^* of L is called *left fundamental reproductivity relation*. Similarly, the *right fundamental reproductivity relation* R^* is defined.

Theorem 2.5. *If $(H,\cdot)$ is a commutative semihypergroup, i.e. the strong commutativity and the strong associativity is valid, then the strong expression of the above L relation:*

$$ux = uy,$$

has the property: $L^*= L$.

Proof. Suppose that two elements x and y of H are L^* equivalent. Therefore, there are $u_1,\dots,u_{n+1}$ elements of U, and $z_1,\dots,z_n$ elements of H, such that

$$u_1x = u_1z_1, u_2z_1 = u_2z_2,\dots,u_nz_{n-1} = u_nz_n, u_{n+1}z_n = u_{n+1}y.$$

From these relations, using the strong commutativity, we obtain

$$u_{n+1}\dots u_2u_1x = u_{n+1}\dots u_2u_1z_1 = u_{n+1}\dots u_1u_2z_1 =$$

$$= u_{n+1}\dots u_2u_1z_2 = \dots = u_{n+1}\dots u_2u_1y$$

Therefore, setting

$$u = u_{n+1}\dots u_2u_1 \in U,$$

we have

$$ux = uy.$$

□

Corollary. Let $(S,\cdot)$ be commutative semigroup which has an element $w \in S$ such that the set wS is finite. Consider the transitive closure L^* of the relation L defined by:

$$xLy \text{ iff there exists } z \in S \text{ such that } zx = zy.$$

Then $< S/L^*,^\circ/\beta^*$ is finite commutative group, where $(^\circ)$ is induced on classes of S/L^*. Open problem: Prove that L^*, is the smallest equivalence: H/L^*, is reproductive.

We present now the small non-degenerate H_v-fields on $(\mathbf{Z}_n,+,\cdot)$ which satisfy the following conditions, appropriate in Santilli's iso-theory:

1. multiplicative very thin minimal,
2. COW (non-commutative),
3. they have the elements 0 and 1, scalars,
4. when an element has inverse element, then this is unique.

Remark that the last condition means than we cannot enlarge the result if it is 1 and we cannot put 1 in enlargement. Moreover we study only the upper triangular cases, in the Cayley table, since the corresponding under, are isomorphic since the commutativity is valid for the underline rings. From the fact that the reproduction axiom in addition is valid, we have always H_v-fields.

Theorem 2.6. *All H_v-fields defined on $(\mathbf{Z}_4,+,\cdot)$, which have non - degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*

The only product which is set is $2\otimes 3 = \{0,2\}$ or $3\otimes 2 = \{0,2\}$.

The fundamental classes are $[0] = \{0,2\}, [1] = \{1,3\}$ and we have

$$(\mathbf{Z}_4,+,\otimes)/\gamma^* \cong (\mathbf{Z}_2,+,\cdot).$$

Theorem 2.7. *All H_v-fields defined on $(\mathbf{Z}_6, +, \cdot)$, which have non - degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases: We have the only one hyperproduct,*

(I) $2 \otimes 3 = \{0, 3\}$ *or* $2 \otimes 4 = \{2, 5\}$ *and*
$3 \otimes 4 = \{0, 3\}$ *or* $3 \otimes 5 = \{0, 3\}$ *or* $4 \otimes 5 = \{2, 5\}$
Fundamental classes: $[0] = \{0, 3\}, [1] = \{1, 4\}, [2] = \{2, 5\}$, *and*
$(\mathbf{Z}_6, +, \cdot)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot)$.

(II) $2 \otimes 3 = \{0, 2\}$ *or* $2 \otimes 3 = \{0, 4\}$ *or* $2 \otimes 4 = \{0, 2\}$ *or* $2 \otimes 4 = \{2, 4\}$ *or*
$2 \otimes 5 = \{0, 4\}$ *or* $2 \otimes 5 = \{2, 4\}$ *or* $3 \otimes 4 = \{0, 2\}$ *or* $3 \otimes 4 = \{0, 4\}$ *or*
$3 \otimes 5 = \{3, 5\}$ *or* $4 \otimes 5 = \{0, 2\}$ *or* $4 \otimes 5 = \{2, 4\}$
Fundamental classes: $[0] = \{0, 2, 4\}, [1] = \{1, 3, 5\}$, *and*
$(\mathbf{Z}_6, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$.

Theorem 2.8. *All H_v-fields defined on $(\mathbf{Z}_9, +, \cdot)$, which have non - degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*
We have the only one hyperproduct,
$2 \otimes 3 = \{0, 6\}$ *or* $\{3, 6\}$, $2 \otimes 4 = \{2, 8\}$ *or* $\{5, 8\}$, $2 \otimes 6 = \{0, 3\}$ *or* $\{3, 6\}$,
$2 \otimes 7 = \{2, 5\}$ *or* $\{5, 8\}$, $2 \otimes 8 = \{1, 7\}$ *or* $\{4, 7\}$, $3 \otimes 4 = \{0, 3\}$ *or* $\{3, 6\}$,
$3 \otimes 5 = \{0, 6\}$ *or* $\{3, 6\}$, $3 \otimes 6 = \{0, 3\}$ *or* $\{0, 6\}$, $3 \otimes 7 = \{0, 3\}$ *or* $\{3, 6\}$,
$3 \otimes 8 = \{0, 6\}$ *or* $\{3, 6\}$, $4 \otimes 5 = \{2, 5\}$ *or* $\{2, 8\}$, $4 \otimes 6 = \{0, 6\}$ *or* $\{3, 6\}$,
$4 \otimes 8 = \{2, 5\}$ *or* $\{5, 8\}$, $5 \otimes 6 = \{0, 3\}$ *or* $\{3, 6\}$, $5 \otimes 7 = \{2, 8\}$ *or* $\{5, 8\}$,
$5 \otimes 8 = \{1, 4\}$ *or* $\{4, 7\}$, $6 \otimes 7 = \{0, 6\}$ *or* $\{3, 6\}$, $6 \otimes 8 = \{0, 3\}$ *or* $\{3, 6\}$,
$7 \otimes 8 = \{2, 5\}$ *or* $\{2, 8\}$,
Fundamental classes: $[0] = \{0, 3, 6\}$, $[1] = \{1, 4, 7\}$, $[2] = \{2, 5, 8\}$, *and*
$(\mathbf{Z}_9, +, \otimes)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot)$.

Theorem 2.9. *All H_v-fields defined on $(\mathbf{Z}_{10}, +, \cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*

(I) *We have the only one hyperproduct,*
$2 \otimes 4 = \{3, 8\}$, $2 \otimes 5 = \{2, 5\}$, $2 \otimes 6 = \{2, 7\}$, $2 \otimes 7 = \{4, 9\}$,
$2 \otimes 9 = \{3, 8\}$,
$3 \otimes 4 = \{2, 7\}$, $3 \otimes 5 = \{0, 5\}$, $3 \otimes 6 = \{3, 8\}$, $3 \otimes 8 = \{4, 9\}$,
$3 \otimes 9 = \{2, 7\}$,
$4 \otimes 5 = \{0, 5\}$, $4 \otimes 6 = \{4, 9\}$, $4 \otimes 7 = \{3, 8\}$, $4 \otimes 8 = \{2, 7\}$,
$5 \otimes 6 = \{0, 5\}$,
$5 \otimes 7 = \{0, 5\}$, $5 \otimes 8 = \{0, 5\}$, $5 \otimes 9 = \{0, 5\}$, $6 \otimes 7 = \{2, 7\}$,

$6 \otimes 8 = \{3, 8\}$,
$6 \otimes 9 = \{4, 9\}$, $7 \otimes 9 = \{3, 8\}$, $8 \otimes 9 = \{2, 7\}$.
Fundamental classes: $[0] = \{0, 5\}$, $[1] = \{1, 6\}$, $[2] = \{2, 7\}$, $[3] = \{3, 8\}$, $[4] = \{4, 9\}$ *and* $(\mathbf{Z}_{10}, +, \otimes)/\gamma^* \cong (\mathbf{Z}_5, +, \cdot)$.

(II) The cases where we have two classes $[0] = \{0, 2, 4, 6, 8\}$ *and* $[1] = \{1, 3, 5, 7, 9\}$*, thus we have fundamental field* $(\mathbf{Z}_{10}, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$*, can be described as follows:*
Taking in the multiplicative table only the results above the diagonal, we enlarge each of the products by putting one element of the same class of the results. We do not enlarge setting the element 1, and we cannot enlarge only the product $3 \otimes 7 = 1$. *The number of those* H_v*-fields is 103.*

Combining the uniting elements procedure with the enlarging theory we can obtain stricter structures or hyperstructures.

Theorem 2.10. *In the ring* $(\mathbf{Z}_n, +, \cdot)$*, with* $n = ms$ *we enlarge the multiplication only in the product of the special elements* $0 \cdot m$ *by setting* $0 \otimes m = \{0, m\}$ *and the rest results remain the same. Then* $(Zn, +, \otimes)$ *is an* H_v*-ring with*

$$(\mathbf{Z}_n, +, \otimes)/\gamma^* \cong (\mathbf{Z}_m, +, \cdot).$$

Proof. First, we remark that the only expressions of sums and products which contain, more than one, elements are the expressions which have at least one time the hyperproduct $0 \otimes m$. Adding to this special hyperproduct the 1, several times we have *modm* equivalence classes. On the other side, since m is a zero divisor, adding or multiplying elements of the same class the results are in one class. Therefore, γ^*-classes form a ring isomorphic to $(\mathbf{Z}_m, +, \cdot)$. □

Remark that we can enlarge other products as well, for example $2 \cdot m$ by setting $2 \otimes m = \{2, m+2\}$, then the result remains the same. In this case 0 and 1 remain scalars.

Corollary. In the ring $(\mathbf{Z}_n, +, \cdot)$, with $n = ps$ where p is prime, we enlarge only the product $0 \cdot p$ by $0 \otimes p = \{0, p\}$ and the rest remain the same. Then $(\mathbf{Z}_n, +, \otimes)$ is very thin H_v-field.

3. $\mathbf{H_v}$-representations

H_v-structures are used in Representation Theory of H_v-groups which can be achieved by generalized permutations [18] or by H_v-matrices [12],

[14], [19], [20], [23], [25], [27], [35]. The representations by generalized permutations can be faced by translations.

Definition 3.1. $\mathbf{H_v}$-matrix (or **h/v-matrix**) is a matrix with entries from an H_v-ring or h.v-ring or h/v-field. The hyperproduct of two H_v-matrices $= (a_{ij})$ and $= (b_{ij})$, of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner, and it is a set of $m \times r$ H_v-matrices. The sum of products of elements of the H_v-ring is considered to be the n-ary circle hope on the hyperaddition. The hyperproduct of H_v-matrices is not WASS.

The problem of the H_v-matrix (or h/v-group) representations is the following:

Definition 3.2. Let $(H, \cdot)$ be H_v-group (or h/v-group). Find H_v-ring (or h/v-ring) $(R, +, \cdot)$, a set M_R=$\{(a_{ij})|a_{ij}$$\in$$R\}$, and a map $T : H \to M_R : h \mapsto T(h)$ such that

$$T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

T is **$\mathbf{H_v}$-matrix (or h/v matrix) representation**. If

$$T(h_1h_2) \subset T(h_1)T(h_2), \forall h_1, h_2 \in H,$$

then $\boldsymbol{T}$ is an *inclusion representation*. If

$$T(h_1h_2) = T(h_1)T(h_2), \forall h_1, h_2 \in H,$$

then $\boldsymbol{T}$ is *good representation* and an induced one T^* for the hypergroup algebra is obtained.

The main theorem of the theory of representations is the following [19], [20]:

Theorem 3.3. *A necessary condition in order to have an inclusion representation T of an h/v-group $(H, \cdot)$ by $n \times n$ h/v-matrices over the h/v-ring $(R, +, \cdot)$ is the following:*
For all classes $\beta^*(x)$, $x \in H$ there must exist elements $a_{ij} \in H, i, j \in \{1, ..., n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij})|a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, ..., n\}\}$$

Thus, inclusion representation $T : H \to M_R : a \mapsto T(a) = (a_{ij})$ induces an homomorphic T^* of H/β^* over R/γ^* by setting $T^*(\beta^*(a)) = [\gamma^*(a_{ij})], \forall\beta^*(a)H/\beta^*$, where the $\gamma^*(a_{ij})R/\gamma^*$ is the ij entry of $T^*(\beta^*(a))$. T^* is called fundamental induced representation of T.

Let T a representation of an h/v-group H by h/v-matrices and $tr_\phi(T(x)) = \gamma^*(Tx_{ii})$ be the fundamental trace, then is called *fundamental character*, the mapping

$$X_T : H \to R/\gamma^* : x \mapsto X_T(x) = tr_\phi(T(x)) = trT^*(x)$$

In representations of H_v-groups there are two difficulties: First to find an H_v-ring or an H_v-field and second, an appropriate set of H_v-matrices.

We now give a definition of a set of h/v-matrices which is closed only under the product of matrices, not under the sum.

Definition 3.4. We call an h/v-matrix $A = (a_{ij}) \in M_{n\times n}$**$\mathbf{U_0}$-matrix** if it is upper triangular where the condition $[a_{11}]\cdot[a_{22}]\cdots[a_{nn}] \neq [0]$, is valid, where $[x]$ is the fundamental class of x. We notice that this set is closed under the product only, not in addition. Thus, it is interesting only when, in the h/v-matrix representations, the product is used. Moreover, we define the product of an element with a class to be the corresponding class.

Theorem 3.5. *In the case of* 3×3 U_0*-matrices the unit* U_0*-matrices are*

$$\mathbf{I} = [1]E_{11} + [1]E_{22} + [1]E_{33}$$

and the inverses of the U_0*-matrix* $A = (a_{ij}) \in M_{3\times 3}$*, are given as follows:*

$$\begin{pmatrix} [a_{11}^{-1} & -(a_{11})^{-1}a_{12}[a_{22}]^{-1} & (a_{11})^{-1}a_{12}(a_{22})^{-1}a_{23}[a_{33}]^{-1} - (a_{11})^{-1}a_{13}[a_{33}]^{-1} \\ 0 & [a_{22}^{-1} & -(a_{22})^{-1}a_{23}[a_{33}]^{-1} \\ 0 & 0 & [a_{33}]^{-1} \end{pmatrix}$$

Proof. Let us denote by E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. First it is clear that all the h/v-matrices of the set $\mathbf{I} = [1]E_{11} + [1]E_{22} + [1]E_{33}$, are unit h/v-matrices in the sense that $(\mathbf{I}\cdot A \cup A\cdot \mathbf{I}) \subset [A] = ([a_{ij}])$.

In order to find the inverse of the given h/v-matrix

$$A = (a_{ij}) = a_{11}E_{11} + a_{22}E_{22} + a_{33}E_{33} + a_{12}E_{12} + a_{13}E_{13} + a_{23}E_{23}$$

we have to find the set of h/v-matrices

$$\mathbf{X} = ([x_{ij}]) = [x_{11}]E_{11}+[x_{22}]E_{22}+[x_{33}]E_{33}+[x_{12}]E_{12}+[x_{13}]E_{13}+[x_{23}]E_{23}$$

such that $A\cdot\mathbf{X} = \mathbf{X}\cdot A = I$, therefore we must have, on the one side and taking into accound that the product of one element with a fundamental class is considered the entire class,

$$a_{11}[x_{11}] = \mathbf{1}, a_{22}[x_{22}] = \mathbf{1}, a_{33}[x_{33}] = \mathbf{1},$$

$$a_{11}[x_{12}] + a_{12}[x_{22}] = 0, a_{11}[x_{13}] + a_{12}[x_{23}] + a_{13}[x_{33}] = 0,$$

$$a_{22}[x_{23}] + a_{23}[x_{33}] = 0.$$

Therefore we have

$$[x_{11}] = [a_{11}]^{-1}, [x_{22}] = [a_{22}]^{-1}, [x_{33}] = [a_{33}]^{-1},$$

and then,

$$[x_{12}] = -(a_{11})^{-1}a_{12}[a_{22}]^{-1}, [x_{23}] = -(a_{22})^{-1}a_{23}[a_{33}]^{-1},$$

$$[x_{13}] = (a_{11})^{-1}a_{12}(a_{22})^{-1}a_{23}[a_{33}]^{-1} - (a_{11})^{-1}a_{13}[a_{33}]^{-1}.$$

□

Example. Consider the h/v-field $(\mathbf{Z}_{10}, +, \otimes)$ where only $3 \otimes 8 = \{4, 9\}$ is a hyperproduct. Let us take the h/v-matrix

$$A = 3E_{11} + E_{22} + 2E_{33} + 6E_{12} + 2E_{13} + 9E_{23}$$

Then from the above formulas we obtain that the set of inverse h/v-matrices is

$$A^{-1} = [2]E_{11} + [1]E_{22} + [3]E_{33} + [3]E12 + [2]E_{13} + [3]E_{23}$$

So, for example, if we take the h/v-matrix

$$A^{-1} = 7E_{11} + 6E_{22} + 8E_{33} + 8E_{12} + 2E_{13} + 3E_{23}$$

we obtain that

$$A \cdot A^{-1} = E_{11} + E_{22} + E_{33} + \{0, 5\}E_{12} + 5E_{23}$$

therefore, it contains a unit h/v-matrix.

4. **Applications**

Last decades H_v-structures have applications in other branches of mathematics and in other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons to mention but a few. The H_v-structure theory is related to fuzzy theory; consequently, hyperstructures can be widely applicable in industry and production, too [1], [4], [5], [9], [36]. Moreover new classes and new hopes were introduced to face the new problems [7], [10], [29], [31], [40].

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The isofields needed, correspond into the hyperstructures were introduced by Santilli & Vougiouklis in 1996 [10], called *e-hyperfields*, [6], [31].

A hyperstructure $(H, \cdot)$ which contain a unique scalar unit e, and for all x, there exists an inverse x^{-1}, i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$, is called *e-hyperstructure.*

Definition 4.1. A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called *e-hyperfield* if the following axioms are valid: $(F, +)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributiv to $(+)$, 0 is absorbing: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$, there exist a multiplicative scalar unit 1: $1 \cdot x = x \cdot 1 = x, \forall x \in F$, and there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers.* If the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Definition 4.2. *The Main e-Construction.* Given a group $(G, \cdot)$, where e is the unit, then we define in G, an extreme large number of hopes $(\otimes)$ as follows:

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

$(G, \otimes)$ is an H_b-group which contains the $(G, \cdot)$, $(G, \otimes)$ is an e-hypergroup. Moreover, if $\forall x, y$ we have $xy = e$, hen $(G, \otimes)$ is a strong e-hypergroup.

Example. Consider the quaternions $\mathbf{Q} = \{1, -1, i, -i, j, -j, k, -k\}$ with $i^2 = j^2 = -1, ij = -ji = k$ and denote $\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}$. We define a lot of hopes $(*)$ by enlarging few products. For example, $(-1) * k = \underline{k}, k * i = \underline{j}$ and $i * j = \underline{k}$. Then $(Q, *)$ is strong e-hypergroup.

We present the Lie-Santilli admissibility on non square matrices [6], [9], [36].

construction 4.3. Let $(L = M_{m \times n}, +)$ be H_v-vector space of $m \times n$ hyper-matrices on H_v-field $(F, +, \cdot)$, $\phi : F \to F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the field F/γ^*. Similarly, let ω_L be the core of the canonical map $\phi' : L \to L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ^*. Take any two subsets $R, S \subseteq L$ then a *Santilli's Lie-admissible hyperalgebra* is obtained by taking the Lie bracket, which is a hope:

$$[,]_{RS} : L \times L \to P(L) : [x, y]_{RS} = xR'y - yS'x = \{xr'y - ys'x | r \in R \text{ and } s \in S\}$$

An application, combining hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [39].

Definition 4.4. In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:

0 ________________ 1

The subjects/participants are asked instead of deciding and checking a grade on the scale, to cut the bar at any point they feel expresses her/his answer to the specific question

The use of the Vougiouklis & Vougiouklis bar instead of a Likert scale has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm.

References

[1] P. Corsini, V. Leoreanu, *Application of Hyperstructure Theory*, Klower Ac. Publ., (2003).

[2] P. Corsini, T. Vougiouklis, *From groupoids to groups through hypergroups*, Rend. Mat. VII, **9**, (1989), 173–181.

[3] B. Davvaz, *On H_v-rings and Fuzzy H_v-ideals*, J. Fuzzy Math.**V.6,N.1**, (1998), 33–42.

[4] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific, 2013.

[5] B. Davvaz, V. Leoreanu-Fotea, *Hyperring Theory and Applications*, Int. Acad. Press, USA, 2007.

[6] B. Davvaz, R.M. Santilli, T. Vougiouklis, Algebra, *Hyperalgebra and Lie-Santilli Theory*, J. Generalized Lie Theory Appl., (2015), **9:2**, 1–5.

[7] Davvaz B., Vougioukli S., Vougiouklis T., *On the multiplicative H_v-rings derived from helix hyperoperations*, Util. Math., **84**, (2011), 53–63.

[8] M. Koskas, *Groupoides demi-hypergroupes et hypergroupes*, J. Math. Pures Appl., **49 (9)**, 1970, 155–192.

[9] R.M. Santilli, *Hadronic Mathematics, Mechanics and Chemistry*, Volumes I, II, III, IV and V, Int. Academic Press, USA, (2007).

[10] R.M. Santilli, T. Vougiouklis, *Isotopies, Genotopies, Hyperstructures and their Applications*, New frontiers in Hyperstructures, Hadronic, (1996), 1–48.

[11] T. Vougiouklis, *Cyclicity in a special class of hypergroups*, Acta Un. Car.-Math. Et Ph., **V.22, N1**, (1981), 3–6.

[12] T. Vougiouklis, *Representations of hypergroups*, Hypergroup algebra, Proc. Convegno: Ipergrouppi, altre strutture, Udine, (1985), 59–73.

[13] T. Vougiouklis, *Generalization of P-hypergroups*, Rend. Circolo Mat. Palermo, Ser.II, **36**, (1987), 114–121.

[14] T. Vougiouklis, *Representations of hypergroups by hypermatrices*, Rivista Mat. Pura Appl., **N 2**, (1987), 7–19.
[15] T. Vougiouklis, *Groups in hypergroups*, Annals Discrete Math.**37**, (1988), 459-4-68.
[16] T. Vougiouklis, *The very thin hypergroups and the S-construction*, Combinatorics'88, Inc. Geom. Comb. Str., **2**, (1991), 471–477.
[17] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, 4^{th} AHA, Xanthi 1990, World Scientific, (1991), 203–211.
[18] T. Vougiouklis, *Representations of hypergroups by generalized permutations*, Algebra Universalis, **29**, (1992), 172–183.
[19] T. Vougiouklis, *Representations of H_v-structures*, Proc. Int. Conf. Group Theory 1992, Timisoara, 1993, 159–184.
[20] T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press Inc, 1994.
[21] T. Vougiouklis, *Some remarks on hyperstructures*, Contemporary Math., Amer. Math. Society, **184**, (199)5, 427–431.
[22] T. Vougiouklis, *H_v-groups defined on the same set*, Discrete Math., **155**, (1996), 259–265.
[23] T. Vougiouklis, *Convolutions on WASS hyperstructures*, Discrete Math., **174**, (1997), 347–355.
[24] T. Vougiouklis, *Enlarging H_v-structures*, Algebras Comb., ICAC'97, Hong Kong, Springer, (1999), 455–463.
[25] T. Vougiouklis, *On H_v-rings and H_v-representations*, Discrete Math., Elsevier, **208/209**, (1999), 615–620.
[26] T. Vougiouklis, *H_v-Lie algebras and h/v-Lie algebras*, Honorary I. Mittas, Aristotle Un. Thessaloniki, (2000), 619–628.
[27] T. Vougiouklis, *Finite H_v-structures and their representations*, Rend. Sem. Mat. Messina, S II, **V9**, (2003), 245–265.
[28] T. Vougiouklis, *The h/v-structures*, J. Discrete Math. Sci. Cryptography, **V.6**, (2003), N.2-3, 235-243.
[29] T. Vougiouklis, *A hyperoperation defined on a groupoid equipped with a map*, Ratio Mat., **N.1**, (2005), 25–36.
[30] T. Vougiouklis, *∂-operations and H_v-fields*, Acta Math. Sinica, English S., **V.23, 6**, (2008), 965–972.
[31] T. Vougiouklis, *The e-hyperstructures*, J. Mahani Math. Research Center, **V.1, N.1**, 2012, 13–28.
[32] T. Vougiouklis, *The Lie-hyperalgebras and their fundamental relations*, Southeast Asian Bull. Math., **V.37(4)**, (2013), 601–614.
[33] T. Vougiouklis, *From H_v-rings to H_v-fields*, Int. J. Alg. Hyp. Appl. **Vol.1, No.1**, 2014, 1–13.
[34] T. Vougiouklis, *Enlarged Fundamentally Very Thin H_v-structures*, J. Algebraic Str. Their Appl. (ASTA), **V.1, No1**, (2014), 11-20.
[35] T. Vougiouklis, *Hypermatrix representations of finite H_v-groups*, European J. Comb., **V.44 B**, (2015), 307–315.
[36] T. Vougiouklis, *Hypermathematics, H_v-structures, hypernumbers, hypermatrices and Lie-Santilli admissibility*, American J. Modern Physics,4(5), (2015), 34–46.

[37] T. Vougiouklis, *On the Hyperstructure Theory*, Southeast Asian Bull. Math., **Vol. 40(4)**, 2016, 603–620.
[38] T. Vougiouklis, *Hypernumbers, Finite Hyper-fields*, AGG, **V.33, N.4**, (2016), 471-490.
[39] T. Vougiouklis, P. Kambakis-Vougiouklis, Bar in Questionnaires, Chinese Business Review, **V.12, N.10**, (2013), 691–697.
[40] T. Vougiouklis, S. Vougiouklis, *Hyper-representations by non square matrices. Helix-hopes*, American J. Modern Physics, **4(5)**, (2015), 52–58.

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