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HH^*- INTUITIONISTIC HEYTING VALUED Ω-ALGEBRA AND HOMOMORPHISM

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Abstract. Intuitionistic Logic was introduced by L. E. J. Brouwer in[1] and Heyting algebra was defined by A. Heyting to formalize the Brouwer's intuitionistic logic[4]. The concept of Heyting algebra has been accepted as the basis for intuitionistic propositional logic. Heyting algebras have had applications in different areas. The co-Heyting algebra is the same lattice with dual operation of Heyting algebra[5]. Also, co-Heyting algebras have several applications in different areas.

In this paper, we introduced the new concept HH^*- Intuitionistic Heyting Valued Ω-Algebra. The purpose of introducing this new concept is to expand the field of researchers' area using both membership degree and non-membership degree. This allows us to get more sensitive results.The HH^*- Intuitionistic Heyting valued set, HH^*- Intuitionistic Heyting valued relation, HH^*- Intuitionistic Heyting valued Ω-algebra and the homomorphism over HH^*- Intuitionistic Heyting valued Ω-algebra were defined.

Key Words: Heyting Valued Algebra, co-Heyting Valued Algebra, Omega Algebra, Intuitionistic Logic.

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1. Introduction

A Heyting algebra is a lattice expanded with a implication operation " $\rightarrow$ ".In 1930, the concept of Heyting algebra introduced by A. Heyting[4] as following,

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Definition 1.1. [4]A Heyting algebra is an algebra $(H, \vee, \wedge, \rightarrow, 0_H, 1_H)$ such that $(H, \vee, \wedge, 0, 1)$ is an lattice and for all $a, b, c \in H$,

$$a \leq b \rightarrow c \Leftrightarrow a \wedge b \leq c$$

$(H, \vee, \wedge, 0_H, 1_H)$ is a Heyting algebra with $\forall a, b \in H$,

$$a \rightarrow b = \bigvee \{c : a \wedge c \leq b, c \in H\}.$$

The notion of co-Heyting algebra for a Heyting algebra defined in [5].

Definition 1.2. [5]A co-Heyting algebra is an algebra $(H^*, \vee, \wedge, \hookrightarrow, 0_{H^*}, 1_{H^*})$ such that $(H^*, \vee, \wedge, 0_{H^*}, 1_{H^*})$ is an lattice and for all $a, b, c \in H^*$,

$$a \hookrightarrow b \leq c \Leftrightarrow a \leq b \vee c$$

$(H^*, \vee, \wedge, 0_{H^*}, 1_{H^*})$ is a co-Heyting algebra with $\forall a, b \in H^*$,

$$a \hookrightarrow b = \bigwedge \{c : a \vee c \geq b, c \in H^*\}.$$

The implication operation in co-Heyting algebra is define $a \rightarrow b = b \hookrightarrow a$.

Let $(H, \vee, \wedge, \rightarrow, 0_H, 1_H)$ be a Heyting algebra and $(H^*, \vee, \wedge, \hookrightarrow, 0_{H^*}, 1_{H^*})$ is a co-Heyting algebra of $(H, \vee, \wedge, \rightarrow, 0_H, 1_H)$.

$L = H \times H^*$ is a lattice with $(x_1, x_2) \leq_L (y_1, y_2) \Leftrightarrow x_1 \leq_H y_1$ and $x_2 \geq_{H^*} y_2$.For $x, y \in L, x = (x_1, x_2)$ and $y = (y_1, y_2)$, the operators $\wedge$ and $\vee$ on $(L, \leq_L)$ are defined as following;

$$\begin{aligned} x \wedge y &= (\min\{x_1, y_1\}, \max\{x_2, y_2\}) \\ x \vee y &= (\max\{x_1, y_1\}, \min\{x_2, y_2\}) \end{aligned}$$

$0_L = (0_H, 1_{H^*})$ and $1_L = (1_H, 0_{H^*})$ are called greatest and least element of L, respectively.

The $\rightarrow_L$ is a binary operation on L with $x \rightarrow_L y = (x_1 \rightarrow_H y_1, y_2 \hookrightarrow_{H^*} x_2)$.

Definition 1.3. A HH^*-Intuitionistic Heyting algebra is an algebra $(L, \vee, \wedge, \rightarrow_L, 0_L, 1_L)$ such that $(L, \vee, \wedge, 0_L, 1_L)$ is an lattice as defined above and for all $a, b, c \in L$,

$$a \leq_L b \rightarrow_L c \Leftrightarrow a \wedge b \leq_L c.$$

The $(L, \vee, \wedge, 0_L, 1_L)$ lattice is a HH^*-Intuitionistic Heyting algebra with $\forall a, b \in L, a = (a_1, a_2)$ and $b = (b_1, b_2)$,

$$\begin{aligned} a &\rightarrow {}_L b = \bigvee \{c : a \wedge c \leq_L b, c \in L\} \\ &= \left(\bigvee \{c_1 : a_1 \wedge c_1 \leq_H b_1, c_1 \in H\}, \bigwedge \{c_2 : b_2 \vee c_2 \geq a_2, c_2 \in H^*\} \right) \end{aligned}$$

Proposition 1.4. *An algebra* $(L, \vee, \wedge, \rightarrow_L, 0_L, 1_L)$ *is a* HH^*-*Intuitionistic Heyting algebra if and only if* $(L, \vee, \wedge, 0_L, 1_L)$ *is an lattice and the following identities hold for all* $a, b, c \in L$,

(1) $a \rightarrow_L a = 1_L$
(2) $a \wedge (a \rightarrow_L b) = a \wedge b$
(3) $b \wedge (a \rightarrow_L b) = b$
(4) $a \rightarrow_L (b \wedge c) = (a \rightarrow_L b) \wedge (a \rightarrow_L c)$

Proof. (1) $\forall\ a \in L$,

$$a \rightarrow_L a = (a_1 \rightarrow_H a_1, a_2 \hookrightarrow_{H^*} a_2) = (1_H, 0_{H^*}) = 1_L$$

(2) From definition it is obtained that,

$$\begin{aligned} a \wedge (a &\rightarrow {}_L b) = (a_1 \wedge (a_1 \rightarrow_H b_1), a_2 \vee (b_2 \hookrightarrow_{H^*} a_2)) \\ &= (a_1 \wedge b_1, a_2 \vee b_2) = a \wedge b \end{aligned}$$

(3)$\forall a, b \in L$,

$$b \wedge (a \rightarrow_L b) = (b_1 \wedge (a_1 \rightarrow_H b_1), b_2 \vee (b_2 \hookrightarrow_{H^*} a_2)) = b$$

(4)$\forall a, b, c \in L$,

$$\begin{aligned} a &\rightarrow {}_L (b \wedge c) = (a_1 \rightarrow_H (b_1 \wedge c_1), (b_2 \wedge c_2) \hookrightarrow_{H^*} a_2) \\ &= ((a_1 \rightarrow_H b_1) \wedge (a_1 \rightarrow_H c_1), (b_2 \hookrightarrow_{H^*} a_2) \vee (c_2 \hookrightarrow_{H^*} a_2)) \\ &= (a \rightarrow_L b) \wedge (a \rightarrow_L c) \end{aligned}$$

□

2. HH^*-Intuitionistic Valued Sets

In this section, firstly we introduced the concept of HH^*-Intuitionistic Valued Set and HH^*-Intuitionistic Valued Function. Then, we defined HH^*-Intuitionistic valued equivalence relation and equivalence class. Some properties of these concepts were examined.

Definition 2.1. Let H be a complete Heyting algebra and H^* be a complete co-Heyting algebra of H. Let X be a universal and $L = H \times H^*$ then HH^*-Intuitionistic valued set is determined with $[=]$ function

$$[=]_L : X \times X \to L, [=]_L (a, b) = ([a = b]_H , [a = b]_{H^*})$$

which satisfy the following conditions.

(1) $[a = b]_L \leq_L [b = a]_L$
(2) $[a = b]_L \wedge [b = c]_L \leq_L [a = c]_L$

If X a universal and $[=]_L$ is a function satisfy the above conditions then X called HH^*-Intuitionistic valued set and it is shown $(X, =_L)$.

Let X be a universal. $u \in X, E(u)$ means the degree of existence the elemet u. For HH^*-Intuitionistic valued set we will use,

$$E(u) = [u \in X]_L .$$

So, $[u \in X]_L = [u = u]_L$.

Definition 2.2. Let A be a HH^*-Intuitionistic valued set. The subset of A is a $s : A \to L$ function with following conditions.

(1) $[x \in s]_L \wedge [x = y]_L \leq_L [y \in s]_L$
(2) $[x \in s]_L \leq_L [x \in A]_L$

Definition 2.3. Let $(X, =_L)$ and $(Y, =_L)$ are HH^*-Intuitionistic valued sets. If $f_L : X \times Y \to L$ function satisfy the following conditions then called HH^*-Intuitionistic valued function and it is shown $f_L : X \to Y$.

F1 $f_L(x, y) \leq_L [x = x]_L \wedge [y = y]_L$
F2 $[x = x']_L \wedge f_L(x, y) \wedge [y = y']_L \leq_L f_L(x', y')$
F3 $f_L(x, y) \wedge f_L(x, y') \leq_L [y = y']_L$
F4 $[x = x]_L \leq_L \bigvee \{f_L(x, y) : y \in Y\}$

Notation 2.4. $f_L(x, y) := [f_L(x) = y]_L = ([f_L(x) = y]_H , [f_L(x) = y]_{H^*})$

Definition 2.5. Let $(X, =_L)$ be an HH^*-Intuitionistic valued set. $I : X \times X \to L$, $I(x, x') = [x = x']_L$ function is called unit function.

Definition 2.6. Let $(X, =_L), (Y, =_L)$ and $(Z, =_L)$ are HH^*-Intuitionistic valued sets and $f_L : X \to Y$, $g_L : Y \to Z$ are HH^*-Intuitionistic valued functions. For $x \in X, z \in Z$,

$$(g \circ f)_L (x, z) = \bigvee \{f_L(x, y) \wedge g_L(y, z) : y \in Y\} .$$

Proposition 2.7. *Let* $(X,=_L), (Y,=_L)$ *and* $(Z,=_L)$ *be* HH^*-*Intuitionistic valued sets and* $f_L : X \to Y$, $g_L : Y \to Z$ *are* HH^*-*Intuitionistic valued functions. The function* $(g \circ f)_L : X \to Z$ *is a* HH^*-*Intuitionistic valued function.*

Proof. (i) Let $x \in X, z \in Z, (g \circ f)_L(x,z) = \bigvee \{f_L(x,y) \wedge g_L(y,z) : y \in Y\}.$

$$\begin{aligned}\bigvee \{f_H(x,y) \wedge g_H(y,z) : y \in Y\} &\leq \bigvee \{[x=x]_H \wedge [y=y]_H \wedge [z=z]_H : y \in Y\} \\ &= [x=x]_H \wedge [z=z]_H \wedge \bigvee \{[y=y]_H : y \in Y\} \\ &\leq [x=x]_H \wedge [z=z]_H\end{aligned}$$

and

$$\begin{aligned}\bigwedge \{f_{H^*}(x,y) \vee g_{H^*}(y,z) : y \in Y\} &\geq \{[x=x]_{H^*} \vee [y=y]_{H^*} \vee [z=z]_{H^*} : y \in Y\} \\ &= [x=x]_{H^*} \vee [z=z]_{H^*} \vee \{[y=y]_{H^*} : y \in Y\} \\ &\geq [x=x]_{H^*} \vee [z=z]_{H^*}\end{aligned}$$

So, $(g \circ f)_L(x,z) \leq_L [x=x]_L \wedge [z=z]_L$

(ii) Let $x, x' \in X$ and $z, z' \in Z$,

$[x=x']_L \wedge (g \circ f)_L(x,z) \wedge [z=z']_L = [x=x']_L \wedge \bigvee \{f_L(x,y) \wedge g_L(y,z) : y \in Y\} \wedge [z=z']_L.$

Firstly,

$[x=x']_H \wedge \bigvee \{f_H(x,y) \wedge g_H(y,z) : y \in Y\} \wedge [z=z']_H$
$= \bigvee \{[x=x']_H \wedge f_H(x,y) \wedge g_H(y,z) \wedge [z=z']_H : y \in Y\}$
$= \bigvee \{[x=x']_H \wedge f_H(x,y) \wedge [y=y]_H \wedge g_H(y,z) \wedge [z=z']_H : y \in Y\}$
$\leq \bigvee \{f_H(x',y') \wedge g_H(y',z') : y \in Y\}$

on the other hand,

$[x=x']_{H^*} \vee \{f_{H^*}(x,y) \vee g_{H^*}(y,z) : y \in Y\} \vee [z=z']_{H^*}$
$= \{[x=x']_{H^*} \vee f_{H^*}(x,y) \vee g_{H^*}(y,z) \vee [z=z']_{H^*} : y \in Y\}$
$= \{[x=x']_{H^*} \vee f_{H^*}(x,y) \vee [y=y]_{H^*} \vee g_{H^*}(y,z) \vee [z=z']_{H^*} : y \in Y\}$
$\geq \{f_{H^*}(x',y') \vee g_{H^*}(y',z') : y' \in Y\}$

Therfore, $[x=x']_L \wedge (g \circ f)_L(x,z) \wedge [z=z']_L \leq (g \circ f)_L(x',z')$

(iii) Let $x \in X$ and $z, z' \in Z$,

$(g \circ f)_L(x,z) \wedge (g \circ f)_L(x,z')$
$= \bigvee \{f_L(x,y) \wedge g_L(y,z) : y \in Y\} \wedge \bigvee \{f_L(x,t) \wedge g_L(t,z') : t \in Y\}$

Now,

$\bigvee \{f_H(x,y) \wedge g_H(y,z) : y \in Y\} \wedge \bigvee \{f_H(x,t) \wedge g_H(t,z') : t \in Y\}$
$= \bigvee \{f_H(x,y) \wedge f_H(x,y) \wedge g_H(y,z) \wedge g_H(y,z') : y \in Y\}$
$= \bigvee \{[y=y]_H \wedge [z=z']_H : y \in Y\}$
$\leq [z=z']_H$

and
$\{f_{H^*}(x,y) \vee g_{H^*}(y,z) : y \in Y\} \vee \{f_{H^*}(x,t) \vee g_{H^*}(t,z') : t \in Y\}$
$= \{f_{H^*}(x,y) \vee f_{H^*}(x,y) \vee g_{H^*}(y,z) \vee g_{H^*}(y,z') : y \in Y\}$
$\geq \{[y=y]_{H^*} \vee [z=z']_{H^*} : y \in Y\}$
$\geq [z=z']_{H^*}$
(iv) Let $x \in X$, then
$\bigvee\{(g\circ f)_L(x,z) : z \in Z\} = \bigvee\{\bigvee\{f_L(x,y) \wedge g_L(y,z) : y \in Y\} : z \in Z\}$

$$
\begin{aligned}
& \bigvee\left\{\bigvee\{f_H(x,y) \wedge g_H(y,z) : y \in Y\} : z \in Z\right\} \\
= \;& \bigvee\left\{\bigvee\{f_H(x,y) : y \in Y\} : z \in Z\right\} \wedge \bigvee\left\{\bigvee\{g_H(y,z) : y \in Y\} : z \in Z\right\} \\
\geq \;& [x=x]_H \wedge \bigvee\left\{\bigvee\{g_H(y,z) : y \in Y\} : z \in Z\right\} \\
= \;& [x=x]_H \wedge \bigvee\left\{\bigvee\{g_H(y,z) : z \in Z\} : y \in Y\right\} \\
\geq \;& [x=x]_H \wedge \bigvee\{[y=y]_H : y \in Y\}
\end{aligned}
$$

and

$$
\begin{aligned}
& \{\{f_{H^*}(x,y) \vee g_{H^*}(y,z) : y \in Y\} : z \in Z\} \\
= \;& \{\{f_{H^*}(x,y) : y \in Y\} : z \in Z\} \vee \{\{g_{H^*}(y,z) : y \in Y\} : z \in Z\} \\
\leq \;& [x=x]_{H^*} \vee \{\{g_{H^*}(y,z) : y \in Y\} : z \in Z\} \\
= \;& [x=x]_{H^*} \vee \{\{g_{H^*}(y,z) : z \in Z\} : y \in Y\} \\
\leq \;& [x=x]_{H^*} \vee \bigvee\{[y=y]_{H^*} : y \in Y\}
\end{aligned}
$$

So, $\bigvee\{(g\circ f)_L(x,z) : z \in Z\} \geq [x=x]_L$ □

Definition 2.8. Let $(X,=_L)$ and $(Y,=_L)$ are HH^*-Intuitionistic valued sets and $f_L : X \to Y$ is HH^*-Intuitionistic valued function.

(1) f_L is a monomorphism. $\Leftrightarrow \forall x, x' \in X, y \in Y$,

$$f_L(x,y) \wedge f_L(x',y) \leq [x=x']$$

(2) $f_L$is a epimorphism. $\Leftrightarrow \forall y \in Y$,

$$[y=y']_L \leq \{f_L(x,y) : x \in X\}$$

Definition 2.9. Let $(X,=_L)$ be a HH^*-Intuitionistic valued sets. $R_L : X \times X \to L$ is called HH^*-Intuitionistic valued equivalence relation if and only if

R1 $R_L(x,y) \wedge [x=x]_L = R_L(x,y), R_L(x,y) \wedge [y=y]_L = R_L(x,y)$
R2 $R_L(x,y) \wedge [x=x']_L \leq R_L(x',y), R_L(x,y) \wedge [y=y']_L \leq R_L(x,y')$

R3 $[x=x]_L \leq R_L(x,x)$:
R4 $R_L(x,y) \leq R_L(y,x)$
R5 $R_L(x,y) \vee R_L(y,z) \leq R_L(x,z)$

Example 2.10. Let $(X,=_L),(Y,=_L)$ are HH^*-Intuitionistic valued sets and $f_L : X \to Y$ is HH^*-Intuitionistic valued function.$\forall x_1, x_2 \in X$,

$$\begin{aligned} C_f(x_1,x_2) &= [f_L(x_1)=f_L(x_2)]_L \\ &= ([f_H(x_1)=f_H(x_2)]_H, [f_{H^*}(x_1)=fH^*(x_2)]_{H^*}) \end{aligned}$$

function is a HH^*-Intuitionistic valued equivalence relation on X.

Definition 2.11. Let R_L be a HH^*-Intuitionistic valued equivalence relation on X. $d_L : X \to L$ is called equivalence class of $R_L \Leftrightarrow$

d1 $d_L(x) \wedge R_L(x,x') \leq d_L(x')$
d2 $d_L(x) \wedge d_L(y) \leq R_L(x,y)$

$d_L(x)$ is the equivalence class of $x \in X$.

Proposition 2.12. *Let R_L be a HH^*-Intuitionistic valued equivalence relation on X and σ_L, τ_L are equivalence class of R_L.*

$$\begin{aligned} \bigvee \{\sigma_L(x) : x \in X\} &= \bigvee \{\tau_L(x) : x \in X\} \text{ and} \\ \sigma_L(x) &\leq \tau_L(x) \Rightarrow \sigma_L = \tau_L \end{aligned}$$

Proof. For $x_0 \in X, \tau_L(x_0) = \bigvee \{\tau_L(x) \wedge \sigma_L(x) : x \in X\}$.

$$\begin{aligned} \bigvee \{\tau_H(x) \wedge \sigma_H(x) : x \in X\} &\leq \bigvee \{R_H(x_0,x) \wedge \sigma_H(x)\} \\ &\leq \sigma_H(x_0) \end{aligned}$$

and

$$\begin{aligned} \{\tau_{H^*}(x) \vee \sigma_{H^*}(x) : x \in X\} &\geq \{R_{H^*}(x_0,x) \vee \sigma_{H^*}(x) : x \in X\} \\ &\geq \sigma_{H^*}(x_0) \end{aligned}$$

□

Proposition 2.13. *Let $(X,=_L),(Y,=_L)$ are HH^*-Intuitionistic valued sets and $f_L : X \to Y$ is HH^*-Intuitionistic valued function.f_L is surjective $\Leftrightarrow \forall y \in Y$,*

$$\bigvee \{[f_L(x)=y] : x \in X\} = [y=y]_L$$

3. HH^*-Intuitionistic Valued $\Omega-$Algebras

To create HH^*-Intuitionistic Valued $\Omega-$Algebra, let Ω be a set of the operations defined as follows.

Let $L = H \times H^*$ such that H is a complete Heyting algebra and $H^*$is a complete co-Heyting algebra of H, so

$$\Omega = \{\omega_L : X^n \times X \to L : \omega \text{ satisfy F1-F4 conditions}\}$$

It means that, if $\omega \in \Omega$, ω is HH^*-Intuitionistic valued function. The concept of HH^*-Intuitionistic valued $\Omega-$algebra can be defined as following;

Definition 3.1. Let $(X, =_L)$ HH^*-Intuitionistic valued set. $A = \langle X, \Omega\rangle$ is called HH^*-Intuitionistic valued $\Omega-$algebra if the following condition satisfy.

For $\omega_L \in \Omega$ and $((x_1, x_2, ..., x_n), c) \in X^n \times X$,
$\bigvee \{\{[x_i \in A]_L \wedge \omega_L ((x_1, x_2, ..., x_n), d) \vee [c = d]_L : i = 1, 2, ..., n\} : d \in X\}$
$\geq \omega_L ((x_1, x_2, ..., x_n), c)$

Example 3.2. Let $A = \langle X, \Omega\rangle$ be a HH^*-Intuitionistic valued $\Omega-$algebra.

$$\{\Theta\} : A \to L, [x \in \{\Theta\}]_L = 1_L$$

is a subset of A.

$$E = \langle \{\Theta\}, \Omega\rangle$$

is a HH^*-Intuitionistic valued $\Omega-$algebra. E is called trivial HH^*-Intuitionistic valued $\Omega-$algebra.

Definition 3.3. Let $A = \langle X, \Omega\rangle$ be a HH^*-Intuitionistic valued $\Omega-$algebra. If $K \subseteq X, B : K \to L$ is HH^*-Intuitionistic valued set, $(B, =_L) \subseteq (A, =_L)$ and for all $\omega_L \in \Omega, \omega_L \downarrow_B$satisfy the (1) then B is HH^*-Intuitionistic valued $\Omega-$subalgebra of A.

Example 3.4. Let $A = \langle X, \Omega\rangle$ be a HH^*-Intuitionistic valued $\Omega-$algebra.$E = \langle \{\Theta\}, \Omega\rangle$ is HH^*-Intuitionistic valued $\Omega-$ subalgebra.

Definition 3.5. Let $A = \langle X, \Omega\rangle$ and $B = \langle Y, \Omega\rangle$ be similar HH^*-Intuitionistic valued $\Omega-$ algebras and $f_L : A \to B$ be HH^*-Intuitionistic valued function. f_L is a HH^*-Intuitionistic valued $\Omega-$ algebra homomorphism$\Leftrightarrow$

H1 $[x = x]_L = [f_L(x) = f_L(x)]_L$
H2 $[x = x']_L \leq [f_L(x) = f_L(x')]_L$
H3 $f_L (\omega_L (x_1, x_2, ..., x_n), y) = \{f_L (x_i, y_i) : y = \omega_L(y_1, y_2, ..., y_n), f_L (x_i, y_i) > 0\}$.

Example 3.6. Let $A = \langle X, \Omega \rangle$ be a HH^*-Intuitionistic valued $\Omega-$algebra and $f_L : E \to A$, $g_L : E \to A$ are HH^*-Intuitionistic valued functions. $\forall x, I(\{\Theta\}, x) = 1_L$ HH^*-Intuitionistic valued $\Omega-$ algebra homomorphism exist. This homomorphism is unique.

Proposition 3.7. *Let A, B, C are similar HH^*-Intuitionistic valued $\Omega-$ algebras. If $f_L : A \to B$, $g_L : B \to C$ are HH^*-Intuitionistic valued $\Omega-$ algebra homomorphisms then $(g \circ f)_L : A \to C$ is a HH^*-Intuitionistic valued $\Omega-$ algebra homomorphism.*

Proof. Let $x_1, x_2, ..., x_n \in A, z \in C$,

i.

$$\begin{aligned} [x = x]_L &= ([x = x]_H, [x = x]_{H^*}) \\ &= ([f_H(x) = f_H(x)]_H, [f_{H^*}(x) = f_{H^*}(x)]_{H^*}) \\ &= [f_L(x) = f_L(x)]_L \end{aligned}$$

ii.

$$\begin{aligned} [x = x']_L &= ([x = x']_H, [x = x']_{H^*}) \\ &\leq ([f_H(x) = f_H(x')]_H, [f_{H^*}(x) = f_{H^*}(x')]_{H^*}) \\ &= [f_L(x) = f_L(x')]_L \end{aligned}$$

iii. $(g \circ f)_L (\omega_L (x_1, x_2, ..., x_n), z) = \bigvee \{f_L (\omega_L (x_1, x_2, ..., x_n), y) \wedge g_L(y, z) : y \in B\}$

Hence,

$$\bigvee \{\{f_H (x_i, y_i) : y = \omega_H (y_1, y_2, ..., y_n), f_H (x_i, y_i) > 0\} \wedge g_H(y, z) : y \in B\}$$

$$= \bigvee \left\{ \begin{array}{c} \{f_H (x_i, y_i) : i = 1, ..., n\} \wedge g_H(\omega_H (y_1, y_2, ..., y_n), z) \\ : y = \omega_H (y_1, y_2, ..., y_n), f_H (x_i, y_i) > 0 \end{array} \right\}$$

$$= \bigvee \left\{ \left\{ \begin{array}{c} f_H (x_i, y_i) \wedge g_H(y_i, z_i) : y = \omega_H (y_1, y_2, ..., y_n), f_H (x_i, y_i) > 0, \\ z = \omega_H (z_1, z_2, ..., z_n), g_H(y_i, z_i) > 0 \end{array} \right\} : y_i \in B \right\}$$

$$= \{\{(g \circ f)_H (x_i, z_i) : z = \omega_H (z_1, z_2, ..., z_n), (g \circ f)_H (x_i, z_i) > 0\} : i = 1, ..., n\}$$

and

$$\{\bigvee \{f_{H^*} (x_i, y_i) : y = \omega_{H^*} (y_1, y_2, ..., y_n), f_{H^*} (x_i, y_i) > 0\} \vee g_{H^*}(y, z) : y \in B\}$$

$$= \left\{ \begin{array}{c} \bigvee \{f_{H^*} (x_i, y_i) : i = 1, ..., n\} \vee g_{H^*} (\omega_{H^*} (y_1, y_2, ..., y_n), z) \\ : y = \omega_{H^*} (y_1, y_2, ..., y_n), f_{H^*} (x_i, y_i) > 0 \end{array} \right\}$$

$$= \left\{ \bigvee \left\{ \begin{array}{c} f_{H^*} (x_i, y_i) \vee g_{H^*} (y_i, z_i) : \\ y = \omega_{H^*} (y_1, y_2, ..., y_n), f_{H^*} (x_i, y_i) > 0, \\ z = \omega_{H^*} (z_1, z_2, ..., z_n), g_{H^*} (y_i, z_i) > 0 \end{array} \right\} : y_i \in B \right\}$$

$$= \bigvee \left\{ \begin{array}{c} (g \circ f)_{H^*} (x_i, z_i) : z = \omega_{H^*} (z_1, z_2, ..., z_n), \\ (g \circ f)_{H^*} (x_i, z_i) > 0, i = 1, ..., n \end{array} \right\}.$$

Now, we obtain that

$$(g \circ f)_L (\omega_L (x_1, x_2, ..., x_n), z)$$
$$= \{(g \circ f)_L (x_i, z_i) : z = \omega_L (z_1, z_2, ..., z_n), (g \circ f)_L (x_i, z_i) > 0\}. \qquad \square$$

Definition 3.8. Let $A = \langle X, \Omega \rangle$ be a HH^*-Intuitionistic valued $\Omega-$algebra and R_L be a HH^*-Intuitionistic valued equivalence relation on X . R_L called HH^*-Intuitionistic valued congruence relation if and only if $\forall \omega_L \in \Omega, x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in X^n$,

$$R_L(\omega_L(x), \omega_L(y)) = \{R_L(x_i, y_i) : i = 1, ..., n\}$$

Example 3.9. Let $(X, =_L), (Y, =_L)$ are HH^*-Intuitionistic valued $\Omega-$algebras and $f_L : X \to Y$ is HH^*-Intuitionistic valued $\Omega-$ algebra homomorphism.$\forall x_1, x_2 \in X, C_f(x_1, x_2)$ is a HH^*-Intuitionistic valued congruence relation on X.

Let $(X, =_L)$ be an HH^*-Intuitionistic valued set and X/R_L is set of the equivalence classes of R_L, HH^*-Intuitionistic valued equivalence relation on X. For, $\tau_L, \sigma_L \in X/R_L, [\tau_L = \sigma_L]_L$ defined as following,

$$[\tau_L = \sigma_L]_L = \bigvee \{\tau_L(x) \vee \sigma_L(x) : x \in X\}$$

So, $(X/R_L, =_L)$ is HH^*-Intuitionistic valued sets.

If $d_1, d_2, ..., d_n, \tau \in X/R_L$ then HH^*-Intuitionistic valued operator on X/R_L defined as follow:

$$\omega_L((d_1, d_2, ..., d_n), \tau) = \left[d_{\omega_L(x_1, x_2, ..., x_n)} = \tau\right]_L$$

Therefore, $a \in X$,

$$d_{\omega_L(x_1, x_2, ..., x_n)}(a) = \{d_{x_i}(a) : i = 1, ..., n\}$$

Theorem 3.10. *Let $A = \langle X, \Omega \rangle$ be a HH^*-Intuitionistic valued $\Omega-$algebra, R_L be a HH^*-Intuitionistic valued equivalence relation on X and $\langle X/R_L, \Omega \rangle$ be a HH^*-Intuitionistic valued $\Omega-$algebra. The function,*

$$f_L(a) = \varphi_a : X \times X/R_L \to L, \varphi_a(x, d_a) = d_a(x) \ \textit{for } a \in X$$

is a HH^*-Intuitionistic valued epimorphism from X to X/R_L and X/R_L is uniquely determined.

Proof. It is clear that, φ_a is surjective and X/R_L is uniquely determined. Now, let $a, x \in X$.

$\varphi_a(x, d_a) = d_a(x) = (\tau_a(x), \sigma_a(x)) \in L$,

$$\begin{aligned} \tau_a(x) &= \bigvee \{R_H(x, y) \wedge [x = a]_H : y \in X\} \\ &\geq \bigvee \{R_H(x, a) \wedge R_H(a, y) \wedge [x = a]_H : y \in X\} \\ &\geq \bigvee \{R_H(a, y) \wedge [x = a]_H : y \in X\} = \tau_x(a) \end{aligned}$$

and

$$\begin{aligned} \sigma_a(x) &= \{R_{H^*}(x,y) \vee [x=a]_{H^*} : y \in X\} \\ &\leq \{R_{H^*}(x,a) \vee R_{H^*}(a,y) \vee [x=a]_{H^*} : y \in X\} \\ &\leq \{R_{H^*}(a,y) \vee [x=a]_{H^*} : y \in X\} = \sigma_x(a) \end{aligned}$$

So, $d_a(x) = d_x(a)$. Furthermore,

$$\begin{aligned} \varphi_{\omega_L(x_1,x_2,\ldots,x_n)}\left(x, d_{\omega_L(x_1,x_2,\ldots,x_n)}\right) &= d_{\omega_L(x_1,x_2,\ldots,x_n)}(x) \\ &= \bigvee \{d_{x_i}(x) : i = 1,\ldots,n\} \\ &= \bigvee \{\varphi_{x_i}(x, d_{x_i}) :: i = 1,\ldots,n\} \end{aligned}$$

φ_a is homomorphism. □

4. Conclusion

Thanks to this extension, we can study algebraic properties of lattice valued sets in broad perspective. We can examine the kind of HH^*-Intuitionistic valued $\Omega-$ algebra homomorphisms, can be defned generated HH^*-Intuitionistic valued $\Omega-$ subalgebra, filters in HH^*-Intuitionistic Valued $\Omega-$Algebra. Furthermore, the concept free HH^*-Intuitionistic Valued $\Omega-$Algebra can be studied.

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