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NUMERICAL SOLUTION OF A TIME-FRACTIONAL INVERSE SOURCE PROBLEM

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Abstract. In this paper, an inverse problem of determining an unknown source term in a time-fractional diffusion equation is investigated. This inverse problem is severely ill-posed. For this reason, a mollification technique is used to obtain a regularized problem. Afterwards, a finite difference marching scheme is introduced to solve this regularized problem. The stability of numerical solution is investigated. Finally, two numerical examples are presented to illustrate the validity and effectiveness of the proposed method.

Key Words: Ill-posed problem, Caputo's fractional derivative, Mollification, Marching scheme.
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1. Introduction

Partial differential equations of fractional order have found many application areas during the last few decades. The fractional order derivatives are non-local and have memory effects, namely in a fractional system the next state depends on its current and all previous states. Thus, fractional differential and integral equations have been used widely to model a range of phenomena in different fields of science. Bioengineering, image and signal processing, fluid and continuum mechanics, heat transfer, control problems are examples of these applications [1–9].

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The special properties of fractional operators motivated researchers to extend some fractional inverse problems for several natural applications in past ten years. These types of problems are suitable for dealing with the models containing some unknown input information. Optical tomography, scattering of waves, heat conduction, machine learning, water and air pollution intensity are examples of these applications [10–15]. The main difficulty in most time fractional inverse problems is their ill-posedness. In present of noisy data, the solution of these problems are not continuously dependent on the input data [13, 14]. Thus, using appropriate regularization methods is necessary to find a stable numerical solution. As a result, algorithms based on some regularization techniques can be useful to find stable solutions for practical problems.

Consider the following inverse source problem of time fractional diffusion equation

$$D_t^{(\alpha)} u(x,t) = u_{xx}(x,t) + F(x,t), \quad 0 < x < 1,\ 0 < t < 1, \tag{1.1}$$

where $F(x,t) = f(x)g(t)$, $g(t)$ is known function and $f(x)$ is unknown in their domains, with the initial condition

$$u(x,0) = u_0(x), \qquad 0 \le x \le 1, \tag{1.2}$$

and the boundary conditions

$$u(0,t) = p(t), \qquad 0 \le t \le 1, \tag{1.3}$$

$$u(1,t) = s(t), \qquad 0 \le t \le 1. \tag{1.4}$$

$D_t^{(\alpha)} u(x,t)$ is the Caputo fractional derivative of order $0 < \alpha < 1$, defined as [1]:

$$D_t^{(\alpha)} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial s}(x,s)ds, \qquad 0 \le t \le 1,$$

where $\Gamma(.)$ is the Gamma function. To determine the set of functions (u, f) in the problem (1.1)-(1.4) we need a additional condition. Here, the condition

$$u_x(0,t) = \phi(t), \qquad 0 \le t \le 1. \tag{1.5}$$

is used. In practice, the input functions $u_0(x), p(t), s(t)$ and $\phi(t)$ are not exact, but some perturbed versions of them are in hand. Thus, at first, we use the mollification regularization method to stabilize the problem. Afterwards, a numerical scheme based on the space marching method will be introduced to approximate the solution of (1.1) and the unknown source term $f(x)$.

2. Mollification method

Let $\delta > 0$, $p > 0$, and

$$A_p = \Big(\int_{-p}^{p} exp(-s^2) ds \Big)^{-1}.$$

The -mollification of an integrable function is based on convolution with the Gaussian kernel

$$\rho_{\delta,p}(t) = \begin{cases} A_p \delta^{-1} exp(-\frac{t^2}{s^2}), & |t| \leq p\delta, \\ 0, & |t| > p\delta. \end{cases}$$

The δ-mollifier $\rho_{\delta,p}$ is a non-negative $C^{\infty}(-p\delta, p\delta)$ function satisfying $\int_{-p\delta}^{p\delta} \rho_{\delta,p}(t)dt = 1$.
Let $g(t) \in L^1(I)$ and $t \in I_\delta = [p\delta, 1 - p\delta]$. The δ-mollification of $g(t)$ is defined by the convolution

$$\mathcal{J}_\delta g(t) = (\rho_\delta * g)(t) = \int_{I_\delta} \rho_\delta(t-s) g(s) ds = \int_{t-p\delta}^{t+p\delta} \rho_\delta(t-s) g(s) ds.$$

The radius of mollification δ is determined automatically by the Generalized Cross Validation (GCV) criteria [16].
Now, let in the inverse problem (1.1)-(1.5), the functions $u_0(x), p(t), s(t)$ and $\phi(t)$ are only known approximately as $u_0^\varepsilon, p^\varepsilon, s^\varepsilon$ and ϕ^ε, such that the infinity norm of the difference between every of these functions and their corresponding approximations are less than a known value ε.
Suppose $\upsilon = \mathcal{J}_\delta u$ is the mollified version of u. So, the regularized problem is formulated as follows

$$D_t^{(\alpha)} \upsilon(x,t) = \upsilon_{xx}(x,t) + f(x)g(t), \quad 0 < x < 1,\ 0 < t < 1, \tag{2.1}$$

$$\upsilon(x,0) = \mathcal{J}_\delta u_0^\varepsilon(x), \quad 0 \leq x \leq 1, \tag{2.2}$$

$$\upsilon(0,t) = \mathcal{J}_{\delta_0} p^\varepsilon(t), \quad 0 \leq t \leq 1, \tag{2.3}$$

$$\upsilon_x(0,t) = \mathcal{J}_{\delta_0'} \phi^\varepsilon(t), \quad 0 \leq t \leq 1, \tag{2.4}$$

where δ, δ_0 and δ_0' are the radii of mollification, and will be chosen using the GCV criteria.

For computation of the source term $f(x)$, from Eq. (2.1), we have

$$f(x) = \frac{D_t^{(\alpha)} \upsilon(x,t) - \upsilon_{xx}(x,t)}{g(t)}.$$

By substituting $t = 0$ at this relation, we obtain

$$f(x) = \frac{D_t^{(\alpha)}\upsilon(x,0) - \upsilon_{xx}(x,0)}{g(0)}, \tag{2.5}$$

where $\upsilon_{xx}(x,0)$ and $g(0)$ are in hand and $D_t^{(\alpha)}\upsilon(x,0)$ will be obtained by using the marching scheme. Finally, the approximation of $f(x)$ is calculated according to (2.5).

3. Marching scheme

In this section, a numerical algorithm is presented to find the solution of (2.1)-(2.5). Let M and N are positive integers. Consider a uniform grid in the unit interval $I = [0,1] \times [0,1]$ as

$$\Omega = \{(x_i = ih, t_n = nk)\ ,\ i = 0,1,...,M;\ n = 0,1,...,N\},$$

in which $Mh = 1$ and $Nk = 1$. In addition, suppose the discrete convolution of the Gaussian kernel $\rho_\delta(t)$ and the grid function υ at the grid point (x_i, t_n) is denoted by R_i^n and let

$$W_i^n = \upsilon_x(ih, nk),\ Q_i^n = D_t^{(\alpha)}\upsilon(ih, nk),\ f_i = f(ih),\ g^n = g(nk).$$

Notice that for $n \in \{1,...,N\}$

$$R_0^n = \mathcal{J}_{\delta_0}p^\varepsilon(nk), \qquad W_0^n = \mathcal{J}_{\delta_0'}\phi^\varepsilon(nk), \qquad Q_0^n = D_t^{(\alpha)}(\mathcal{J}_{\delta_0}p^\varepsilon(nk)),$$

and for $i \in \{0,...M\}$

$$R_i^0 = \mathcal{J}_\delta u_0^\varepsilon(ih), \qquad T_i = D^2(\mathcal{J}_\delta u_0^\varepsilon(ih)),$$

where $D^2 := D_+D_-$ is second-order finite differences operator. Now we approximate the partial differential equation of fractional order in system (2.1)-(2.5) by the finite difference schemes

$$R_{i+1}^n = R_i^n + hW_i^n, \tag{3.1}$$

$$W_{i+1}^n = W_i^n + hQ_i^n - hf_ig^n, \tag{3.2}$$

$$Q_{i+1}^n = D_t^{(\alpha)}(\mathcal{J}_{\delta_{i+1}}R_{i+1}^n), \tag{3.3}$$

$$f_{i+1} = \frac{1}{g^0}(Q_{i+1}^0 - T_{i+1}), \tag{3.4}$$

where $i = 0,1,...,M-1$ and $n = 0,1,...,N$.

The computation of Caputo's fractional derivative $D_t^{(\alpha)}(\mathcal{J}_{\delta_i}R_i^n)$ in the presence of noisy data is an ill-posed problem [17]. Suppose $q^\varepsilon(t)$ is a perturbed version of the exact function $q(t)$. To approximate $\mathcal{J}_\delta(D_t^{(\alpha)}q^\varepsilon)$

on a uniform partition K of the unit interval, we follow the mollification technique proposed in [17]. Let D_0 be centered finite difference operator, D_+ be forward finite difference operator and $\mathbf{Q}$ be the discrete version of q. Then the discrete solution $(D_t^{(\alpha)}\mathbf{Q}^\varepsilon)_\delta$ in the grid points, will be as

$$\begin{aligned}\left(D^{(\alpha)}\mathbf{Q}^\varepsilon\right)_\delta(t_1) &= D_+(\mathcal{J}_\delta\mathbf{Q}^\varepsilon)(t_1)W_1,\\ \left(D^{(\alpha)}\mathbf{Q}^\varepsilon\right)_\delta(t_2) &= D_+(\mathcal{J}_\delta\mathbf{Q}^\varepsilon)(t_1)W_2 + D_+(\mathcal{J}_\delta\mathbf{Q}^\varepsilon)(t_2)W_1,\end{aligned}$$

and

$$\begin{aligned}\left(D^{(\alpha)}\mathbf{Q}^\varepsilon\right)_\delta(t_j) &= D_+(\mathcal{J}_\delta\mathbf{Q}^\varepsilon)(t_1)W_j\\ &+\sum_{i=2}^{j-1} D_0(\mathcal{J}_\delta\mathbf{Q}^\varepsilon)(t_i)W_{j-i+1} + D_+(\mathcal{J}_\delta\mathbf{Q}^\varepsilon)(t_j)W_1,\end{aligned}$$

where $j = 3, 4, ..., n$ and the quadrature weights $W_j = W_j(\alpha, t_j)$ are integrated exactly with values

$$\begin{aligned}W_1 &= \frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\Big(\frac{\Delta t}{2}\Big)^{1-\alpha},\\ W_i &= \frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\Big[\Big((2i+1)\frac{\Delta t}{2}\Big)^{1-\alpha} - \Big((2i-1)\frac{\Delta t}{2}\Big)^{1-\alpha}\Big],\end{aligned}$$

for $i = 2, 3, ..., j-1$ and

$$W_j = \frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\Big[j\Delta t - \Big[\Big(j-\frac{1}{2}\Big)\Delta t\Big]^{1-\alpha}\Big].$$

4. Stability analysis of the algorithm

Theorem 4.1. *(Stability of the algorithm) Suppose $|R_i|$, $|W_i|$, $|Q_i|$ are maximum values of $|R_i^n|$, $|W_i^n|$, $|Q_i^n|$, where $n = 0, 1, ..., N$. For the marching scheme, there exist two constants θ_1 and θ_2, such that*

$$\max\{|R_M|, |W_M|, |Q_M|, |f_M|\} \le \theta_1 \max\{|R_0|, |W_0|, |Q_0|, |f_0|\} + \theta_2.$$

Proof. Let $M_g = \max_n\{|g^n|\}$. By using (3.1) and (3.2), we have

$$|R_{i+1}^n| \le (1+h)\max\{|R_i^n|, |W_i^n|\}, \tag{4.1}$$

$$|W_{i+1}^n| \le (1+hM_g)\max\{|W_i^n|, |Q_i^n|, |f_i|\}. \tag{4.2}$$

From (3.3) and by applying theorem (4.6) in [16], we have

$$|Q_{i+1}^n| = |D_t^{(\alpha)}(\mathcal{J}_{\delta_{i+1}} R_{i+1}^n)| = \Big|\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{D_0(\mathcal{J}_{\delta_{i+1}} R_{i+1}^n)}{(t-s)^\alpha} ds\Big|$$

$$\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{4A_p \|R_{i+1}^n\|_\infty}{\delta_{i+1}|(t-s)^\alpha|} ds = \frac{4A_p(nk)^{1-\alpha}\|R_{i+1}^n\|_\infty}{\delta_{i+1}\Gamma(2-\alpha)}. \tag{4.3}$$

Let $\bar{\delta} = \min_i\{\delta_i\}$. From (4.1) and (4.3), we have

$$|Q_{i+1}^n| \leq \frac{4A_p(nk)^{1-\alpha}(1+h)}{\bar{\delta}\Gamma(2-\alpha)} \max\{|R_i^n|.|W_i^n|\}. \tag{4.4}$$

Also, let $m_g = \min_n\{|g^n|\}$ and $M_T = \max_i\{|T_i|\}$. From (3.4) and (4.4), we have

$$|f_{i+1}| \leq \frac{4A_p(nk)^{1-\alpha}(1+h)}{\bar{\delta} m_g \Gamma(2-\alpha)} \max\{|R_i^n|, |W_i^n|\} + \frac{M_T}{m_g}. \tag{4.5}$$

Also, let

$$C_1 = \max\Big\{1, \frac{4A_p(nk)^{1-\alpha}}{\bar{\delta}\Gamma(2-\alpha)}, \frac{4A_p(nk)^{1-\alpha}}{\bar{\delta} m_g\Gamma(2-\alpha)}\Big\},$$

$$C_2 = \max\Big\{1, M_g, \frac{4A_p(nk)^{1-\alpha}}{\bar{\delta}\Gamma(2-\alpha)}, \frac{4A_p(nk)^{1-\alpha}}{\bar{\delta} m_g\Gamma(2-\alpha)}\Big\}.$$

From (4.1)-(4.5), we obtain

$$\max\{|R_{i+1}|, |W_{i+1}|, |Q_{i+1}|, |f_{i+1}|\} \leq (C_1 + hC_2) \max\{|R_i|, |W_i|, |Q_i|, |f_i|\} + \frac{M_T}{m_g}.$$

Iterating this inequality M times, we have

$$\max\{|R_M|, |W_M|, |Q_M|, |f_M|\} \leq (C_1 + hC_2)^M \max\{|R_0|, |W_0|, |Q_0|, |f_0|\} + \tau\frac{M_T}{m_g}.$$

where $\tau = \sum_{i=0}^{M-1}(C_1 + hC_2)^i$. This inequality implies

$$\max\{|R_M|, |W_M|, |Q_M|, |f_M|\} \leq C_1^M \exp\Big(\frac{C_2}{C_1}\Big) \max\{|R_0|, |W_0|, |Q_0|, |f_0|\} + \tau\frac{M_T}{m_g}.$$

Letting $\theta_1 = C_1^M \exp\Big(\frac{C_2}{C_1}\Big)$ and $\theta_2 = \tau\frac{M_T}{m_g}$ complete the proof of stability. □

5. Numerical examples

In this section, two examples are solved to test the ability of the proposed algorithm. To simulate the data for the inverse problem, some random noises are added to the data resulted from the additional functions. Suppose that the maximum level of noise in the data functions is ε . Then, for generating noisy data, we use the formula

$$r_n^{\varepsilon} = r(t_n) + \varepsilon_n,$$

where the ε_n are Gaussian random variables with variance $\sigma^2 = \varepsilon^2$.

Example 5.1. Consider problem (1.1)-(1.5) with $g(t) = 1$, $u_0(x) = \sin(x) + \cos(2\alpha\pi x), p(t) = 0$ and $\phi(t) = 0$. The exact solution of this problem is

$$u(x,t) = E_\alpha(-t^\alpha)\sin(x) + \cos(2\alpha\pi x),$$

where $E_\alpha(.)$ is the Mittag-Leffer function and

$$f(x) = 4\pi^2\alpha^2\cos(2\alpha\pi x).$$

Let $M = 150$ and $N = 150$. Figure 1, Figure 2 and Figure 3 display the exact and numerical solutions of this problem for several values of α when $\varepsilon = 0.05$. Figure 4 shows the comparison between the exact and the computed solutions of $f(x)$ with regularization and without regularization when $\varepsilon = 0.1$ and $\alpha = 0.6$. Furthermore, Figure 5 shows the exact and the estimated solutions to $f(x)$ for numerous values of α when $\varepsilon = 0.01$.

Example 5.2. Consider problem (1.1)-(1.5) with $g(t) = \dfrac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha}+6$, $u_0(x) = -x^3 + 1, p(t) = 1$ and $\phi(t) = t^2$. The exact solution of this problem is $u(x,t) = xt^2 - x^3 + 1$ and $f(x) = x$.

Figure 6 and Figure 7 display the absolute error function when $\alpha = 0.45, M = 150, N = 150$ and $\varepsilon = 0.05, 0.1$. Figure 8 shows the comparison between the exact and the computed solutions to $f(x)$ with regularization and without regularization when $\varepsilon = 0.1$ and $\alpha = 0.25$. Also, Figure 9 shows the exact and numerical approximations to $f(x)$ for several values of ε. Finally, Figure 10 indicates the absolute errors of numerical approximations for $f(x)$.

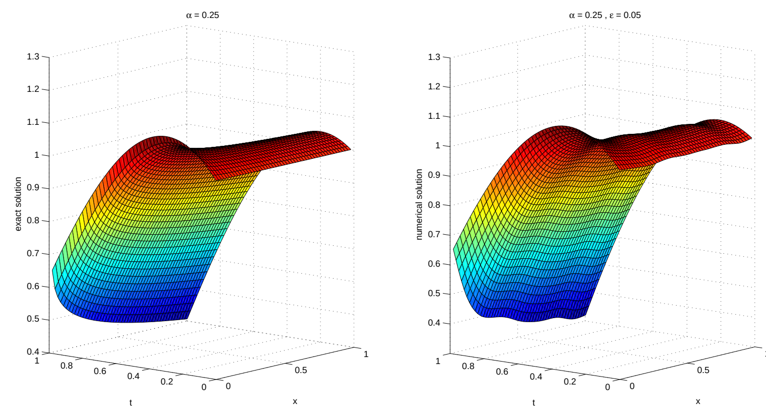

FIGURE 1. The exact and numerical solution of Example 5.1 when $\alpha = 0.25$ and $\varepsilon = 0.05$.

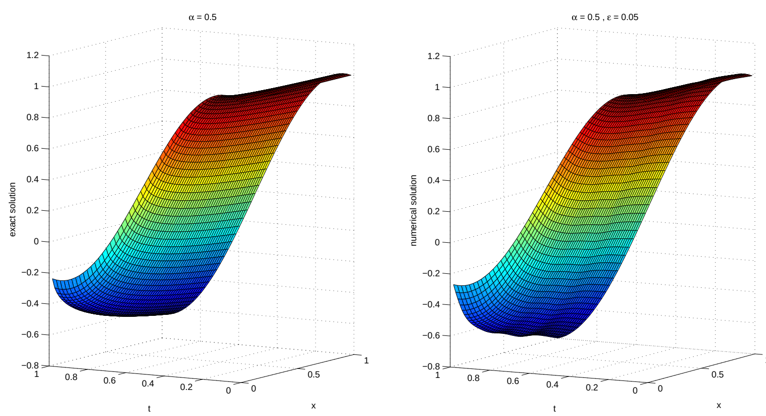

FIGURE 2. The exact and numerical solution of Example 5.1 when $\alpha = 0.5$ and $\varepsilon = 0.05$.

6. Conclusion

In this article, an inverse problem of the time-fractional diffusion equation have been investigated with unknown source term. Since the problem was ill-posed, a mollification method was applied on the data to get an equivalent regularized problem. The approximate solution of this problem was derived by using a space marching finite difference scheme.

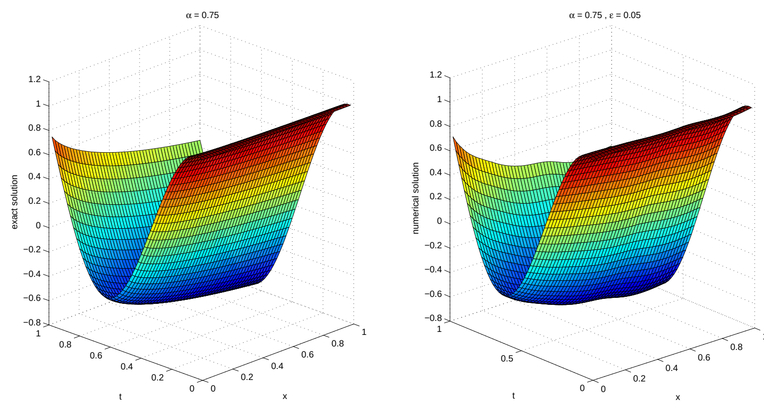

FIGURE 3. The exact and numerical solutions of Example 5.1 when $\alpha = 0.75$ and $\varepsilon = 0.05$.

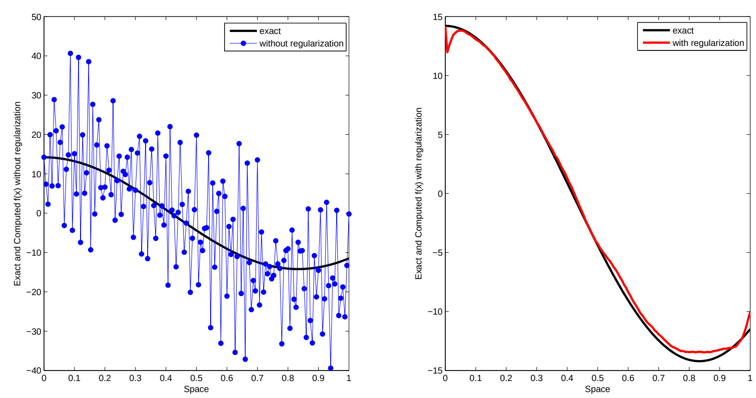

FIGURE 4. The exact and numerical values of $f(x)$ for Example 5.1 when $\alpha = 0.6$ and $\varepsilon = 0.1$.

Also, stability of the method was proved. At the end, two numerical implementations were presented to investigate the validity of the proposed method. The numerical results verify stability and accuracy of the algorithm in the presence of noise.

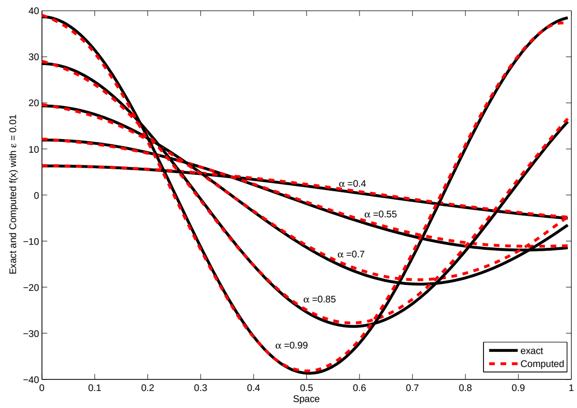

FIGURE 5. The exact and numerical approximations to $f(x)$ in Example 5.1 for various values of α when $\varepsilon = 0.01$.

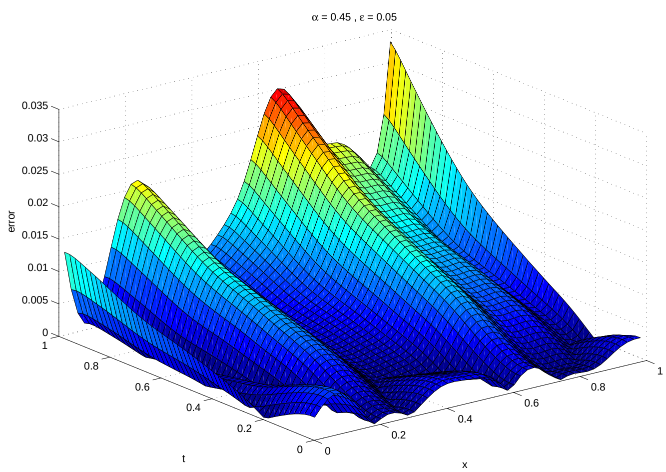

FIGURE 6. The absolute error function for Example 5.2 when $\alpha = 0.45$ and $\varepsilon = 0.05$.

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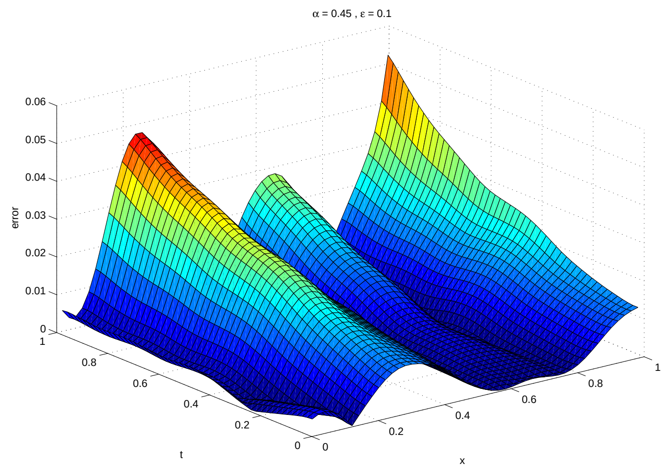

FIGURE 7. The absolute error function for Example 5.2 when $\alpha = 0.45$ and $\varepsilon = 0.1$.

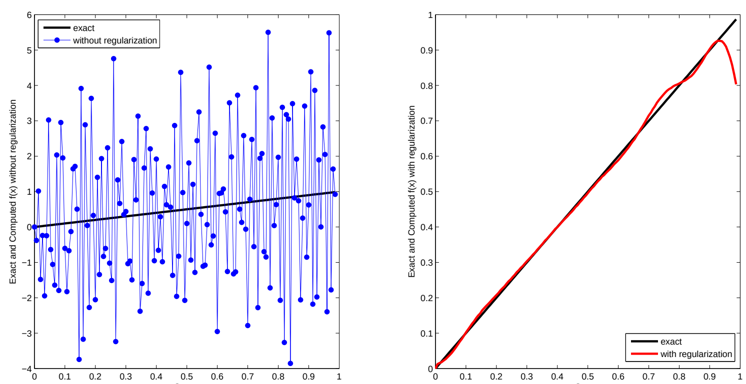

FIGURE 8. The exact and numerical values of $f(x)$ in Example 5.2 when $\alpha = 0.25$ and $\varepsilon = 0.1$.

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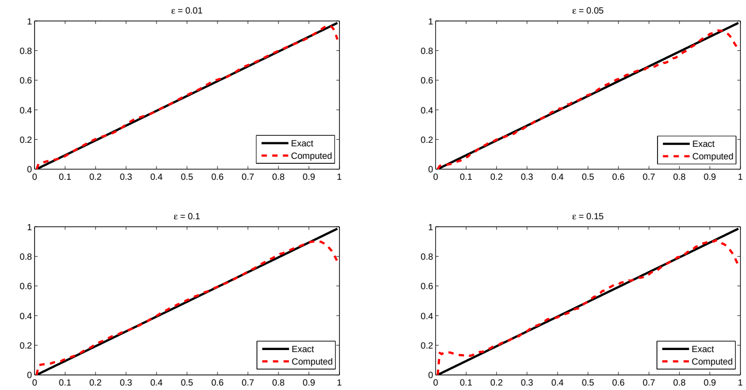

FIGURE 9. The exact and numerical approximations to $f(x)$ in Example 5.2 when $\alpha = 0.5$.

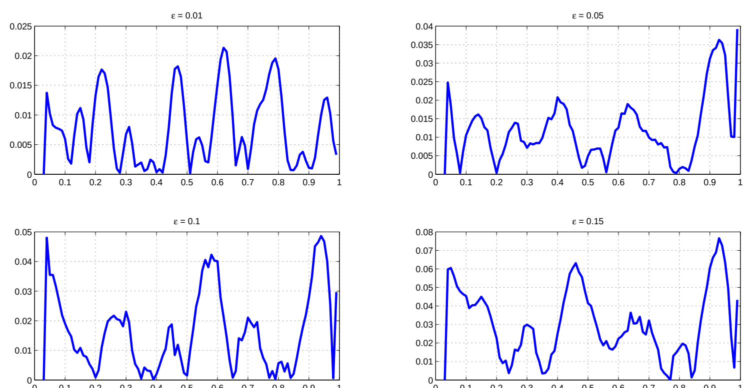

FIGURE 10. Absolute error of numerical approximations $f(x)$ in Example 5.2 when $\alpha = 0.5$.

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