

ONE-STAGE UNCERTAIN LINEAR OPTIMIZATION

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ABSTRACT. Uncertainty is one of the intrinsic features of natural phenomenon and optimization is not an exception. Parameters in an optimization problem may be inaccurately presented, or due to variation after solving the problem, the current optimal solution may not be optimal or feasible for the new data. One of the approaches towards the uncertainty is the uncertainty theory initiated by Liu in 2007 and completed in 2011. This theory describes the uncertainty originated from human reasoning in mathematical axiomatic words. In this paper, we consider one-stage uncertain linear optimization problem, where the parameters linearly depend on several independent uncertain variables. A solution method is presented which is similar to the branch and bound method for the integer linear program. The presented methodology is explained by simple illustrative examples.

Key Words: Uncertainty Theory, One-stage uncertain optimization problem, Linear Optimization.

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1. INTRODUCTION

Linear Optimization (LO) has been studied from different points of view and has proved its ability in modeling a vast range of practical problem in various fields. It is numerically tractable its theory is almost complete and powerful solvers have been developed. One of the basic assumptions in an LO problem is certainty of the parameters while this is not always the case in practice. In some cases, parameters are not estimated exactly

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and sometimes they are changed after solving the problem. Consequently the obtained optimal solution may not be optimal or feasible more.

Let us consider an LO problem as

$$(1.1) \quad \min_x \left\{ \sum_{j=1}^n c_j x_j : \sum_{j=1}^n a_{ij} x_j \leq b_i, i = 1, \dots, m, x_j \geq 0, j = 1, \dots, n \right\}.$$

Using sensitivity analysis and parametric programming is a traditional approach and there are many published results with different points of views (See e.g. [5, 4, 12]). In sensitivity analysis, the effect of substitution of a parameter with another value is investigated. If this parameter is in the objective function, the optimality may be affected. While the change in the right-hand-side data may result the current optimal solution being infeasible. When this substitution occurs in the right-hand-side of constraints, the current optimal solution might be neither feasible nor optimum. Additionally, the goal of parametric programming is finding a region for parameters, where a specific feature of the problem is invariant. For instance, it may be the interval where the optimal basis is invariant when c_1 varies. When variation occurs in the right hand side of constraints, the optimal solution may be change while the optimal basis is the same. Observe that in sensitivity analysis and parametric programming, new value of parameters is still considered exact values.

Another point of view to handle the indeterminacy is Fuzzy Theory [10]. Here, parameters of the problem are considered as fuzzy numbers and eventually a crisp linear program is produced and solved [9]. The main weak point of fuzzy models is that, the fuzzy number used in the model is not known and applying different fuzzy numbers may result in different and sometimes contradictory results.

Robust optimization could be another approach for managing uncertainty. In this method, it is assumed that data belongs to a set but without any specific probability distribution [6]. For example, the objective function coefficient vector might be in an ellipsoid. The vector is close to the border as the same possibility as it could be next to the centroid. A special case is the interval LO problem [8]. In this case, each parameter varies independently in an interval, while in general they might be bound to each other in some way. In the interval LO, for example, objective function coefficient vector is in a hyper-rectangular instead of an ellipsoid. We must note that the intervals themselves are identified by an expert and they are fixed afterward. Thus, a sort of uncertainty still affects the problem. In the robust LO problem, the model reduces to a crisp one to

produce an optimal solution such that for all parameters in these intervals, it is still optimal, while the notion of optimality is somehow different.

One may want to address the uncertainty using probability theory. In such an approach, a probability distribution is associated with each uncertain parameter. For example, c_1 might have a uniform distribution and thus its expected value can be considered instead. One of the main deficiencies of this approach is that, for exact estimation of appropriate probability distribution, enough number of historical data is necessary. Moreover, the obtained optimal solution is not the desired one in practice and it could be only a kind of assurance during a decision making process. In stochastic linear programming, uncertainty may appear in satisfying of constraints, and a lower bound for the probability of the satisfaction could be considered [2]. For example, instead of a linear constraint in (1.1), the following is replaced.

$$g_i(x) = P\left(\sum_{j=1}^n a_{ij}x_j \leq b_i\right) \geq \alpha_i,$$

where $0 \leq \alpha_i \leq 1$ and is selected by the decision maker. Observe that $g_i(x) = P(\sum_{j=1}^n a_{ij}x_j \leq b_i)$ is nonlinear in general and even it is not convex in some cases. This will add a higher rate complexity to the problem.

In an LO problem, when there is no enough credible data to estimate the probability distribution of the underlying indeterminate data or the problem violates the fuzzy rules, one may have no choice but relying on the human reasoning. Uncertainty theory has been initiated in 2007 [6] and then refined in 2011 [7] is an axiomatic paradigm to answer the uncertainty originated from the expert's opinion.

An uncertain problem can be defined as one-stage or multi-stage problem. In a one-stage uncertain problem, all decisions are made in one step, while in multi-stage uncertain problems, some decisions must be made before any realization of the uncertain phenomenon. Some other decisions are taken after the realization of the uncertain phenomenon stage by stage. LO in an uncertain environment has been investigated in [3]. The author proposed two different models and all input data are considered as independent uncertain variables. The results also were demonstrated on simple examples. Two-stage uncertain linear optimization has been studied in [11]. The authors considered this problem in several cases and proposed a solution approach. In this paper, we consider one-stage uncertain LO problem. It is assumed that, in addition to some crisp constraints, there are some other uncertain constraints. These later constraints are linearly

dependent on several uncertain variables. The uncertain variables are all considered to be linear since the methodology is independent of the type of uncertain variable. We also assume that the belief that an uncertain constraint holds with minimum belief rate prescribed by the decision maker. The methodology for solving such problems is represented and illustrated using some concrete examples.

This paper is organized as follows. In Section 2, basic concepts of uncertainty theory are reviewed. The one-stage uncertain LO problem is presented in Section 3. A computational algorithm is devised in Section 4. Final Section includes some concluding remarks.

2. UNCERTAINTY THEORY

Let us review some basic prerequisite concepts of the uncertainty theory. Let Γ be a nonempty set, and L consists of subsets of Γ be a σ -algebra over Γ . The pair (Γ, L) is called a measurable space, and each member $\Lambda \in L$ of this measurable space is called an event. An uncertain measure is a function M from L to $[0, 1]$ which satisfies the following axioms.

- (Normality Axiom) For the universal set Γ , $M\{\Gamma\} = 1$.
- (Duality Axiom) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event Λ .
- (Subadditivity Axiom) $M\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}$ for all countable sequence of events $\Lambda_1, \Lambda_2, \dots$

The value of $M\{\Lambda\}$ indicates the belief degree to the event “We believe Λ will happen”. The triplet (Γ, L, M) is called an uncertainty space. Liu in [6] has defined an additional axiom to the uncertain measure which distinguishes the uncertainty theory from the probability theory.

- (Product Axiom) Let (Γ_k, L_k, M_k) , $k = 1, 2, \dots$ be uncertainty spaces. The product uncertain measure M is an uncertain measure satisfying

$$M\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\},$$

Where Λ_k is an event in Γ_k and $k = 1, 2, \dots$

A function $f : (\Gamma, L) \rightarrow \mathbb{R}$ is measurable if for any Borel set B in \mathbb{R} , the inverse image of f is a member of the σ -Algebra L . This means that $f^{-1}(B) = \{\gamma : f(\gamma) \in B\} \in L$. All continuous functions and monotone functions are measurable functions. An uncertain variable ξ is a

measurable function from (Γ, L, M) to \mathbb{R} . For an uncertain variable, the uncertainty distribution Φ is defined by $\Phi(x) = M\{\xi \leq x\}$.

There are several uncertainty distributions in the literature. Since our results are independent of the distribution type, we only remind the linear uncertain distribution and refer to [6] for the others.

A linear uncertain distribution is denoted by $L(a, b)$ and defined as

$$\Phi(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b. \end{cases}$$

where $a < b$ are two real numbers. A regular uncertainty distribution $\Phi(x)$ is a continuous and strictly increasing function that satisfy $0 < \Phi(x) < 1$, $\lim_{x \rightarrow -\infty} \Phi(x) = 0$, and $\lim_{x \rightarrow +\infty} \Phi(x) = 1$. For a regular uncertainty distribution $\Phi(x)$ on $(0, 1)$, well-defined inverse function $\Phi^{-1}(\alpha)$ exists and is called the inverse uncertainty distribution of ξ . Inverse uncertainty distribution of the linear variable $L(a, b)$ is $\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b$.

Uncertain variables $\xi_1, \xi_2, \dots, \xi_n$ are called independent if and only if for arbitrary Borel sets B_1, B_2, \dots, B_n , it holds

$$M\left\{\bigcap_{i=1}^n \xi_i \in B_i\right\} = \bigwedge_{i=1, \dots, n} M\{\xi_i \in B_i\}.$$

An uncertain variable ξ is nonnegative when $M\{\xi < 0\} = 0$, and it is positive when $M\{\xi \geq 0\} = 1$. Analogous definitions can be provided for nonpositive and negative uncertain variables.

Theorem 2.1. [6] *Let $\xi_1, \xi_2, \dots, \xi_n$ be uncertain variables. Further, let f be a measurable real-valued function. Then $f(\xi_1, \xi_2, \dots, \xi_n)$ be an uncertain variable.*

Observe that there is no direct way to compute the value of a function that depends on an uncertain variable, specially when it appears in a constraint. One useful alternative in to apply the α -chance counterpart. In this situation, satisfying such constraint is replaced by a minimum belief degree that an uncertain constraint holds. Such a constraint then can be easily calculated using the inverse uncertainty distribution as stated in the following theorem.

Theorem 2.2. [6] *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \dots, \Phi_n$, respectively. Consider the function $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$ is strictly increasing with respect*

to $\xi_1, \xi_2, \dots, \xi_m$, $m \leq n$, and strictly decreasing with respect to ξ_m, \dots, ξ_n . Then $M\{f(\xi_1, \xi_2, \dots, \xi_n) \leq 0\} \geq \alpha$, if and only if

$$f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)) \leq 0.$$

3. ONE-STAGE UNCERTAIN LINEAR OPTIMIZATION

A one-stage uncertain LO can be defined in a general form as

$$(3.1) \quad \min\{c^T x \mid T(\xi)x \geq h(\xi), x \in X\},$$

where $\xi : \Omega \rightarrow \mathbb{R}^r$ is an uncertain vector on the uncertain space (Γ, L, M) . Here, $T(\xi)$ is an uncertain matrix of $m \times n$ dimension and $h(\xi) \in \mathbb{R}^s$ is an uncertain vector depending on the uncertain vector ξ . In general, it is assumed that $T(\xi), h(\xi)$ is independent of the decision variable x . This means that our decision does not affect uncertain input data. Let

$$(3.2) \quad \zeta(x, \xi) := T(\xi)x - h(\xi).$$

Further, let

$$(3.3) \quad T(\xi) = T + \sum_{j=1}^r T_j \xi_j, h(\xi) = h + \sum_{j=1}^r h_j \xi_j.$$

Eq. (3.3) can be rewritten as $\zeta_i(x, \xi) := t_i^T(\xi)x - h_i(\xi)$, $i = 1, \dots, m$, where t_i^T is the i -th row of $T(\xi)$. Therefore

$$(3.4) \quad \begin{aligned} \zeta_i(x, \xi) &:= t_i^T x - h_i + \sum_{j=1}^r t_{ij}^T \xi_j x - \sum_{j=1}^r h_{ij} \xi_j \\ &= \sum_{k=1}^n t_{ik} x_k - h_i + \sum_{j=1}^r \xi_j \left(\sum_{k=1}^n t_{ijk} x_k - h_{ij} \right). \end{aligned}$$

Using the simplifying notations

$$\bar{d}_i = \sum_{k=1}^n t_{ik} x_k - h_i \text{ and } d_{ij} = \sum_{k=1}^n t_{ijk} x_k - h_{ij},$$

we have

$$(3.5) \quad \zeta_i(x, \xi) := \bar{d}_i + \sum_{j=1}^r \xi_j d_{ij}.$$

The α -chance form of problem (3.1) is

$$(3.6) \quad \min\{c^T x \mid M\{t_i^T(\xi)x - h_i(\xi) \geq 0\} \geq \alpha, i = 1, \dots, m, x \in X\}.$$

Using (3.5), constraints of Problem (3.6) is converted to the following form.

$$(3.7) \quad M\left\{\sum_{j=1}^r \xi_j(-d_{ij}) \leq \bar{d}_i\right\} \geq \alpha, i = 1, \dots, m.$$

Let us define $v^+ = \begin{cases} 0, v < 0 \\ v, v \geq 0 \end{cases}$ and $v^- = \begin{cases} -v, v \leq 0 \\ 0, v > 0 \end{cases}$. Based on Theorem 2.2, Eq. (3.7) holds if and only if

$$(3.8) \quad \sum_{j=1}^r (-d_{ij})^+ \Phi_j^{-1}(\alpha) - \sum_{j=1}^r (-d_{ij})^- \Phi_j^{-1}(1 - \alpha) \leq \bar{d}_i, i = 1, \dots, m.$$

Considering these facts, Problem (3.1) is simplified as a crisp LO problem

$$(3.9) \quad \min\{c^T x \mid \sum_{j=1}^r (-d_{ij})^+ \Phi_j^{-1}(\alpha) - \sum_{j=1}^r (-d_{ij})^- \Phi_j^{-1}(1 - \alpha) \leq \bar{d}_i, \\ i = 1, \dots, m, x \in X\}.$$

In the sequel, we solve Problem (3.9), step by step. First, we consider that the problem only has one uncertain constraint, i.e., $i = 1$.

3.1. Case $r = 1$. We first consider that the problem depends on only one uncertain variable, $\xi = (\xi_1)$. Therefore, Problem (3.9) is specialized as

$$(3.10) \quad \min\{c^T x \mid M\{(-d_1)\xi_1 \leq \bar{d}\} \geq \alpha, x \in X\}.$$

Recall that x varies in X , and consequently d_1 might be positive (non-negative) or negative, depending on the vector x . We consider then two subcases.

- (1) Let $d_1 \geq 0$. We first add this constraint to the problem. Moreover, $(-d_1)^+ = 0$, $(-d_1)^- = d_1$. Therefore, $M\{(-d_1)\xi_1 \leq \bar{d}\} \geq \alpha$ if and only if $-(d_1)\Phi^{-1}(1 - \alpha) \leq \bar{d}$. Considering the uncertain variable as a linear as $\xi_1 = L(a, b)$, we have the following equivalent crisp constraint

$$(3.11) \quad -d_1(b + (a - b)\alpha) \leq \bar{d}$$

Therefore, Problem (3.9) is reduced to

$$(3.12) \quad \min\{c^T x \mid -d_1(b + (a - b)\alpha) \leq \bar{d}, d_1 \geq 0, x \in X\}.$$

- (2) Let $d_1 \leq 0$. We add this constraint to the problem. Further, $(-d_1)^+ = -d_1, (-d_1)^- = 0$. Then, $M\{(-d_1)\xi_1 \leq \bar{d}\} \geq \alpha$, if and only if $-(d_1)\Phi^{-1}(\alpha) \leq \bar{d}$. With the assumption that the uncertain variable is linear as $\xi_1 = L(a, b)$, we have the following equivalent crisp constraint.

$$(3.13) \quad -d_1(a + (b - a)\alpha) \leq \bar{d}.$$

Therefore, the problem (3.7) is replaced with the following problem.

$$(3.14) \quad \min\{c^T x \mid -d_1(a + (b - a)\alpha) \leq \bar{d}, d_1 \leq 0, x \in X\}.$$

Solving two problems (3.12) and (3.14) for a given belief rate α may result to several outcomes. If both problems are infeasible, then the problem has no solution for the given α . Reducing the value of belief degree α may result to a feasible and consequently optimal solution. If at least one of these problems has finite optimal solution, then the one-stage uncertain LO has an optimal solution as well. If both problems have optimal solution, then the optimal solution of the one-stage uncertain LO is obtained with one of the Problems (3.12) and (3.14).

Let us explain the methodology with a simple example.

Example 3.1. Consider the following problem. In this problem, we considered $\alpha = 0.9$.

$$(3.15) \quad \begin{aligned} \min \quad & -2x_1 - x_2 \\ \text{s.t.} \quad & M\{(3x_1 - x_2 + 2)\xi_1 \leq 2x_2 - 3\} \geq 0.9 \\ & x_1 + x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0, \end{aligned}$$

where $\xi_1 = L(1, 3)$. Here, $-d_1 = 3x_1 - x_2 + 2$ and $\bar{d} = 2x_2 - 3$. Considering $d_1 \geq 0$, the following LO problem produces.

$$(3.16) \quad \begin{aligned} \min \quad & -2x_1 - x_2 \\ \text{s.t.} \quad & 3x_1 - x_2 + 2 \leq 0 \\ & 3.6x_1 - 3.2x_2 \leq -5.4 \\ & x_1 + x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

An optimal solution of this problem is $(x_1, x_2) = (0.25, 2.75)$ and the objective value is $z = -3.25$.

If $d_1 \leq 0$, we have the following LO problem.

$$(3.17) \quad \begin{array}{ll} \min & -2x_1 - x_2 \\ \text{s.t.} & 3x_1 - x_2 + 2 \geq 0 \\ & 8.4x_1 - 4.8x_2 \leq -8.6 \\ & x_1 + x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

An optimal solution of Problem (3.17) is $(x_1, x_2) = (0.439, 2.561)$, with the objective value $z = -3.439$. Comparison of Problems (3.16) and (3.17) reveals that the optimal solution of the one-stage uncertain LO problem (3.15) is $(x_1, x_2) = (0.439, 2.561)$.

3.2. Case $r = 2$. In this case, we suppose that the input data depends on two independent uncertain variables ξ_1 and ξ_2 . We still assume that the problem has a single uncertain constraint. Corresponding problem is

$$(3.18) \quad \min\{c^T x \mid M\{\xi_1(-d_1) + \xi_2(-d_2) \leq \bar{d}\} \geq \alpha, x \in X\}.$$

Depending on the values of d_1 and d_2 (both varies with x) four subproblem are identified.

- (1) Let $x \in X$ is such that $d_1 \geq 0$ and $d_2 \geq 0$. We will add these inequities to the problem. Moreover

$$(3.19) \quad M\{-\xi_1(-d_1)^- - \xi_2(-d_2)^- \leq \bar{d}\} \geq \alpha,$$

holds if and only if

$$(3.20) \quad -d_1\Phi_1^{-1}(1-\alpha) + d_2\Phi_2^{-1}(1-\alpha) \leq \bar{d}.$$

With the assumption $\xi_1 = L(a_1, b_1)$ and $\xi_2 = L(a_2, b_2)$, (21) is identical with

$$(3.21) \quad -d_1(b_1 + (a_1 - b_1)\alpha) - d_2(b_2 + (a_2 - b_2)\alpha) \leq \bar{d}.$$

Therefore, Problem (3.18) reduces to

$$(3.22) \quad \min\{c^T x \mid -d_1(b_1 + (a_1 - b_1)\alpha) - d_2(b_2 + (a_2 - b_2)\alpha) \leq \bar{d}, \\ d_1 \geq 0, d_2 \geq 0, x \in X\}.$$

Without further details, crisp LO problems in other three cases are as follows.

- (1) With the assumption $d_1 \leq 0$ and $d_2 \geq 0$, we have the following problem.

$$(3.23) \quad \min\{c^T x \mid -d_1(a_1 + (b_1 - a_1)\alpha) - d_2(b_2 + (a_2 - b_2)\alpha) \leq \bar{d}, \\ d_1 \leq 0, d_2 \geq 0, x \in X.\}$$

- (2) With the assumption $d_1 \geq 0$ and $d_2 \leq 0$, we have the following problem.

$$(3.24) \quad \min\{c^T x \mid -d_1(b_1 + (a_1 - b_1)\alpha) - d_2(a_2 + (b_2 - a_2)\alpha) \leq \bar{d}, \\ d_1 \geq 0, d_2 \leq 0, x \in X.\}$$

- (3) With the assumption $d_1 \leq 0$ and $d_2 \leq 0$, we have the following problem.

$$(3.25) \quad \min\{c^T x \mid -d_1(a_1 + (b_1 - a_1)\alpha) - d_2(a_2 + (b_2 - a_2)\alpha) \leq \bar{d}, \\ d_1 \geq 0, d_2 \leq 0, x \in X.\}$$

Solving these problems produces the optimal solution of the Problem (3.18). If none of the Problems (3.22)-(3.25) is feasible, the Problem (3.18) has no solution for the belief rate α . Feasible solution may be possible with smaller values of α . When some of these problems has optimal solution, then the solution of (3.18) is identical with the one with smallest objective value. Let us explain the result via a simple example.

Example 3.2. Consider the following one-stage uncertain LO problem. In all subproblems, we fixed the belief rate at $\alpha = 0.9$.

$$(3.26) \quad \begin{array}{ll} \min & -2x_1 - x_2 \\ \text{s.t.} & M\{(x_1 - x_2)\xi_1 + (-2x_1 + x_2 - 2)\xi_2 \leq 2x_2 - 3\} \geq 0.9 \\ & x_1 + x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0, \end{array}$$

where $\xi_1 = L(1, 3)$ and $\xi_2 = L(2, 4)$. In the first three cases, the problems have optimal solutions as follows, while the optimization problem in the final case is infeasible. The objective values and optimal solutions are as follows.

- (1) $(x_1, x_2) = (1.5, 1.5)$ and $z = -4.5$.
- (2) $(x_1, x_2) = (0.333, 2.667)$ and $z = -3.333$.
- (3) $(x_1, x_2) = (3, 0)$ and $z = -6$.

Therefore, for the belief rate $\alpha = 0.9$, optimal solution of problem (3.26) is produced by the third case as $(x_1, x_2) = (3, 0)$ and $z = -6$.

As these two cases reveal, when there is only a single uncertain constraint with r independent uncertain variables, the solution approach leads to 2^r subproblems, with possible infeasibility of some. In the next section, we present an algorithm for finding the solution when more than one uncertain constraint exists.

4. COMPUTATIONAL ALGORITHM

Let us organize the solution approach as an algorithm. The suggested algorithm runs like the branch and bound algorithm in the integer LO problem. To explain the algorithm, we restrict it to the case $r = 2$, with multiple uncertain constraints. It can be easily generalized for arbitrary number of uncertain variables. We also assume that the two uncertain variables are linear as $\xi_1 = L(a_1, b_1)$ and $\xi_2 = L(a_2, b_2)$. In this way, we have the following uncertain problem.

$$(4.1) \quad \min\{c^T x \mid M\{-\xi_1(-d_{1i}) - \xi_2(-d_{2i}) \leq \bar{d}_i\} \geq \alpha, i = 1, \dots, m, x \in X\}.$$

Starting Step. Solve the problem ignoring uncertain constraints. If this problem is infeasible, *Stop*. The uncertain problem (4.1) has no solution for any belief rate α .

Branching Step.

- (1) Set $i = 1$.
- (2) Depending on the values of d_{1i} and d_{2i} , consider 4 subproblems. To each subproblem, three additional constraints are added to the problem in the parent node.
- (3) Solve each produced problem.
- (4) If one of these subproblems is infeasible, cut the branching of that node at the subsequent branching step.
- (5) Save all existing optimal solutions and objective values.
- (6) If $i = m$, go to the next step, otherwise set $i := i + 1$, and go to 2.

Decision Making Step

- (1) If there is no saved optimal solution and object value, *Stop*. The problem (4.1) has no solution.
- (2) Find the smallest objective value. The optimal solution of the corresponding subproblem is the optimal solution of (4.1).

For more details let us mention one of the subproblems as

$$(4.2) \quad \min\{c^T x \mid -d_{11}(b_1 + (a_1 - b_1)\alpha) - d_{21}(b_2 + (a_2 - b_2)\alpha) \leq \bar{d}_1, \\ d_{11} \geq 0, d_{21} \geq 0, x \in X\},$$

First Level Constraints	Node Name	Second Level Constraints	Optimal Solution	Objective Value
$d_{11} \geq 0, d_{12} \geq 0$	C_{11}	$d_{21} \geq 0, d_{22} \geq 0$	$x = (0, 0)$	$z = 0$
	C_{12}	$d_{21} \geq 0, d_{22} \leq 0$	$x = (0.32, 0.32)$	$z = -0.32$
	C_{13}	$d_{21} \leq 0, d_{22} \geq 0$	Infeasible	N.A.
	C_{14}	$d_{21} \leq 0, d_{22} \leq 0$	Infeasible	N.A.
$d_{11} \geq 0, d_{12} \leq 0$	C_{21}	$d_{21} \geq 0, d_{22} \geq 0$	Infeasible	N.A.
	C_{22}	$d_{21} \geq 0, d_{22} \leq 0$	Infeasible	N.A.
	C_{23}	$d_{21} \leq 0, d_{22} \geq 0$	Infeasible	N.A.
	C_{24}	$d_{21} \leq 0, d_{22} \leq 0$	Infeasible	N.A.
$d_{11} \leq 0, d_{12} \geq 0$	C_{31}	$d_{21} \geq 0, d_{22} \geq 0$	$x = (1, 0)$	$z = -2$
	C_{32}	$d_{21} \geq 0, d_{22} \leq 0$	$x = (0.75, 0.25)$	$z = -1.75$
	C_{33}	$d_{21} \leq 0, d_{22} \geq 0$	$x = (3, 0)$	$z = -6$
	C_{34}	$d_{21} \leq 0, d_{22} \leq 0$	$x = (1.57, 0.52)$	$z = -3.652$

TABLE 1. Numerical results in Example 4.1.

and in a sample subproblem to the problem (4.2) in level $i = 2$, the following three constraints will be added.

$$(4.3) \quad -d_{11}(b_1 + (a_1 - b_1)\alpha) - d_{21}(b_2 + (a_2 - b_2)\alpha) \leq \bar{d}_1, d_{11} \geq 0, d_{21} \geq 0,$$

We assert that when the uncertain problem (4.1), has no feasible solution, the decision maker may desire to reduce the value of belief rate to have a feasible and consequently an optimal solution.

To illustrate the algorithm, we continue with two simple examples.

Example 4.1. Consider the following problem. We fixed the belief rate at $\alpha = 0.9$.

$$(4.4) \quad \begin{aligned} & \min && -2x_1 + x_2 \\ & s.t. && M\{(x_1 - x_2)\xi_1 + (-2x_1 + x_2 - 2)\xi_2 \leq 2x_2 - 3\} \geq 0.9 \\ & && M\{(x_1 + x_2 - 1)\xi_1 + (-x_1 + 3x_2)\xi_2 \leq x_1 - x_2 + 2\} \geq 0.9 \\ & && x_1 + x_2 \leq 3, x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

where $\xi_1 = L(1, 3)$ and $\xi_2 = L(2, 4)$. The results are depicted in Table 1. Here, optimization problem in some nodes are infeasible, specially when at the first level $d_{11} \leq 0, d_{12} \leq 0$, which is dropped from the table. As the results reveal, an optimal solution of (4.4) is $(x_1, x_2) = (3, 0)$, with the objective value $z = -6$, identified by the subproblem in Node C_{33} .

First Level Constraints	Node Name	Second Level Constraints	Optimal Solution	Objective Value
$d_{11} \geq 0, d_{12} \geq 0$	C_{11}	$d_{21} \geq 0, d_{22} \geq 0$	$x = (-1, 0)$	$z = 1$
	C_{12}	$d_{21} \geq 0, d_{22} \leq 0$	$x = (0, -2)$	$z = -6$
	C_{13}	$d_{21} \leq 0, d_{22} \geq 0$	$x = (-1, 0)$	$z = 1$
	C_{14}	$d_{21} \leq 0, d_{22} \leq 0$	$x = (-1, 0)$	$z = 1$
$d_{11} \geq 0, d_{12} \leq 0$	C_{21}	$d_{21} \geq 0, d_{22} \geq 0$	$x = (-1, 0)$	$z = 1$
	C_{22}	$d_{21} \geq 0, d_{22} \leq 0$	$x = (-0.7, 0.1)$	$z = 1$
	C_{23}	$d_{21} \leq 0, d_{22} \geq 0$	$x = (-1.118, 0.236)$	$z = 1.829$
	C_{24}	$d_{21} \leq 0, d_{22} \leq 0$	$x = (-1.092, 0.184)$	$z = 1.642$
$d_{11} \leq 0, d_{12} \geq 0$	C_{31}	$d_{21} \geq 0, d_{22} \geq 0$	$x = (0, -2)$	$z = -6$
	C_{32}	$d_{21} \geq 0, d_{22} \leq 0$	$x = (3.69, -9.379)$	$z = -31.82$
	C_{33}	$d_{21} \leq 0, d_{22} \geq 0$	$x = (0, 0.333)$	$z = 1$
	C_{34}	$d_{21} \leq 0, d_{22} \leq 0$	$x = (0, 0.333)$	$z = 1$
$d_{11} \leq 0, d_{12} \leq 0$	C_{41}	$d_{21} \geq 0, d_{22} \geq 0$	$x = (0, 0.333)$	$z = 1$
	C_{42}	$d_{21} \geq 0, d_{22} \leq 0$	$x = (0, 0.247)$	$z = 0.74$
	C_{43}	$d_{21} \leq 0, d_{22} \geq 0$	$x = (0, 0.396)$	$z = 1.188$
	C_{44}	$d_{21} \leq 0, d_{22} \leq 0$	$x = (0, 0.396)$	$z = 1.188$

TABLE 2. Numerical results in Example 4.2.

Example 4.2. Consider the following problem. We fixed the belief rate at $\alpha = 0.95$.

(4.5)

$$\begin{aligned}
 \min \quad & -x_1 + 3x_2 \\
 \text{s.t.} \quad & M\{2x_1\xi_1 + (-x_1 + 3x_2 - 1)\xi_2 \leq x_1 + x_2\} \geq 0.95 \\
 & M\{(-x_1 + 2x_2 - 2)\xi_1 + (2x_1 + x_2 + 3)\xi_2 \leq 3x_1 - 2x_2 + 3\} \geq 0.95 \\
 & x_1 + x_2 \leq 3
 \end{aligned}$$

where $\xi_1 = L(2, 4)$ and $\xi_2 = L(1, 2)$. The results are depicted in Table 2. As the results show, all subproblems have optimal solution, and an optimal solution of (4.5) is $(x_1, x_2) = (3.69, -9.379)$, with the objective value $z = -31.82$, identified by the subproblem in Node C_{32} .

5. CONCLUDING REMARKS

In this paper, we considered a one-stage LO problem, where some of its constraints data is uncertain and satisfying these constraints are desired with a prescribed belief rate. A computation algorithm was presented and the methodology is examined on some simple examples. Though the complexity of the algorithm was not studied, obviously it can be improved in the implementation by effective computational codes. For example, saving the final basis of a subproblem in each node can reduce the complexity of solving the subsequent subproblems. Moreover, using the depth-first

strategy in visiting the nodes could further reduce the computational efforts. The methodology can be generalized for other optimization problems other than LO. One may also consider a bounding strategy in the pruning of the branches.

REFERENCES

- [1] A. Ben-Tal and A. Nemirovski, *Robust solutions of linear programming problems contaminated with uncertain data*, Mathematical programming, **88**(2000), 411-424.
- [2] J.R. Birge and F. Louveaux, *Introduction to stochastic programming*, Springer Science & Business Media,(2011).
- [3] A. Ghaffari-Hadigheh, *Two linear programming models in uncertain environment (in Persian)*, Journal of Operational Research in Its Applications, **50**, 37-51.
- [4] A. Ghaffari-Hadigheh and N. Mehanfar, *Matrix Perturbation and Optimal Partition Invariancy in Linear Optimization*, Asia-Pacific Journal of Operational Research, **32**(2015).
- [5] A. Ghaffari-Hadigheh and Terlaky Tams, *Sensitivity analysis in linear optimization: Invariant support set intervals*, European Journal of Operational Research, **169**(2006), 1158-1175.
- [6] B. Liu, *Uncertainty theory*, Springer, (2007).
- [7] B. Liu, *Uncertainty theory: A branch of mathematics for modeling human uncertainty*, Springer, **300**(2011).
- [8] G.H. Huang and M.F. Cao, *Analysis of Solution Methods for Interval Linear Programming*, Journal of Environmental Informatics, **17**(2011).
- [9] H. Naseri, *Fuzzy Simplex Methods for Fuzzy uncertain linear programming*, Operations Research and Applications, **4**(2013).
- [10] H. Tanaka and K. Asai, *Fuzzy linear programming problems with fuzzy numbers*, Fuzzy Sets and Systems, **13**(1984),1-10.
- [11] M. Zheng, Y. Yi, Z. Wang and J.F. Chen, *Study on two-stage uncertain programming based on uncertainty theory*, Journal of Intelligent Manufacturing, (2017), 633-642.
- [12] Roos, Cornelis, Terlaky, Tams and Vial, *Interior point algorithms for linear optimization*, Springer Science, (2005).

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