*Journal of Hyperstructures* 7 *(2) (2018), 94-103.*
*ISSN: 2322-1666 print/2251-8436 online*

# QUOTIENT BRK-ALGEBRAS

K. VENKATESWARLU AND GIRUM AKLILU

Abstract. In this paper we introduce the notion of quotient BRK-algebras and also investigate the properties of these algebras. Further we establish first isomorphism theorem for the special subclass of BRK-algebras namely anti-symmetric BRK-algebras.

**Key Words:** BRK-algebra, congruences, translation ideal, anti-symmetric BRK- algebra.
**2010 Mathematics Subject Classification:** Primary: 06F35; Secondary 08A35.

## 1. Introduction

Two classes of abstract algebras namely BCK and BCI algebras were initially introduced by Y. Imai and K. Iseki [1, 2]. It is known that BCK is a proper subclass of BCI algebras. Subsequently many researchers introduced and studied extensively on generalizations of BCK/BCI-algebras namely BCH-algebras by Hu and LI [3], Q- algebras by J.Negger and etl [4], BRK-algebras by Ravi Kumar in [5] . The order of generalization is as follows BCK/BCI/BCH/Q/BRK-algebras.

In this paper, we investigate the study of BRK- algebras by introducing the notion of homomorphism, congruence and translation ideals. The process of constructing a quotient BRK-algebra is in the usual way but not with simply ideal. A stronger condition has been introduced on ideal called translation ideal to obtain a quotient BRK-Algebra. Further we gave an example (see 4.6) that an ordinary ideal does not give a congruence. Also we introduce a sub class of BRK- algebra called anti -symmetric BRK-algebra which is a common subclass of the two

Received: 12 January 2018, Accepted: 11 April 2018. Communicated by Irina Cristea;
∗Address correspondence to K. VENKATESWARLU; E-mail:drkvenkateswarlu@gmail.com.

distinct algebras namely BRK-algebra and BH-algebra. Finally we conclude this paper by establishing the first isomorphism theorem for the anti-symmetric BRK-algebras, which are a subclass of BRK-algebras.

## 2. Preliminaries

We collect certain definitions and examples from the existing literature.

**Definition 2.1.** Let $X = (X, *, 0)$ be an algebra of type $(2, 0)$. Then $X$ is called:

(1) BCH-algebra ([3]) if it satisfies
 (1) $x * x = 0$,
 (2) $x * y = 0$ and $y * x = 0$ imply $x = y$,
 (3) $(x * y) * z = (x * z) * y$

(2) a Q-algebra ([4]) if it satisfies (1), (3) and
 (4) $x * 0 = x$.

(3) a BH-algebra ([6]) if it satisfies (1), (2) and (4).

**Definition 2.2.** ([5]). An algebra $(X, *, 0)$ of type $(2, 0)$ is called a BRK-algebra if it satisfies the following axioms

(1) $x * 0 = x$,
(2) $(x * y) * x = 0 * y$

for all $x, y \in X$.

*Example* 2.3. ([5]) Let $X = R \setminus \{-n\}$, $0 \neq n \in Z^+$ where $R$ is the set of real numbers and $Z^+$ is the set of positive integers.If we define a binary operation $*$ on $X$ by

$$x * y := \frac{n(x - y)}{n + y},$$

then $(X, *, 0)$ is a BRK algebra.

*Example* 2.4. ([5]) Let $X = \{0, 1, 2\}$ be a set with the following Cayley table:

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 2 | 2 |
| 1 | 1 | 0 | 0 |
| 2 | 2 | 0 | 0 |

Then $(X, *, 0)$ is a BRK algebra.

**Theorem 2.5.** *([5]) In any BRK-algebra $X$ the following holds for any $x, y \in X$,*

(1) $x * x = 0$
(2) $0 * (x * y) = (0 * x) * (0 * y)$
(3) $x * y = 0$ *implies* $0 * x = 0 * y$

**Definition 2.6.** Let $X$ be a BRK-algebra and let $I$ be nonempty subset of $X$. Then

(1) $I$ is called a subalgebra of $X$ if $x * y \in I$ for all $x, y \in I$
(2) $I$ is called an ideal of $X$ if for any $x, y \in X$:
  (i) $0 \in I$,
  (ii) $x * y \in I$ and $y \in I$ imply $x \in I$.
(3) $I$ is called closed ideal of $X$ if it is both an ideal and a subalgebra.

*Remark* 2.7. In general an ideal of a BRK-algebra may not be a subalgebra and vice-versa.

*Example* 2.8. Let $X = (Z, -, 0)$ be the BRK-algebra of set of integers under subtraction. Then the set I of all non negative integer forms an ideal which is not a subalgebra. Indeed $0 \in I$ and if $x, y - x \in I$ then $0 \leq x$ and $0 \leq y - x$ which imply $0 \leq y$ and hence $y \in I$ ( in this case the relation $\leq$ is the usual order of real numbers. Thus I is an ideal of X. Clearly $I$ is not a subalgebra as it is not closed under subtraction.

## 3. Translation Ideals, Homomorphisms

We introduce the notion of translation ideals in a BRK-algebra.

**Definition 3.1.** An ideal $I$ of a BRK algebra $X$ is called translation ideal if it satisfies the condition $x * y, y * x \in I \Rightarrow (x * z) * (y * z), (z * x) * (z * y) \in I$.

*Example* 3.2. Let $X = (Z, -, 0)$ be the BRK-algebra of set of integers under the usual subtraction of real numbers. Then the set of all non negative integers forms a translation ideal of $X$.

The following two examples demonstrate the existence of an ideal which is not translation.

*Example* 3.3. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 0 | 1 | 0 |

Then $(X, *, 0)$ is a BRK-algebra. Clearly $I = \{0\}$ is an ideal but not translation since $3 * 1, 1 * 3 \in I$ but $(3 * 2) * (1 * 2) = 1 \notin I$.

*Example* 3.4. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 |
| 2 | 2 | 3 | 0 | 2 |
| 3 | 3 | 3 | 0 | 0 |

Then $(X, *, 0)$ is a BRK-algebra and $I = \{0, 1\}$ is an ideal of $X$ which is not translation (as $0 * 1, 1 * 0 \in I$ but $(2 * 0) * (2 * 1) = 2 \notin I$).

*Remark* 3.5. In general a translation ideal may not be closed (see Example 3.2).

**Definition 3.6.** Let X and Y be BRK-algebras. A mapping $f : X \longrightarrow Y$ is called a homomorphism from $X$ into $Y$ if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

A homomorpism $f$ is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorpism. Two BRK-algebras $X$ and $Y$ are said to be isomorphic, written $X \cong Y$, if there exists an isomorphism $f : X \longrightarrow Y$. For any homomorphism $f : X \longrightarrow Y$ the set $\{x \in X : f(x) = 0\}$ is called kernel of $f$, denoted by $Ker(f)$ and the set $\{f(x) : x \in X\}$ is called the image of $f$, denoted by $Imf$.

**Lemma 3.7.** *Let $f : X \longrightarrow Y$ be homomorphism of BRK-algebras. Then*

(1) $f(0) = 0$
(2) $x * y = 0$ *implies* $f(x) * f(y) = 0$.

*Proof.*
(1) $f(0) = f(0 * 0) = f(0) * f(0) = 0$.
(2) If $x * y = 0$, then $f(x * y) = f(0)$ which implies $f(x) * f(y) = 0$. □

**Theorem 3.8.** *Let $f : X \longrightarrow Y$ be homomorphism of BRK-algebras.*

i. *If $S$ is a subalgebra of $X$, then $f(S)$ a subalgebra of $Y$*
ii. *If $K$ is a subalgebra of $Y$, then $f^{-1}(K)$ is a subalgebra of $X$ containing $Kerf$*
iii. *If $I$ is an ideal of $X$ and $f$ is injective, then $f(I)$ is an ideal of $f(X)$.*
iv. *If $J$ is an ideal of $Y$, then $f^{-1}(J)$ is an ideal of $X$.*
v. *If $I$ is a translation ideal of $X$ and $f$ is injective, then $f(I)$ is a translation ideal of $f(X)$.*
vi. *If $J$ is a translation ideal of $Y$, then $f^{-1}(J)$ is a translation ideal of $X$.*

*Proof.* Straightforward □

**Corollary 3.9.** *If $f : X \longrightarrow Y$ is homomorphism of BRK algebras, then $Kerf$ is a closed ideal of $X$ and $Imf$ is a subalgebra of $Y$.*

*Remark* 3.10. For any BRK-homomorphism $f$,

(1) $kerf = \{0\}$ may not imply $f$ is injective.
For instance, let $X = (X, *, 0)$ be the BRK-algebra where $X = \{0, 1, 2\}$ and $*$ is given by the Cayley table:

| * | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 2 | 2 |
| 1 | 1 | 0 | 0 |
| 2 | 2 | 0 | 0 |

Let $f : X \longrightarrow X$ be defined by $f(0) = 0$ and $f(1) = f(2) = 2$. clearly $f$ is a homomorphism with $kerf = \{0\}$ but $f$ is not injective.

(2) $kerf$ may not be a translation ideal of $X$.
For instance, consider the BRK-algebra $X$ in example 3.3. Clearly the identity map $id : X \longrightarrow X$ is a homomorphism, but $kerf = \{0\}$ is not a translation ideal.

## 4. Quotient BRK-Algebra

In this section we will study the quotient algebra of BRK-algebra. We define congruence relation on BRK-algebra as in the usual way.

**Definition 4.1.** An equivalence relation $\theta$ on a BRK-algebra $X$ is called a congruence relation if it has a compatibility property:

$(x, y) \in \theta$ and $(u, v) \in \theta$ imply $(x*u, y*v) \in \theta$ for all $x, y, u, v \in X$.

Given a congruence relation $\theta$ on a BRK-algebra $X$, we use the notation $\theta_x$ for the equivalence class determined by $x$ i.e. $\theta_x = \{y \in X : (y, x) \in \theta\}$ and $X/\theta$ for the quotient set $\{\theta_x : x \in X\}$.

**Theorem 4.2.** *Let $X$ be a BRK-algebra. Define $*$ on the quotient set $X/\theta$ by $\theta_x * \theta_y = \theta_{x*y}$. Then $(X/\theta, *, \theta_0)$ is a BRK-algebra which is called a quotient BRK-algebra induced by the congruence $\theta$.*

*Proof.* Since $\theta$ is a congruence relation $*$ is well defined. For any $\theta_x, \theta_y \in X/\theta$ we have

(1) $\theta_x * \theta_0 = \theta_{x*0} = \theta_0$, and
(2) $(\theta_x * \theta_y) * \theta_x = \theta_{x*y} * \theta_x = \theta_{(x*y)*x} = \theta_{0*y} = \theta_0 * \theta_y$.

Thus $(X/\theta, *, \theta_0)$ is a BRK-algebra. □

**Theorem 4.3.** *Let $X/\theta = (X/\theta, *, \theta_0)$ be the quotient BRK-algebra induced by a congruence $\theta$. Then $\theta_0$ is a closed ideal of $X$.*

*Proof.* Since $(0, 0) \in \theta$, $0 \in \theta_0$. Suppose $x, y * x \in \theta_0$, then $(x, 0) \in \theta$ and $(y * x, 0) \in \theta$. Now from $(y, y) \in \theta$ and $(x, 0) \in \theta$ we have $(y * x, y) \in \theta$. Also from $(y * x, y) \in \theta$ and $(y * x, 0) \in \theta$ we get $(0, y) \in \theta$ and hence by symmetry we have $(y, 0) \in \theta$ i.e. $y \in \theta_0$. Therefor $\theta_0$ is an ideal of $X$. Next, if $x, y \in \theta_0$ then $(x, 0) \in \theta$ and $(y, 0) \in \theta$ and hence $(x * y, 0) \in \theta$ i.e. $x * y \in \theta_0$. Thus it is closed. □

Now we construct a congruence relation on $X$ via translation ideal.

**Theorem 4.4.** *Let $I$ be a translation ideal of a BRK-algebra X. Define a relation $\backsim$ on $X$ by $x \backsim y \Leftrightarrow x * y, y * x \in I$. Then $\backsim$ is a congruence relation on $X$ which is called the congruence relation of $X$ induced by a translation ideal $I$.*

*Proof.* For any $x \in X$, since $x * x = 0 \in I$, we have $x \backsim x$ hence it is reflexive. From the definition of $\backsim$ it is clear that $\backsim$ is symmetric. If $x \backsim y$ and $y \backsim z$ then $x * y, y * x, y * z, z * y \in I$. Now $x * y, y * x \in I$ implies $(x * z) * (y * z) \in I$. But then since $I$ is an ideal and $y * z \in I$, $x * z \in I$. Similarly $z * x \in I$. Thus $\backsim$ is transitive. Hence $\backsim$ is an equivalence relation. Next suppose $x \backsim y$ and $u \backsim v$. But then since $I$ is a translation ideal we have $x * y, y * x \in I \Rightarrow (x * u) * (y * v) \in I \Rightarrow ((x * u) * (y * v)) * ((y * u) * (y * v)) \in I$ and hence $(x * u) * (y * v) \in I$ (as I is an ideal and $(y * u) * (y * v) \in I$). Similarly we can show that $(y * v) * (x * u) \in I$. Thus $\backsim$ is a congruence relation. □

*Remark* 4.5. In the above theorem if we take an arbitrary ideal instead of translation , the relation may not be congruence.

*Example* 4.6. Consider the BRK-algebra $X$ and its ideal $I$ in Example3.4. Define $\sim$ on $X$ by $x \sim y \Leftrightarrow x * y, y * x \in I$. But then $2 \sim 2$ and $0 \sim 1$ but $2 * 0 = 2 \nsim 3 = 2 * 1$ as $2 * 3 = 2 \notin I$ hence not a congruence relation.

Now for any translation ideal $I$ of a BRK-algebra $X$ we use the notation $I_x$ for the equivalence class determined by $x$ and $X/I$ for the set of all equivalence classes of X for the congruence relation of $X$ induced by $I$. Clearly $I_x = \{y \in X : x \sim y\}$ and $X/I = \{I_x : x \in X\}$.

**Corollary 4.7.** *Let $X$ be a BRK-algebra and $I$ be a translation ideal of $X$. Define $*$ on $X/I$ by $I_x * I_y = I_{x*y}$, for all $x, y \in X$. Then $(X/I, *, I_0)$ is a BRK algebra.*

**Definition 4.8.** $(X/I, *, I_0)$ is called the quotient BRK-algebra of $X$ determined by a translation ideal $I$

*Remark* 4.9. Let X be a BRK-algebra. Then

(1) in general for a translation ideal I of X, $I_0$ may not be equal to $I$ and
(2) a translation ideal I of X is closed if and only if $I_0 = I$.

## 5. ANTI-SYMMETRIC BRK-ALGEBRA

In this section we introduce a new subclass of BRK-algebra called anti-symmetric BRK-algebra and we will estabilish the homomorpism theorem for this subclass.

**Definition 5.1.** A BRK-algebra $X$ is called an anti-symmetric BRK-algebra if it satisfies (2): $x * y = 0$ and $y * x = 0$ imply $x = y$ for any $x, y \in X$.

The following examples illustrate such algebra exist.

*Example* 5.2. Let $X = \{0, 1, 2\}$ be a set with Cayley table:

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 2 | 2 |
| 1 | 1 | 0 | 0 |
| 2 | 2 | 0 | 0 |

Then $(X, *, 0)$ is a BRK- algebra which is not anti-symmetric BRK-algebra as $2 * 1 = 1 * 2 = 0$ but $1 \neq 2$.

*Example* 5.3. Let $X = \{0, 1, 2, 3\}$ be a set with Cayley table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 |
| 2 | 2 | 1 | 0 | 1 |
| 3 | 3 | 2 | 3 | 0 |

Then $(X, *, 0)$ is an anti-symmetric BRK-algebra which is not BCH as $(3 * 1) * 1 = 0 \neq 2 = (3 * 2) * 1$.

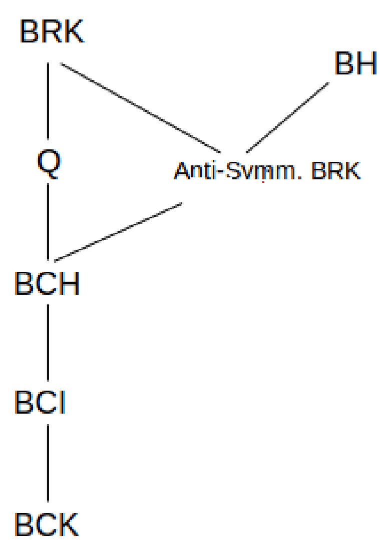
*Example* 5.4. Let $X = \{0, 1, 2, 3\}$ be a set with Cayley table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 3 | 0 | 2 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 2 | 0 | 3 |
| 3 | 3 | 3 | 1 | 0 |

Then $(X, *, 0)$ is a BH-algebra which is not an anti-symmetric BRK-algebra as $(2 * 3) * 2 = 1 \neq 2 = 0 * 3$.

*Remark* 5.5. Every anti-symmetric BRK-algebra is a BH- algebra but not the converse.

The following figure shows the relationship of the algebras.



**Theorem 5.6.** *Let $X$ and $Y$ be anti-symmetric BRK-algebra and $f : X \longrightarrow Y$ be BRK-homomorhism. Then $f$ is injective if and only if $kerf = \{0\}$.*

*Proof.* Obviously if $f$ is injective then clearly $Kerf = \{0\}$. On the other hand, suppose that $x, y \in X$ and $f(x) = f(y)$. Then $f(x * y) = f(x) * f(y) = f(x) * f(x) = 0$. Hence $x * y \in Kerf$ and so $x * y = 0$. Similarly we have $y * x = 0$. Thus $x = y$ and hence $f$ is injective. □

**Theorem 5.7.** *Let $X$ and $Y$ be anti-symmetric BRK-algebra and $f : X \longrightarrow Y$ be BRK-homomorhism. Then $Kerf$ is a translation ideal of $X$.*

*Proof.* Since $Kerf$ is an ideal of $X$, it is enough to show that it is translation. If $x * y, y * x \in kerf$, then $f(x) * f(y) = f(y) * f(x) = 0 \Rightarrow f(x) = f(y)$. But then for any $z \in X$ $f((x * z) * (y * z)) = (f(x) * f(z)) * (f(y) * f(z)) = (f(x) * f(z)) * ((f(x) * f(z)) = 0$ which implies $(x * z) * (y * z) \in kerf$. Similarly $(z * x) * (z * y) \in kerf$. Thus $Kerf$ is a translation ideal. □

**Theorem 5.8.** *Let $X$ and $Y$ be anti-symmetric BRK-algebra and $f : X \longrightarrow Y$ be a BRK homomorphism. If $I = Kerf$, then $X/I \cong Imf$.*

*Proof.* As I is a translation ideal of X, $X/I$ is a BRK algebra. Define a mapping $\alpha : X/I \longrightarrow Imf$ by $\alpha(I_x) = f(x)$. Then

(1) $\alpha$ is well defined. Suppose $I_x = I_y$ for some $I_x, I_y \in X/I$. Then
$I_x = I_y \Rightarrow x * y, y * x \in I \Rightarrow f(x * y) = f(y * x) = 0$
$\Rightarrow f(x) * f(y) = f(y) * f(x) = 0 \Rightarrow f(x) = f(y) \Rightarrow \alpha(I_x) = \alpha(I_y)$.
Thus $\alpha$ is well defined.

(2) $\alpha$ a homomorpism. For any $I_x, I_y \in X/I$ we have
$\alpha(I_x * I_y) = \alpha(I_{x*y}) = f(x * y) = f(x) * f(y) = \alpha(I_x) * \alpha(I_y)$.
Therefor $\alpha$ is a homomorphism.

(3) $\alpha$ is injecive. Let $\alpha(I_x) = \alpha(I_y)$ for some $I_x, I_y \in X/I$. Then
$\alpha(I_x) = \alpha(I_y) \Rightarrow f(x) = f(y) \Rightarrow f(x) * f(y) = f(y) * f(x) = 0$
$\Rightarrow f(x * y) = f(y * x) = 0 \Rightarrow x * y, y * x \in I \Rightarrow I_x = I_y$.
Hence $\alpha$ is one to one.

(4) $\alpha$ is onto. Indeed let $y$ be any element in $Imf$. But then there exists $x \in X$ such that $f(x) = y$. Now $I_x \in X/I$ and

$$\alpha(I_x) = f(x) = y \text{ and hence } \alpha \text{ is onto.}$$

Hence $\alpha$ is an isomorphism and $X/I \cong Imf$. □

**Acknowledgments**

The authors thank the referee for his/her valuable comments and suggestions in improving this paper. Moreover the second author would like to thank the authorities of Dr. Lankapalli Bullyya College for providing the necessary facilities to carry out the research.

## References

[1] Y. Imai and K. Iseki : *On axiom systems of propositional calculi. XIV*, Proceedings of the Japan Academy.,vol.42, pp. 19-22(1966)

[2] K. Iseki: *An algebra related with a propositional calculus*,Proceedings of the Japan Academy., vol.42, pp. 26-29 (1966)

[3] Q. P. Hu and X. Li: *On BCH algebras* Mathematics Seminar Notes, vol. 11, pp. 313-320, (1983)

[4] J. Negger, S. S. Ahn, and H. S. Kim: *On Q algebras*, International Journal of Mathematics and Mathematical Sciences, vol. 27, no. 12, pp. 749-757, (2001)

[5] Ravi Kumar Bandaru: *On BRK Algebras*, International Journal of Mathematics and Mathematical Sciences vol. 2012, Article ID 952654

[6] Y. B. Jun, E. H. Roh, and H. S. Kim: *On BH algebras*, Scientiae Mathematicae Japonica, vol. 1, no. 3, pp. 347-354 (1998)

[7] C. B. Kim and H. S. Kim: *On BM algebras*, Scientiae Mathematicae Japonica, vol 63, no. 3, pp. 421-427 (2006)

[8] H. Yisheng, : *BCI algebra*, Science press, China, pp. 1-92 (2006)

[9] Andrzej Walendziak: *On BM-algebra and related topics*, Mathematica Slovaca . (2014)

**K.Venkateswarlu**
Department of Mathematics, Dr. Lankapalli Bullayya College Visakhapatnam, India
Email: drkvenkateswarlu@gmail.com

**Girum Aklilu Abebe**
Department of Mathematics, Addis Ababa University Addis Ababa , Ethiopia
Email: girum.akililu@gmail.com