Journal of Hyperstructures 7 *(2) (2018), 104-123.*
ISSN: 2322-1666 print/2251-8436 online

UNIFORMITIES ON OCP-POLYGROUPS

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Abstract. A topological polygroup where every open subset is a complete part is called OCP-polygroup. Different uniform structures are built on OCP-polygroups as well as on their quotients and studied some related results including Roelcke uniformity.

Key Words: Topological polygroup, OCP-polygroup, left OCP-polygroup uniformity, right OCP-polygroup uniformity, two-sided OCP-polygroup uniformity, Roelcke uniformity.
2010 Mathematics Subject Classification: Primary: 20N20; Secondary: 22A30.

1. Introduction

During last three decades the theory of hypergroups introduced by Marty [26] has drawn much attention of fellow mathematicians especially those who have beautiful algebraic mind. Later it was studied by Dresher and Ore [15], Koskas [25], Corsini[10], Corsini and Leoreanu [11], Mittas [30], Vougiouklis [36], Davvaz[12], Freni[16], Tallini [35], and many others. In [20], Hoskova-Mayerova studied various kinds of continuity of hyperoperations, namely pseudocontinuous, strongly pseudocontinuous and continuous hyperoperations. Application of hypergroups have mainly appeared in special subclasses. For example, polygroups which are certain subclasses of hypergroups are studied in [23] by Ioulidis and are used to study color algebra [6, 7, 8, 9]. Quasi-canonical hypergroups (called "polygroups" by Comer) were introduced in [5], as a generalization of canonical hypergroups, introduced in [30]. There exists a rich

Received: 16 February 2018, Accepted: 23 April 2018. Communicated by Jianming Zhan;
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bibliography: publications appeared within 2013 can be found in "Polygroup Theory and Related Systems" by Davvaz [12]. This book contains the principle definitions endowed with examples and the basic results of the theory. Also see [1, 3, 13, 24, 28, 29]. Till now, only a few papers treated the notion of topological hyperstructures, for example see [2, 17, 18, 20, 32, 33, 34]. Heidari et al.[17, 18] introduced the concepts of topological hypergroups and topological polygroups. In [33], Shadkami et al. presented some facts about complete parts in polygroups and they used these facts to obtain some new results in topological polygroups and they introduced the concept of cp-resolvable topological polygroups. In [34], Shadkami et al. considered a topological polygroup and established various relations between complete parts and open sets and they investigated the properties of big subsets in a topological polygroup. In [32], Singha et al. introduced topological complete hypergroup, topological regular hypergroup and investigated some of their properties.

Now, this paper extends the theory of hypergroups by putting up different uniform structures along with Roelcke uniformity on a special class of topological polygroups as well as on their quotients.

2. Topological and Hyperalgebraic warmup

Let H be a nonempty set. A function $\circ : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the family of nonempty subsets of H, is called a *hyperoperation* and the ordered couple $(H, \circ)$, as in general case, is called a *hypergroupoid*. If A and B are two nonempty subsets of a hypergroupoid $(H, \circ)$ and $x \in H$, then

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \; x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, $x \circ (y \circ z) = (x \circ y) \circ z$ and it is called a *quasihypergroup* if *reproduction axiom* holds, that is for every $x \in H$, we have $x \circ H = H = H \circ x$. The couple $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup. A nonempty subset K of a hypergroup $(H, \circ)$ is called a *subhypergroup* if K is itself a hypergroup. In other words, a nonempty subset K of a hypergroup $(H, \circ)$ is a subhypergroup if (1) for all $a, b \in K \Rightarrow a \circ b \subseteq K$ and (2) for all a in K, $a \circ K = K = K \circ a$.

For $n > 1$, β_n is a relation on a semihypergroup H defined as follows:

$$a \beta_n b \Leftrightarrow \exists \; (x_1, x_2, ..., x_n) \in H^n \text{ such that } \{a, b\} \subseteq \prod_{i=1}^{n} x_i,$$

and let $\beta = \bigcup_{i=1}^{\infty} \beta_i$, where $\beta_1 = \{(x,x) : x \in H\}$ is the diagonal relation on H. Clearly, this relation is reflexive and symmetric. Koskas [25] introduced β^*−the transitive closure of β which coincides [16] with β on hypergroups. The relation β^* is called the *fundamental relation* on H and H/β^* is called the *fundamental group*. Let $(H,\circ)$ be a semihypergroup and A be a nonempty subset of H. Then, A is said to be a complete part of H if for any nonzero natural number n and for all $a_1, a_2, ..., a_n$ of H, the following implication holds:

$$A \cap \prod_{i=1}^{n} a_i \neq \phi \Rightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

Let $(H_1,\circ)$ and $(H_2,*)$ be two hypergroups. A map $f : H_1 \to H_2$ is called

(1) a *homomorphism* if for all x, y of H, we have $f(x \circ y) \subseteq f(x) * f(y)$;
(2) a *good homomorphism* if for all x, y of H, we have $f(x \circ y) = f(x) * f(y)$.

A *polygroup* [6, 12], which is a very special kind of hypergroup, is a system $P = \langle P, \circ, e, ^{-1} \rangle$, where $e \in P$, $^{-1}$ is a unitary operation on P, $\circ : P \times P \to \mathcal{P}^*(P)$ and the following axioms hold for all $x, y, z \in P$:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$;
(2) $e \circ x = x \circ e = \{x\}$;
(3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

The following elementary facts about polygroups follow easily from the axioms:

$$e \in x \circ x^{-1} \cap x^{-1} \circ x,\ e^{-1} = e,\ (x^{-1})^{-1} = x,\ \text{and}\ (x \circ y)^{-1} = y^{-1} \circ x^{-1}.$$

Let A be a nonempty subset of P. Then, $A^{-1} = \{x^{-1} : x \in A\}$. A nonempty subset K of a polygroup P is a *subpolygroup* of P if

(1) $a, b \in K$ implies $a \circ b \subseteq K$,
(2) $a \in K$ implies $a^{-1} \in K$.

A subpolygroup K of a polygroup P is *normal* in P if $a \circ K = K \circ a$, for all $a \in P$. Let $\langle P_1, \cdot, e_1, ^{-1} \rangle$ and $< P_2, *, e_2, ^{-I} \rangle$ be two polygroups. Then, the product $P_1 \times P_2$ with respect to the hyperoperation hyperoperation $\circ$ defined by

$$(x_1, y_1) \circ (x_2, y_2) = \{(x, y) : x \in x_1 \cdot x_2, y \in y_1 * y_2\},$$

is known as *direct hyperproduct* of P_1 and P_2, interestingly the direct hyperproduct $P_1 \times P_2$ is a polygroup [12]. Let $\langle P_1, \circ, e_1, ^{-1} \rangle$ and $\langle P_2, *, e_2, ^{-I} \rangle$ be two polygroups. A good homomorphism $f : P_1 \to P_2$

is said to be a *strong homomorphism* if $f(e_1) = e_2$ [12].

Now let us state some more definitions and results which will be used as ready references.

Proposition 2.1. [18] *Let $\langle P_1, \circ, e_1, ^{-1} \rangle$ and $\langle P_2, *, e_2, ^{-I} \rangle$ be two polygroups and $f : P_1 \to P_2$ be a strong homomorphism. Then, $f(x^{-1}) = [f(x)]^{-I}$, for all $x \in P_1$.*

Lemma 2.2. [20] *Let (H, τ) be a topological space, then the family $\mathcal{B}$ consisting of all $S_V = \{U \in \mathcal{P}^*(H) : U \subseteq V\}, V \in \tau$ is a base for a topology on $\mathcal{P}^*(H)$. This topology is denoted by τ^*.*

Definition 2.3. [17] Let $(H, \circ)$ be a hypergroup and (H, τ) be a topological space. Then, the system $(H, \circ, \tau)$ is called a *topological hypergroup* if with respect to the product topology on $H \times H$ and the topology τ^* on $\mathcal{P}^*(H)$

(1) the mapping $(x, y) \mapsto x \circ y$ from $H \times H \to \mathcal{P}^*(H)$ and
(2) the mapping $(x, y) \mapsto x/y$ from $H \times H \to \mathcal{P}^*(H)$
are continuous, where $x/y := \{z \in H : x \in z \circ y\}$.

Definition 2.4. [18] Let $P = \langle P, \circ, e, ^{-1} \rangle$ be a polygroup and (P, τ) be a topological space. Then, the system $P = \langle P, \circ, e, ^{-1}, \tau\rangle$ is called a *topological polygroup* if the mappings $\circ : P \times P \to \mathcal{P}^*(P)$ and $^{-1} : P \to P$ are continuous.

An open subset U of a topological polygroup P is said to be symmetric if $U^{-1} = U$.

Theorem 2.5. [18] *Let P be a topological polygroup. Then, every subpolygroup K of P with the relative topology is a topological polygroup.*

Let $\langle P, \circ, e, ^{-1}, \tau\rangle$ be a topological polygroup and K be a normal subpolygroup of P. Let π be the natural mapping $x \mapsto x \circ K$ of P onto P/K. Then, $(P/K, \overline{\tau})$ is a topological space, where $\overline{\tau}$ is the quotient topology induced by π. That is for every subset X of P, $\{x \circ K : x \in X\}$ is an open subset of P/K if and only if $\pi^{-1}(\{x \circ K : x \in X\})$ is an open subset of P. X/K denote the set $\{x \circ K : x \in X\}$ for every subset X of P.

Theorem 2.6. [18] *Let K be a normal subpolygroup of topological polygroup P and every open subset of P be a complete part. Then, $\langle P/K, \odot, K, ^{-I} \rangle$ is a topological polygroup, where $x \circ K \odot y \circ K = \{z \circ K : z \in x \circ y\}$ and $(x \circ K)^{-I} = x^{-1} \circ K$.*

Let X be a nonempty set. Then, the set $\Delta_X = \{(x,x) \in X \times X : x \in X\}$ denotes the diagonal of X.
Let X, Y, Z be nonempty sets. For $M \subseteq X \times Y$, $N \subseteq Y \times Z$, $A \subseteq X$ define

(1) $M \circ N = \{(x,z) \in X \times Z :$ there exists y $\in Y$ such that $(x,y) \in M$ and $(y,z) \in N\}$;
(2) $M^{-1} = \{(y,x) \in Y \times X : (x,y) \in M\}$;
(3) $M[A] = \{y \in Y :$ there exists $x \in A$ such that $(x,y) \in M\}$.

If $A = \{a\}$, then write $M[a]$ instead of $M[\{a\}]$.
Now, recall the definition of uniformity on a set.

Definition 2.7. [31] Let X be a nonempty set and $\mathcal{U} \subseteq \mathcal{P}(X \times X)$, $\mathcal{U} \neq \phi$, $\mathcal{U}$ is called a uniformity (or a uniform structure) on X if

(U1) $\Delta_X \subseteq M$ for all $M \in \mathcal{U}$;
(U2) $M^{-1} \in \mathcal{U}$ for all $M \in \mathcal{U}$;
(U3) If $M \in \mathcal{U}$ and $N \in \mathcal{P}(X \times X)$ such that $M \subseteq N$, then $N \in \mathcal{U}$;
(U4) $M \cap N \in \mathcal{U}$ for all $M, N \in \mathcal{U}$;
(U5) For all $M \in \mathcal{U}$ there exists $N \in \mathcal{U}$ such that $N \circ N \subseteq M$.

Let X be a topological space and $\mathcal{U}$ be a uniformity on the underlying set X. $\mathcal{U}$ is said to be a *compatible uniformity* on X if the topology induced by $\mathcal{U}$ on X coincides with the underlying topology on X. The definition of compatibility can also be reformulated as follows.

For every $U \in \mathcal{U}$ and $x \in X$, put $U[x] = \{y \in X : (x,y) \in U\}$. The set $U[x]$ is called the *U-ball* with center at x. The uniformity $\mathcal{U}$ on X is compatible with X if $U[x]$ is a neighborhood of x in X for all $x \in X$ and $U \in \mathcal{U}$, and the family of all U-balls forms a neighborhood base for the underlying topology on X.

Throughout this paper, "neighborhood" stands for "open neighborhood".

3. Uniform structures on a special kind of polygroups and on their quotients

Let P be a topological polygroup and $\mathcal{N}_s(e)$ be the family of open symmetric neighborhoods of e in P. For an element $V \in \mathcal{N}_s(e)$, define three subsets O_V^l, O_V^r and O_V of $P \times P$ as follows:

$$O_V^l = \{(x,y) \in P \times P : x^{-1} \circ y \subseteq V\}, \tag{3.1}$$

$$O_V^r = \{(x,y) \in P \times P : x \circ y^{-1} \subseteq V\}, \tag{3.2}$$

$$O_V = O^l_V \cap O^r_V. \tag{3.3}$$

Denote the diagonal of $P \times P$ by Δ_P. A subset B of $P \times P$ is called symmetric if $(y,x) \in B$ whenever $(x,y) \in B$. So we are getting

Lemma 3.1. *The sets O^l_V, O^r_V and O_V are open symmetric entourages of the diagonal Δ_P in $P \times P$, for each $V \in \mathcal{N}_s(e)$.*

To introduce three natural uniform structures on P, one more auxiliary fact need to be defined in the following way.

Given two subsets A and B of $P \times P$, the composition $A \oplus B$ of A and B is defined by

$$A \oplus B = \{(x,z) \in P \times P : (x,y) \in A \text{ and } (y,z) \in B \text{ for some } y \in P\}.$$

For $A \subseteq P \times P$ and an integer $n \geq 1$, define inductively a set $nA \subseteq P \times P$ by letting $1A = A$, $2A = A \oplus A$ and in general $(n+1)A = nA \oplus A$, for each $n \geq 1$.

Lemma 3.2. *Suppose P be a topological polygroup, U and V are elements of $\mathcal{N}_s(e)$ in P, $n \in \mathbb{N}$, $V^n \subseteq U$. Then, $nO^l_V \subseteq O^l_U$, $nO^r_V \subseteq O^r_U$ and $nO_V \subseteq O_U$.*

Proof. For $n = 1$, the result holds trivially, so assume that $n \geq 2$. Let $(x,y) \in nO^l_V = O^l_V \oplus O^l_V \oplus \cdots \oplus O^l_V (n \text{ times})$, then there exist elements $z_1, z_2, \cdots, z_{n-1} \in P$ such that $(z_i, z_{i+1}) \in O^l_V$ for each $i = 0, 1, 2, \cdots, n-1$, where $z_0 = x$ and $z_n = y$, Hence, $z_i^{-1} \circ z_{i+1} \subseteq V$ if $0 \leq i \leq n$, it follows that

$$\begin{aligned} &x^{-1} \circ y \subseteq \Pi_{i=0}^{n-1}(z_i^{-1} \circ z_{i+1}) \subseteq V^n \subseteq U, \\ &\Rightarrow (x,y) \in O^l_U. \end{aligned}$$

So, $nO^l_V \subseteq O^l_U$ and the other parts. □

Definition 3.3. A topological polygroup is said to be an *OCP-polygroup* if every open subset of it is a complete part.

Example 3.4. Let P be a polygroup and β^* be the fundamental relation on P. Then, $\tau = \{\bigcup\limits_{u \in U} \beta^*(u) | U \subseteq P\} \cup \{\phi\}$ is a topology on P and $(P, \circ, e, ^{-1}, \tau)$ is an OCP-polygroup.

Example 3.5. Consider the set of integers $\mathbb{Z}$ and define the hyperoperation $\circ$ on it as follows:

For every $m \in \mathbb{Z}$, $m \circ 0 = m$ and if $m, n \in \mathbb{Z} \setminus \{0\}$, then

$$m \circ n = \begin{cases} 2\mathbb{Z} & \text{if } m+n \in 2\mathbb{Z}; \\ (2\mathbb{Z})^c & \text{otherwise.} \end{cases}$$

Also let $\tau = \{\phi, 2\mathbb{Z}, (2\mathbb{Z})^c, \mathbb{Z}\}$. Then, τ is a topology on $\mathbb{Z}$, and $(\mathbb{Z}, \circ, 0, -, \tau)$ is an OCP-polygroup, where the unitary operation $-$ is the ordinary negation.

Now define three natural uniformities on a given OCP-polygroup P. Consider the following families:

$$\mathcal{B}^l_P = \{O^l_V : V \in \mathcal{N}_s(e)\}, \tag{3.4}$$

$$\mathcal{B}^r_P = \{O^r_V : V \in \mathcal{N}_s(e)\}, \tag{3.5}$$

$$\mathcal{B}_P = \{O_V : V \in \mathcal{N}_s(e)\}, \tag{3.6}$$

where $\mathcal{N}_s(e)$ denotes the family of open symmetric neighborhoods containing e in P. By Lemma 3.1, each of the families $\mathcal{B}^l_P, \mathcal{B}^r_P$ and $\mathcal{B}_P$ consists of open entourages of Δ_P in $P \times P$. Denote by $\mathcal{D}_P$ the family of symmetric subsets of $P \times P$. Finally, put

$$\mathcal{V}^l_P = \{D \in \mathcal{D}_P : O^l_V \subseteq D \text{ for some } V \in \mathcal{N}_s(e)\}, \tag{3.7}$$

$$\mathcal{V}^r_P = \{D \in \mathcal{D}_P : O^r_V \subseteq D \text{ for some } V \in \mathcal{N}_s(e)\}, \tag{3.8}$$

$$\mathcal{V}_P = \{D \in \mathcal{D}_P : O_V \subseteq D \text{ for some } V \in \mathcal{N}_s(e)\}, \tag{3.9}$$

It is clear from the above definitions that $\mathcal{V}^l_P \subseteq \mathcal{V}_P$ and $\mathcal{V}^r_P \subseteq \mathcal{V}_P$.

The next theorem explains the role of the above six families.

Theorem 3.6. *For any OCP-polygroup P, the families $\mathcal{V}^l_P, \mathcal{V}^r_P$ and $\mathcal{V}_P$ are uniformities on the space P with respective bases $\mathcal{B}^l_P, \mathcal{B}^r_P$ and $\mathcal{B}_P$. Each of the three uniformities is compatible with P.*

Proof. Let's verify the first claim of the theorem for the family $\mathcal{V}^l_P$ by showing that it satisfies the following five conditions:

(U1) $\Delta_P \subseteq O$ for each $O \in \mathcal{V}^l_P$;
(U2) If $O \in \mathcal{V}^l_P$, then $O^{-1} \in \mathcal{V}^l_P$;
(U3) If $O \in \mathcal{V}^l_P$ and $O \subseteq W \in \mathcal{D}_P$, then $W \in \mathcal{V}^l_P$;
(U4) If $O_1, O_2 \in \mathcal{V}^l_P$, then $O_1 \cap O_2 \in \mathcal{V}^l_P$;
(U5) For every $O \in \mathcal{V}^l_P$, there is $W \in \mathcal{V}^l_P$ such that $2W \subseteq O$.

Clearly, (U1), (U2) follow from the Lemma 3.1.

To prove (U3), let $O \in \mathcal{V}^l_P$ and $O \subseteq W \in \mathcal{D}_P$, then there exists $V \in \mathcal{N}_s(e)$ such that

$$O^l_V \subseteq O \;\Rightarrow\; O^l_V \subseteq O \subseteq W \in \mathcal{D}_P \;\Rightarrow\; W \in \mathcal{V}^l_P.$$

To verify (U4), take $O_1, O_2 \in \mathcal{V}^l_P$. Then, there exist $V_1, V_2 \in \mathcal{N}_s(e)$ such that $O^l_{V_1} \subseteq O_1$ and $O^l_{V_2} \subseteq O_2$. Put $V = V_1 \cap V_2$, then $V \in \mathcal{N}_s(e)$ and $O^l_V \in \mathcal{V}^l_P$. Now $O^l_V \subseteq O^l_{V_1} \cap O^l_{V_2} \subseteq O_1 \cap O_2$ and $O_1 \cap O_2 \in \mathcal{D}_P$. So (U3) implies that $O_1 \cap O_2 \in \mathcal{V}^l_P$.

To show (U5), let $O \in \mathcal{V}^l_P$, then there exists $U \in \mathcal{N}_s(e)$ such that $O^l_U \subseteq O$. Choose $V \in \mathcal{N}_s(e)$ satisfying $V^2 = V \circ V \subseteq U$ [18]. Then, $W = O^l_V \in \mathcal{V}^l_P$ and Lemma 3.2 implies that $2W \subseteq O^l_U \subseteq O$. Hence, $\mathcal{V}^l_P$ is a uniformity on P. From (3.4), (3.7) it follows that $\mathcal{B}^l_P$ is a base for the uniformity $\mathcal{V}^l_P$. Similarly, the families $\mathcal{V}^r_P$ and $\mathcal{V}_P$ are uniformities on P with respect to the bases $\mathcal{B}^r_P$ and $\mathcal{B}_P$, respectively.

Now let us show that the above three uniformities are compatible with P. To show it for the family $\mathcal{V}^l_P$, let $O \in \mathcal{V}^l_P$ and $x \in P$. Then, there exists $V \in \mathcal{N}_s(e)$ such that $O^l_V \subseteq O$. This implies that $O^l_V[x] \subseteq O[x]$. Since V is a complete part, (3.1) implies that $O^l_V[x] = x \circ V$, which is open in P. Thus, $x \in x \circ V \subseteq O[x]$ and Hence, $O[x]$ is a neighborhood of x in P and the family $\{O[x] : O \in \mathcal{V}^l_P\}$ is a neighborhood base for P at x. This shows that the uniformity $\mathcal{V}^l_P$ is compatible with P.

Similarly, the same can be shown for the family $\mathcal{V}^r_P$.

Here, $O_V[x] = x \circ V \cap V \circ x$, for all $V \in \mathcal{N}_s(e)$ and $x \in P$. Since $x \circ V \cap V \circ x$ is open in P, the uniformity $\mathcal{V}_P$ is compatible with P. □

Let's call $\mathcal{V}^l_P, \mathcal{V}^r_P$ and $\mathcal{V}_P$ the *left OCP-polygroup uniformity, right OCP-polygroup uniformity* and the *two-sided OCP-polygroup uniformity* on P, respectively.

Let P be an OCP-polygroup and H be a subpolygroup of P. Then, every open subset of H is a complete part (by Lemma 3.11[33]). So, one can think of left uniformity $\mathcal{V}^l_{P,H}$ on the polygroup H, which consists of the intersections $V \cap (H \times H)$, with $V \in \mathcal{V}^l_P$. Similarly, H inherits from P the right and two-sided induced uniformities denoted by $\mathcal{V}^r_{P,H}$ and $\mathcal{V}_{P,H}$, respectively.

Next result shows that these three induced uniformities coincide with the actual uniformities on H.

Proposition 3.7. *The equalities $\mathcal{V}^l_{P,H} = \mathcal{V}^l_H$, $\mathcal{V}^r_{P,H} = \mathcal{V}^r_H$ and $\mathcal{V}_{P,H} = \mathcal{V}_H$ hold for each subpolygroup H of an OCP-polygroup P.*

Proof. It is sufficient to verify the equality $\mathcal{V}^l_{P,H} = \mathcal{V}^l_H$, leaving others for similar verification. Let $V \in \mathcal{N}_s(e)$ and put $U = V \cap H$. Then,

$$\begin{aligned} O^l_V \cap (H \times H) &= \{(x,y) \in H \times H : x^{-1} \circ y \subseteq V\} \\ &= \{(x,y) \in H \times H : x^{-1} \circ y \subseteq U\} \\ &= O^l_U. \end{aligned}$$

Since U is an open symmetric neighborhood of the identity e, it follows that $O^l_U \in \mathcal{V}^l_H$.

Finally, since the sets O^l_V form a base for the left OCP-polygroup uniformity $\mathcal{V}^l_P$ on P, hence the uniformities $\mathcal{V}^l_H$ and $\mathcal{V}^l_{P,H}$ coincide. □

The following result shows the relation between $\mathcal{V}^l_P, \mathcal{V}^r_P$ and $\mathcal{V}_P$.

Theorem 3.8. *For every OCP-polygroup P, the two-sided uniformity $\mathcal{V}_P$ is the coarsest uniformity on P finer than each of the uniformities $\mathcal{V}^l_P$ and $\mathcal{V}^r_P$.*

Proof. $O_V = O^l_V \cap O^r_V$, for each $V \in \mathcal{N}_s(e)$, then from (3.7), (3.8) and (3.9) it follows that $\mathcal{V}_P$ is finer than $\mathcal{V}^l_P$ and $\mathcal{V}^r_P$.

Now, suppose that $\mathcal{U}$ is a uniformity on P finer than both $\mathcal{V}^l_P$ and $\mathcal{V}^r_P$. Let $O \in \mathcal{V}_P$, then there exists $V \in \mathcal{N}_s(e)$ such that $O_V \subseteq O$. Since $\mathcal{U}$ is finer than both $\mathcal{V}^l_P$ and $\mathcal{V}^r_P$, it follows that there exist $U_1, U_2 \in \mathcal{U}$ such that $U_1 \subseteq O^l_V$ and $U_2 \subseteq O^r_V$. Now $U = U_1 \cap U_2 \in \mathcal{U}$ and $U \subseteq O^l_V \cap O^r_V = O_V \subseteq O$. Therefore, $\mathcal{U}$ is finer than $\mathcal{V}_P$ and hence $\mathcal{V}_P$ is the coarsest uniformity finer than $\mathcal{V}^l_P$ and $\mathcal{V}^r_P$. □

Definition 3.9. Let P be a topological polygroup. A subset A of P is said to be invariant if $x \circ A \circ x^{-1} = A$, for each $x \in P$.

Definition 3.10. A topological polygroup P is said to be balanced if it has a local base at the identity consisting of invariant sets.

Next result characterizes balanced OCP-polygroups.

Lemma 3.11. *An OCP-polygroup P is balanced if and only if for every neighborhood U of the identity e in P, there exists a neighborhood V of e such that $x \circ V \circ x^{-1} \subseteq U$, for each $x \in P$.*

Proof. If P is balanced, then the result follows immediately.

For the converse, let U be a neighborhood of e in P. Then, by the hypothesis there exists a neighborhood O of e such that $x \circ O \circ x^{-1} \subseteq U$, for each $x \in P$. Then, the set $V = \bigcup_{x \in P} [x \circ O \circ x^{-1}]$ is open in P and

$V \subseteq U$. For $y \in P$, we obtain

$$\begin{aligned} y \circ V \circ y^{-1} &= \bigcup_{x \in P} [y \circ (x \circ O \circ x^{-1}) \circ y^{-1}] \\ &= \bigcup_{x \in P} [(y \circ x) \circ O \circ (y \circ x)^{-1}] \\ &= \bigcup_{\substack{t \in y \circ x \\ x \in P}} [t \circ O \circ t^{-1}] = V \end{aligned}$$

This shows that V is invariant and so P has a base of open invariant sets at e. $\square$

Lemma 3.11 and Theorem 2.19 [18] induces

Corollary 3.12. *Every compact OCP-polygroup is balanced.*

The following theorem indicates a class of polygroups where the aforementioned uniformities become identical.

Theorem 3.13. *For an OCP-polygroup P, the uniformities $\mathcal{V}_P^l$, $\mathcal{V}_P^r$ and $\mathcal{V}_P$ coincide if and only if the polygroup P is balanced. Therefore, the three uniformities coincide for every compact OCP-polygroup.*

Proof. First suppose that P is balanced. Let $\mathcal{N}$ be the family of open, symmetric, invariant neighborhoods of the identity e in P. By the assumptions, $\mathcal{N}$ is a local base for P at e. Therefore, the families $\beta^l = \{O_V^l : V \in \mathcal{N}\}$ and $\beta^r = \{O_V^r : V \in \mathcal{N}\}$ are bases for the uniformities $\mathcal{V}_P^l$ and $\mathcal{V}_P^r$, respectively. Since each $V \in \mathcal{N}$ is an invariant, symmetric subset of P, it follows that

$$\begin{gathered} (x, y) \in O_V^l \Rightarrow x^{-1} \circ y \subseteq V \Rightarrow y^{-1} \circ x = (x^{-1} \circ y)^{-1} \subseteq V^{-1} = V \Rightarrow \\ x \circ y^{-1} \subseteq x \circ y^{-1} \circ x \circ x^{-1} \subseteq x \circ V \circ x^{-1} = V \Rightarrow (x, y) \in O_V^r. \end{gathered}$$

Therefore, $O_V^l \subseteq O_V^r$. Similarly, one can show that $O_V^l \supseteq O_V^r$. Hence, $O_V^l = O_V^r$ for each $V \in \mathcal{N}$. This shows that the uniformities $\mathcal{V}_P^l$ and $\mathcal{V}_P^r$ have the same base $\beta^l = \beta^r$. Therefore, the uniformities $\mathcal{V}_P^l$ and $\mathcal{V}_P^r$ coincide. Hence, the Theorem 3.8 implies that the two-sided uniformity $\mathcal{V}_P$ on P coincides with each of the uniformities $\mathcal{V}_P^l$ and $\mathcal{V}_P^r$.

For the converse, suppose that $\mathcal{V}_P^l = \mathcal{V}_P^r$. Since $\mathcal{B}_P^l$ and $\mathcal{B}_P^r$ are bases for $\mathcal{V}_P^l$ and $\mathcal{V}_P^r$, respectively, for each $U \in \mathcal{N}_s(e)$ there exists $V \in \mathcal{N}_s(e)$ such that $O_V^l \subseteq O_U^r$. This shows $x^{-1} \circ y \subseteq V \Rightarrow x \circ y^{-1} \subseteq U$ for $x, y \in P$. In other words, $x \circ V \subseteq U \circ x$ for each $x \in P$. This implies $x \circ V \circ x^{-1} \subseteq U \circ x \circ x^{-1} = U$ for each $x \in P$ (by Proposition 2.1 [33]). Therefore, P is balanced by Lemma 3.11.

The last assertion follows from Corollary 3.12. $\square$

Consider the additive abelian topological group $\mathbb{R}$ of reals, then the three natural uniformities of $\mathbb{R}$ coincide. Denote each of them by $\mathcal{U}$.

Definition 3.14. Let P be an OCP-polygroup. A real valued function f on P is called left uniformly continuous if f is a uniformly continuous mapping of $(P, \mathcal{V}_P^l)$ to $(\mathbb{R}, \mathcal{U})$, i.e., for every $\epsilon > 0$, there exists $O \in \mathcal{V}_P^l$ such that $|f(y) - f(x)| < \epsilon$ whenever $(x, y) \in O$.

Similarly, f is called right uniformly continuous if f is a uniformly continuous mapping of $(P, \mathcal{V}_P^r)$ to $(\mathbb{R}, \mathcal{U})$. If f is both left and right uniformly continuous on P, then it is called uniformly continuous on P

The next lemma is immediate after the above definition.

Lemma 3.15. *A real valued function f on an OCP-polygroup is left uniformly continuous if and only if, for every $\epsilon > 0$, there exists a neighborhood V of the identity in P such that $|f(t) - f(x)| < \epsilon$ for all $t \in x \circ V$ and $x \in P$. Similarly, f is right uniformly continuous if and only if, for every $\epsilon > 0$, there exists a neighborhood W of the identity in P such that $|f(t) - f(x)| < \epsilon$ for all $t \in W \circ x$ and $x \in P$.*

Proof. The proofs are straightforward from the fact that $(x, y) \in O_V^l \Leftrightarrow y \in x \circ V$ and $(x, y) \in O_W^r \Leftrightarrow y \in W \circ x$, since V, W are complete parts of P. $\square$

An immediate application of this lemma is

Proposition 3.16. *Every continuous real-valued function on a compact OCP-polygroup is uniformly continuous.*

Proof. Let P be a compact OCP-polygroup and $f : P \to \mathbb{R}$ be a continuous function. Since the left and right uniformities on P coincide (by Theorem 3.13), it is sufficient to prove that f is left uniformly continuous on P. Let $\epsilon > 0$ be arbitrary real number. For each $x \in P$, choose a neighborhood U_x of the identity e in P so that $|f(y) - f(x)| < \epsilon/2$, whenever $y \in x \circ U_x$. Then, there exists a neighborhood V_x of e such that $V_x \circ V_x \subseteq U_x$ [18]. Now $\{x \circ V_x\}_{x \in P}$ is an open cover of the compact polygroup P, there exist $x_1, x_2, \cdots, x_n \in P$ such that $P = \bigcup\limits_{i=1}^{n} x_i \circ V_{x_i}$.

Put $V = \bigcap\limits_{i=1}^{n} V_{x_i}$.

Let $y \in P$, then $y \in x_k \circ V_{x_k}$ for some $k \in \{1, 2, \cdots, n\}$. This implies that $|f(y) - f(x_k)| < \epsilon/2$. Now if $x \in y \circ V$, then $x \in (x_k \circ V_{x_k}) \circ V = x_k \circ (V_{x_k} \circ V) \subseteq x_k \circ (V_{x_k} \circ V_{x_k}) \subseteq x_k \circ U_{x_k}$ and hence $|f(x) - f(x_k)| < \epsilon/2$.

Thus, $|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(y) - f(x_k)| < \epsilon/2 + \epsilon/2 = \epsilon$, whenever $x \in y \circ V$. This shows that f is left uniformly continuous on P (by Lemma 3.15). $\square$

Definition 3.17. Let $f : P \to H$ be a mapping of OCP-polygroups P and H. f is said to be left uniformly continuous if f is uniformly continuous as a mapping of the uniform space $(P, \mathcal{V}^l_P)$ to $(H, \mathcal{V}^l_H)$. i.e., for every $O^l_U \in \mathcal{V}^l_H$ there exists $O^l_V \in \mathcal{V}^l_P$ such that $(f(x), f(y)) \in O^l_U$ whenever $(x, y) \in O^l_V$.

Similarly, f is said to be right uniformly continuous if f is uniformly continuous as a mapping of the uniform space $(P, \mathcal{V}^r_P)$ to $(H, \mathcal{V}^r_H)$. If f is both left and right uniformly continuous, then it is called a uniformly continuous mapping.

Proposition 3.18. *Let $\langle P_1, \circ, e_1, ^{-1}, \tau_1\rangle$ and $\langle P_2, *, e_2, ^{-I}, \tau_2\rangle$ be two OCP-polygroups and $f : P_1 \to P_2$ be a continuous strong homomorphism. Then, f is uniformly continuous.*

Proof. It is sufficient to prove that f is left uniformly continuous, then the right uniform continuity follows by similar arguments. Let $O^l_U \in \mathcal{V}^l_{P_2}$, where U is an open symmetric neighborhood of the identity e_2 in P_2. Since f is continuous and $f(e_1) = e_2$, there exists a symmetric neighborhood V of e_1 in P_1 such that $f(V) \subseteq U$. Let $(x, y) \in O^l_V$, i.e., $x^{-1} \circ y \subseteq V$. Then, Proposition 2.1 implies $[f(x)]^{-I} * f(y) = f(x^{-1} \circ y) \subseteq f(V) \subseteq U$ and hence $(f(x), f(y)) \in O^l_U$. This completes the proof. $\square$

Proposition 3.19. *Let $\langle P_1, \cdot, e_1, ^{-1}, \tau_1\rangle$ and $\langle P_2, *, e_2, ^{-I}, \tau_2\rangle$ be two topological polygroups. Then, the maps*

(1) $((x_1, y_1), (x_2, y_2)) \mapsto (x_1, y_1) \circ (x_2, y_2)$ *from* $(P_1 \times P_2) \times (P_1 \times P_2)$ *to* $\mathcal{P}^*(P_1 \times P_2)$*;*

(2) $(x, y) \mapsto (x^{-1}, y^{-I})$ *from* $(P_1 \times P_2)$ *to* $(P_1 \times P_2)$*;*

are continuous with respect to the product topology τ on $P_1 \times P_2$ induced from τ_1, τ_2.

Proof. (1) Let $W \in \tau$ and $(x_1, y_1) \circ (x_2, y_2) \subseteq W$. i.e., $(x_1 \cdot x_2) \times (y_1 * y_2) \subseteq W$. Since $W \in \tau$, there exist $U \in \tau_1$ and $V \in \tau_2$ such that $U \times V = W$ and $x_1 \cdot x_2 \subseteq U$, $y_1 * y_2 \subseteq V$. Now $x_1 \cdot x_2 \subseteq U$ implies there exist $U_1, U_2 \in \tau_1$ containing x_1, x_2, respectively such that $U_1 \cdot U_2 \subseteq U$ and $y_1 * y_2 \subseteq V$ implies there exist $V_1, V_2 \in \tau_2$ containing y_1, y_2, respectively such that $V_1 * V_2 \subseteq V$. Hence, $U_1 \times V_1$ and $U_2 \times V_2$ belong to τ containing (x_1, y_1) and (x_2, y_2), respectively such that $(U_1 \times V_1) \circ (U_2 \times V_2) =$

$(U_1 \cdot U_2) \times (V_1 * V_2) \subseteq U \times V = W$. This shows the continuity of the hyperproduct $\circ$.

(2) The continuity of the map $(x, y) \mapsto (x^{-1}, y^{-I})$ on $P_1 \times P_2$ follows from the continuity of the maps $x \mapsto x^{-1}$ on P_1 and $x \mapsto x^{-I}$ on P_2. $\square$

Corollary 3.20. *Let $\langle P_1, \cdot, e_1, ^{-1}, \tau_1\rangle$ and $\langle P_2, *, e_2, ^{-I}, \tau_2\rangle$ be two OCP-polygroups. Then, their direct hyperproduct $P_1 \times P_2$ is an OCP-polygroup.*

Proof. The direct hyperproduct $P_1 \times P_2$ is a topological polygroup after the Proposition 3.19.

To show the open subsets of $P_1 \times P_2$ are complete parts, let $W \in \tau$ and for $n \in \mathbb{N}$, $\Pi_{i=1}^{n}(x_i, y_i) \cap W \neq \phi$, where $x_i \in P_1$, $y_i \in P_2$ and τ is the product topology induced from τ_1, τ_2. Then, there exist $U \in \tau_1$, $V \in \tau_2$ such that $W = U \times V$. Now $\Pi_{i=1}^{n}(x_i, y_i) = \Pi_{i=1}^{n}x_i \times \Pi_{i=1}^{n}y_i$ implies that $\Pi_{i=1}^{n}x_i \cap U \neq \phi$ and $\Pi_{i=1}^{n}y_i \cap V \neq \phi$. Since U and V are complete parts, $\Pi_{i=1}^{n}x_i \subseteq U$ and $\Pi_{i=1}^{n}y_i \subseteq V$. This shows that $\Pi_{i=1}^{n}(x_i, y_i) = \Pi_{i=1}^{n}x_i \times \Pi_{i=1}^{n}y_i \subseteq U \times V = W$. Hence, W is a complete part and $P_1 \times P_2$ is an OCP-polygroup. $\square$

Recall the fact that if $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces, then the product of $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ is a uniform space $(Z, \mathcal{W})$ with the underlying set $Z = X \times Y$ and the uniformity $\mathcal{W}$ on Z whose base consists of the sets

$$W_{U,V} = \{((x, y), (x', y')) \in Z \times Z : (x, x') \in U, (y, y') \in V\} \tag{3.10}$$

where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. The uniformity $\mathcal{W}$ is called the product of $\mathcal{U}$ and $\mathcal{V}$ and is written as $\mathcal{W} = \mathcal{U} \times \mathcal{V}$.

Proposition 3.21. *Let $\langle P, \circ, e_P, ^{-1}, \tau_P\rangle$ and $\langle H, *, e_H, ^{-I}, \tau_H\rangle$ be two OCP-polygroups. Then, the left(right, two-sided) OCP-polygroup uniformity of the product polygroup $P \times H$ coincides with the product of the left(right, two-sided) OCP-polygroup uniformities of P and H.*

Proof. Let's prove the proposition only for left OCP-polygroup uniformity, leaving others for similar verification. Let $Z = P \times H$, the sets of the form $U \times V$ form a base of the neighborhoods at the identity element (e_P, e_H) of Z, where $U \in \mathcal{N}_s(e_P)$ and $V \in \mathcal{N}_s(e_H)$. Now a basic entourage of the diagonal in $Z \times Z$ has the form $O^l_{U,V} = \{((x, y), (x_1, y_1)) \in Z \times Z : x^{-1} \cdot x_1 \subseteq U, y^{-1} * y_1 \subseteq V\}$, where $U \in \mathcal{N}_s(e_P)$ and $V \in \mathcal{N}_s(e_H)$. This shows that the set $O^l_{U,V}$ coincides with the set W_{U^*,V^*} defined in (3.10), where $U^* = O^l_U \in \mathcal{V}^l_P$ and $V^* = O^l_V \in \mathcal{V}^l_H$.

Since the sets U^* and V^* with $U \in \mathcal{N}_s(e_P)$ and $V \in \mathcal{N}_s(e_H)$, form a base for the uniformities $\mathcal{V}_P^l$ and $\mathcal{V}_H^l$ respectively, the corresponding sets W_{U^*,V^*} form a base for the product uniformity $\mathcal{W} = \mathcal{V}_P^l \times \mathcal{V}_H^l$ on Z. Therefore, $O_{U,V}^l = W_{U^*,V^*}$ implies that the uniformities $\mathcal{V}_Z^l$ and $\mathcal{W}$ coincide. $\square$

Besides the above mentioned three uniformities, every OCP-polygroup admits a fourth uniformity which is called *Roelcke uniformity.* Let's begin with a lemma which will be used in the sequel.

Lemma 3.22. *In a topological polygroup P, the map $f : P \times P \times P \to \mathcal{P}^*(P)$ defined by $f(x, y, z) = x \circ y \circ z$ for all $x, y, z \in P$, is continuous.*

Proof. Let $x, y, z \in P$ and U be an open set containing $x \circ y \circ z$. Now, $x \circ y \circ z = \bigcup\limits_{v \in y \circ z} x \circ v \subseteq U$, i.e., $x \circ v \subseteq U$ for every $v \in y \circ z$. By the continuity of the map $(x, y) \mapsto x \circ y$, there exist open sets U_x, U_v containing x, v respectively such that $U_x \circ U_v \subseteq U$. Take $W = \bigcup\limits_{v \in y \circ z} U_v$, then W is an open set containing $y \circ z$. Again, by the continuity of the map $(x, y) \mapsto x \circ y$, there exist open sets U_y, U_z containing y, z respectively such that $U_y \circ U_z \subseteq W$. Now, $U_x \circ U_v \subseteq U$ for each $v \in y \circ z$ implies $U_x \circ (\bigcup\limits_{v \in y \circ z} U_v) \subseteq U$, i.e., $U_x \circ W \subseteq U$. This implies that $U_x \circ U_y \circ U_z \subseteq U$, which proves the result. $\square$

Let $\mathcal{N}_s(e)$ be the family of open symmetric neighborhoods of the identity e in an OCP-polygroup P. For an element $V \in \mathcal{N}_s(e)$, let

$$O_V^t = \{(x, y) \in P \times P : y \in V \circ x \circ V\}. \tag{3.11}$$

Here O_V^t is an open symmetric entourage of the diagonal in $P \times P$. Now define two families $\mathcal{B}_P^t$ and $\mathcal{V}_P^t$ as follows:

$$\mathcal{B}_P^t = \{O_V^t : V \in \mathcal{N}_s(e)\}, \tag{3.12}$$

$$\mathcal{V}_P^t = \{D \in \mathcal{D}_P : O_V^t \subseteq D \text{ for some } V \in \mathcal{N}_s(e)\}, \tag{3.13}$$

where $\mathcal{D}_P$ is the family of symmetric subsets of $P \times P$.

Theorem 3.23. *For any OCP-polygroup P, the family $\mathcal{V}_P^t$ is a uniformity compatible with P and $\mathcal{B}_P^t$ is a base for $\mathcal{V}_P^t$. Moreover, $\mathcal{V}_P^t$ is the finest uniformity on P coarser than each of the uniformities $\mathcal{V}_P^l$ and $\mathcal{V}_P^r$.*

Proof. Let's first verify that $\mathcal{V}_P^t$ is a uniformity on P. Here (U1) and (U2) holds trivially. To verify (U3), let $O \in \mathcal{V}_P^t$ and $O \subseteq W \in \mathcal{D}_P$. Then, there exists $V \in \mathcal{N}_s(e)$ such that $O_V^t \subseteq O$ and this implies that $O_V^t \subseteq O \subseteq W$ and hence $W \in \mathcal{V}_P^t$. To verify (U4), let O_1, $O_2 \in O_V^t$. Then, there exist $V_1, V_2 \in \mathcal{N}_s(e)$ such that $O_{V_1}^t \subseteq O_1$ and $O_{V_2}^t \subseteq O_2$. Put $V = V_1 \cap V_2$, then $V \in \mathcal{N}_s(e)$ and $O_V^t \in \mathcal{V}_P^t$. Now $O_V^t \subseteq O_{V_1}^t \cap O_{V_2}^t \subseteq O_1 \cap O_2$ and $O_1 \cap O_2 \in \mathcal{D}_P$. So by (U3), $O_1 \cap O_2 \in \mathcal{V}_P^t$. To verify (U5), let $O \in \mathcal{V}_P^t$. Then, there exists $U \in \mathcal{N}_s(e)$ such that $O_U^t \subseteq O$. Now choose $V \in \mathcal{N}_s(e)$ such that $V^2 = V \circ V \subseteq U$ and put $W = O_V^t$. We show $2W \subseteq O$. Let (x, z), $(z, y) \in W$. Then, $z \in V \circ x \circ V$ and $y \in V \circ z \circ V$. Hence, $y \in V \circ (V \circ x \circ V) \circ V = V^2 \circ x \circ V^2 \subseteq U \circ x \circ U \Rightarrow (x, y) \in O_U^t \subseteq O \Rightarrow 2W \subseteq O$.
Therefore, $\mathcal{V}_P^t$ is a uniformity on P.

To show the uniformity $\mathcal{V}_P^t$ is compatible with P, let $O \in \mathcal{V}_P^t$ and $x \in P$. Then, there exists $V \in \mathcal{N}_s(e)$ such that $O_V^t \subseteq O$. Now, $O_V^t[x] \subseteq O[x]$. From (3.11) it follows that $O_V^t[x] = V \circ x \circ V$, which is open in P and $x \in V \circ x \circ V \subseteq O[x]$. Hence, $O[x]$ is a neighborhood of x in P. Now suppose U be an open set and $x \in U$. Then, by the continuity of the map $f : P \times P \times P \to \mathcal{P}^*(P)$, defined by $f(x, y, z) = x \circ y \circ z$ (Lemma 3.22), there exists an open symmetric neighborhood V of the identity e in P such that $f(V \times \{x\} \times V) \subseteq U$, i.e., $V \circ x \circ V \subseteq U$. Therefore, $O_V^t[x] \subseteq U$ and hence the family $\{O[x] : O \in \mathcal{V}_P^t\}$ is a neighborhood base for P at the point x. This shows that the uniformity $\mathcal{V}_P^t$ is compatible with P.

Since $O_V^l \subseteq O_V^t$ and $O_V^r \subseteq O_V^t$ for each $V \in \mathcal{N}_s(e)$, it follows that $\mathcal{V}_P^t$ is coarser than $\mathcal{V}_P^l$ and $\mathcal{V}_P^r$. Now suppose that $\mathcal{U}$ is a uniformity on P such that $\mathcal{U} \subset \mathcal{V}_P^l$ and $\mathcal{U} \subset \mathcal{V}_P^r$. Let O be an arbitrary element of $\mathcal{U}$. Then, there exists $O_1 \in \mathcal{U}$ such that $O_1 + O_1 \subseteq O$. Since $\mathcal{U}$ is coarser than $\mathcal{V}_P^l$ and $\mathcal{V}_P^r$, there exists $V \in \mathcal{N}_s(e)$ such that $O_V^l \subseteq O_1$ and $O_V^r \subseteq O_1$. Let $x \in P$ and $v, w \in V$. Then, for each $y \in v \circ x$ and for each $p \in x \circ w$, $(y, x) \in O_V^r$ and $(x, p) \in O_V^l$ and it follows that $(y, p) \in O_V^r + O_V^l \subseteq O_1 + O_1 \subseteq O$. But here $y \in v \circ x \Rightarrow x \in v^{-1} \circ y$, which implies $p \in v^{-1} \circ y \circ w \subseteq V \circ y \circ V$, i.e., $(y, p) \in O_V^t$. Hence, $\mathcal{V}_P^t$ is finer than $\mathcal{U}$. □

Theorem 3.24. *For an OCP-polygroup P the following conditions are equivalent.*

(a) $\mathcal{V}_P^l = \mathcal{V}_P^r$;
(b) $\mathcal{V}_P^t = \mathcal{V}_P$;

(c) $\mathcal{V}_P^t = \mathcal{V}_P^l = \mathcal{V}_P^r = \mathcal{V}_P$;
(d) P *is balanced.*

Proof. Clearly (c) implies (d).The Roelcke uniformity $\mathcal{V}_P^t$ on P is coarser than each of the uniformities $\mathcal{V}_P^l$ and $\mathcal{V}_P^r$ (Theorem 3.23), also the two-sided uniformity $\mathcal{V}_P$ is finer than each of the uniformities $\mathcal{V}_P^l$ and $\mathcal{V}_P^r$ (Theorem 3.8). Therefore, (b) implies (a). Theorem 3.13 implies (a)$\Leftrightarrow$(d). Thus, it remains to show that (d)$\Rightarrow$(c). Suppose that P is balanced. Then, $\mathcal{V}_P^l = \mathcal{V}_P^r = \mathcal{V}_P$(by Theorem 3.13). So, it is only required to verify the equality $\mathcal{V}_P^t = \mathcal{V}_P^r$. Since $\mathcal{V}_P^t$ is coarser than $\mathcal{V}_P^r$, it suffices to show that $\mathcal{V}_P^r \subseteq \mathcal{V}_P^t$. Let $O \in \mathcal{V}_P^r$. Then, one can choose without loss of generality that $O = O_V^r$, for some $V \in \mathcal{N}_s(e)$. Since P is balanced, it has a local base consisting of open symmetric invariant neighborhoods of e. Hence, there exists an invariant set $U \in \mathcal{N}_s(e)$ such that $U \circ U \subseteq V$. Now if $(x, y) \in O_U^t$, then $y \in U \circ x \circ U \subseteq U \circ x \circ U \circ x^{-1} \circ x = U \circ U \circ x \subseteq V \circ x$, i.e., $(x, y) \in O_V^r$. This implies $O_V^t \subseteq O_V^r$ and hence $\mathcal{V}_P^r \subseteq \mathcal{V}_P^t$. □

Now, let's turn our investigation towards uniform structures for quotient spaces of OCP-polygroups by its subpolygroups and for quotient polygroups.

Let P be a topological polygroup and K be a subpolygroup of it. Then, $(P/K, \overline{\tau})$ is a topological space, where $P/K = \{x \circ K : x \in P\}$ and $\overline{\tau}$ is the quotient topology induced by π.

Theorem 3.25. *Let $\langle P, \circ, e, ^{-1}, \tau\rangle$ be an OCP-polygroup and K be a subpolygroup of it. Let π be the natural mapping $x \mapsto x \circ K$ of P onto P/K. Then,*

(1) *π is continuous;*
(2) *the quotient topology $\overline{\tau}$ is the finest topology on P/K with respect to which π is continuous.*

Furthermore, if K is a normal subpolygroup of P, then

(3) *π is a strong homomorphism.*

Proof. (1) π is continuous by the definition of quotient topology.

(2) Let τ' be any other topology on P/K with respect to which π is continuous. Let $\mathcal{O} \in \overline{\tau}$. Then, there exists some open subset V of P such that $\mathcal{O} = V/K$ ([18]). Here, $\pi^{-1}(\mathcal{O}) = \pi^{-1}(V/K) = K \circ V$, which is open in P (by Lemma 2.13 [18]). But by the definition of quotient topology, all such $\mathcal{O}$'s are open in quotient topology. This shows that the quotient topology $\overline{\tau}$ is finer than τ'.

(3) Now, suppose K be a normal subpolygroup. Then, the quotient

space P/K is a topological polygroup with respect to the operations defined in the Theorem 2.6. Here, $f(e) = e \circ K = K$, the identity in P/K. Also, for $x, y \in P$, $\pi(x \circ y) = \{z \circ K : z \in x \circ y\} = x \circ K \odot y \circ K = \pi(x) \odot \pi(y)$. Hence, π is a strong homomorphism. $\square$

Theorem 3.26. *Let K be a subpolygroup of an OCP-polygroup P and $\pi : P \to P/K$ be the natural mapping. Then, the family $\mathcal{B}^r_{P/K} = \{E^r_V : V \in \mathcal{N}_s(e)\}$ is a base for a right uniformity $\mathcal{E}^r_{P/K}$ of the quotient space P/K, where $E^r_V = \{(\pi(x), \pi(y)) : y \in V \circ x\}$ and $\mathcal{E}^r_{P/K}$ is compatible with P/K.*

Moreover, if for every $U \in \mathcal{N}(e)$ there exists $W \in \mathcal{N}(e)$ such that $K \circ W \subseteq U \circ K$, then the family $\mathcal{B}^l_{P/K} = \{E^l_V : V \in \mathcal{N}_s(e)\}$ is a base for a left uniformity $\mathcal{E}^l_{P/K}$ on P/K compatible with P/K, where $E^l_V = \{(\pi(x), \pi(y)) : y \in x \circ V\}$ and $\mathcal{N}(e)$ is the family of all neighborhoods of the identity e.

Proof. Let's prove the results for the family $\mathcal{E}^l_{P/K}$, leaving the other for similar verification.

The family $\mathcal{B}^l_{P/K}$ is a base for $\mathcal{E}^l_{P/K}$, so it suffices to check (U1), (U2), (U4) and (U5) for the family $\mathcal{B}^l_{P/K}$ instead of the family $\mathcal{E}^l_{P/K}$. Since the map $\pi : P \to P/K$ is open ([18]), E^l_V are open symmetric entourages of the diagonal in the space $P/K \times P/K$. So (U1), (U2) holds. (U4) is evident. To prove (U5), let $V \in \mathcal{N}_s(e)$. Then, there exists $O \in \mathcal{N}_s(e)$ such that $O \circ O \subseteq V$ (Theorem 3.6 [18]). Also, by the given condition there exists $W \in \mathcal{N}_s(e)$ satisfying $W \subseteq O$ and $K \circ W \subseteq O \circ K$. We show $2E^l_W \subseteq E^l_V$.

Suppose that $x, y, y_1, z \in P$ such that $y \in x \circ W$, $z \in y_1 \circ W$ and $\pi(y) = \pi(y_1)$, i.e., $(\pi(x), \pi(y))$, $(\pi(y_1), \pi(z)) \in E^l_W$. Then, $(\pi(x), \pi(z)) \in 2E^l_W$. We claim that$(\pi(x), \pi(z)) \in E^l_V$. For, $\pi(y) = \pi(y_1) \Rightarrow y \circ K = y_1 \circ K \Rightarrow y_1 \in y \circ K \subseteq x \circ W \circ K$. Then, $z \in y_1 \circ W \subseteq x \circ W \circ K \circ W \subseteq x \circ W \circ O \circ K \subseteq x \circ O \circ O \circ K \subseteq x \circ V \circ K$. This implies that $\pi(z) = \pi(p)$ for some $p \in x \circ V$. This shows that $2E^l_W \subseteq \{(\pi(x), \pi(p)) : x \in P, p \in x \circ V\} = E^l_V$.

Finally, to show the uniformity $\mathcal{E}^l_{P/K}$ is compatible with P/K, let $x \circ K \in P/K$. Then, $x \circ K = \pi(x) \in \pi(x \circ V) \subseteq E^l_V[\pi(x)]$, where $V \in \mathcal{N}_s(e)$. Since the map π is open of P onto P/K ([18]), $E^l_V[\pi(x)]$ is a neighborhood of $x \circ K$ in P/K. For the converse, let O be a neighborhood of $x \circ K$ in P/K. Then, there exists $V \in \mathcal{N}_s(e)$ such that $\pi(x \circ V) \subseteq O$.

Also by the given condition choose $W \in \mathcal{N}_s(e)$ such that $K \circ W \subseteq V \circ K$. So, it follows that $x \circ K = \pi(x) \in E^l_W[\pi(x)] \subseteq \pi(x \circ V) \subseteq O$. For, let $\pi(y) \in E^l_W[\pi(x)]$, for some $y \in P$. Then, $y \in x \circ W \subseteq x \circ K \circ W \subseteq x \circ V \circ K$, this implies $\pi(y) \in \pi(\pi^{-1}\{p \circ K : p \in x \circ V\}) = \{p \circ K : p \in x \circ V\} = \pi(x \circ V)$. Therefore, the quotient topology on P/K is coarser than the topology on P/K induced by the uniformity $\mathcal{E}^l_{P/K}$. And (2) of Theorem 3.25 shows that the two topologies on P/K coincide. □

We conclude with the following

Theorem 3.27. *Let K be a normal subpolygroup of an OCP-polygroup P and P/K be the topological quotient polygroup with respect to the operations defined as on Theorem 2.6. Then, the right uniformity $\mathcal{V}^r_{P/K}$ of the topological polygroup P/K coincides with the right uniformity $\mathcal{E}^r_{P/K}$ on P/K when in the latter case P/K is considered as the quotient space of P. Also, the same is true for the uniformities $\mathcal{V}^l_{P/K}$ and $\mathcal{E}^l_{P/K}$ on P/K.*

Proof. Let $\pi : P \to P/K$ be the quotient map. Take an arbitrary element $U \in \mathcal{N}_s(e)$, where $\mathcal{N}_s(e)$ is the family of all open symmetric neighborhoods of e. Since the map π is an open strong homomorphism, $\pi(U) = V$ is an open symmetric neighborhood of K in P/K. So, it is sufficient to verify that $E^r_U = O^r_V$, where E^r_U is defined as in Theorem 3.26 and $O^r_V = \{(s,t) \in P/K \times P/K : s \odot t^{-I} \subseteq V\}$. Let $x, y \in P$ and $y \in U \circ x$, i.e., $x \circ y^{-1} \subseteq U$. Then, Proposition 2.1 implies $\pi(x) \odot [\pi(y)]^{-I} = \pi(x \circ y^{-1}) \subseteq \pi(U) = V$. This implies that $E^r_U \subseteq O^r_V$. For the converse, let $(z,t) \in P/K$ and $(z,t) \in O^r_V$. Then, there exist $x, y \in P$ such that $\pi(x) = z$ and $\pi(y) = t$ and $(\pi(x), \pi(y)) \in O^r_V$, which implies that $\pi(y) \in V \odot \pi(x)$. We claim that $y \in U \circ x$. For, suppose $y \notin U \circ x$, then $\pi(y) \notin \pi(U \circ x) = \pi(U) \odot \pi(x) = V \odot \pi(x)$, a contradiction. Therefore, $(\pi(x), \pi(y)) \in E^r_U$, i.e., $O^r_V \subseteq E^r_U$. In the same way the result can be verified for the families $\mathcal{V}^l_{P/K}$ and $\mathcal{E}^l_{P/K}$. □

Acknowledgments

The authors are grateful to the University of North Bengal, India for the encouragement towards research works.

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