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A WEIGHTED ALGORITHM TO SOLVE THE CONFORMABLE TIME FRACTIONAL REACTION-DIFFUSION-CONVECTION PROBLEM

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Abstract. A simple algorithm is applied in this paper to solve the conformable time fractional reaction-diffusion-convection problem (CTFRDCP) with varriable coefficients. The aim of applying this algorithm is to overcome the inability of the differential transform method to solve such problems. The differential transform method is implemented twice. Once with initial condition, again with boundary conditions. A convex combination of two solutions is considered as solution of the problem.

Key Words: Time fractional heat conduction problem, Fractional differential transform method.

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1. Introduction

The fractional heat equation includes a fractional derivative with respect to space and/or time variable. By replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0,1)$ to the standard heat equation, we have a time fractional heat equation [1]. For example, models that describe heat conduction in materials with non-standard structure, such as porous materials, ploymers and so on, use derivatives of fractional-order [2, 3]. Many articles have been written to obtain analytical and numerical solutions of fractional heat conduction equation [5, 6, 7, 8, 9]. A method that gives the exact solutions or approximate

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solutions by power series, namely, differential transform method (DTM), proposed by Zhou [10] for solving some boundary value problems in ordinary dierential equations. To solve some PDEs, the 2D DTM was proposed by Chen and Ho [11]. An alternative technique which is similar to DTM and derived from the power series expansion, named reduced differential transform method (RDTM), proposed by Keskin and Oturanc [12] to solve linear and nonlinear PDEs. In this paper, by applying the fractional power series expansions where proposed by Abdeljawad [13], the RDTM is adapted to conformable fractional derivative [14] to solve CTFRDCP of the form

(1.1)
$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + a_0(x)u(x,t) + a_1(x)\frac{\partial u(x,t)}{\partial x} + a_2(x)\frac{\partial^2 u(x,t)}{\partial x^2} - f(x,t) = 0,$$
$$(x,t) \in [0,L] \times [0,T],$$

with the conditions

$$u(x,0) = \phi(x), \qquad 0 \le x \le L, \tag{1.2}$$

$$u(0,t) = g(t), \qquad 0 \le t \le T, \tag{1.3}$$

$$u_x(0,t) = h(t), \qquad 0 \le t \le T, \tag{1.4}$$

where $a_0(x), a_1(x), a_2(x), f(x,t), \phi(x), g(t)$ and $h(t)$ are given functions and $\dfrac{\partial^\alpha u}{\partial t^\alpha}$ is the conformable time fractional derivative of order α defined in [14].

The arrangement of this paper is in the following plan: In section 2, the conformable fractional derivative is reviewed. In Section 3, the RDTM is given based on the conformable fractional derivative and a weighted algorithm is introduced. Finally, in Section 4, some test problems are solved in order to show the ability and efficiency of the algorithm.

2. The conformable fractional derivative

In this section, some necessary definitions and mathematical preliminaries of the conformable fractional derivative required for our work are reviewed.

Definition 2.1. [14] Given a function $f : [0,\infty) \to \mathbb{R}$. Then, the conformable fractional derivative of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0, \alpha \in (0,1)$. If f is α-differentiable in some $(0,a), a > 0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

$T_\alpha(f)(t)$ satisfies all the properties in the following theorem.

Theorem 2.2. [14] *Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t > 0$, then*
i. $T_\alpha(c_1 f + c_2 g)(t) = c_1 T_\alpha(f)(t) + c_2 T_\alpha(g)(t)$.
ii. $T_\alpha(t^\beta) = \beta t^{\beta-\alpha}$ *for all* $\beta \in \mathbb{R}$.
iii. $T_\alpha(C) = 0$ *for all constant* C.
iv. $T_\alpha(fg)(t) = f(t)T_\alpha(g)(t) + g(t)T_\alpha(f)(t)$.
v. $T_\alpha(\frac{f}{g})(t) = \frac{g(t)T_\alpha(f)(t) - f(t)T_\alpha(g)(t)}{g^2(t)}$.
Specially for certain function we have [14]
1) $T_\alpha(\sin \frac{1}{\alpha}t^\alpha) = \cos \frac{1}{\alpha}t^\alpha$.
2) $T_\alpha(\cos \frac{1}{\alpha}t^\alpha) = \sin \frac{1}{\alpha}t^\alpha$.
3) $T_\alpha(e^{\frac{1}{\alpha}}t^\alpha) = e^{\frac{1}{\alpha}}t^\alpha$.

Theorem 2.3. [13] *Assume f is an infinitely α-differentiable function, for some $0 < \alpha \leq 1$ at a neighborhood of a point t_0. Then f has the fractional power series expansion:*

$$f(t) = \sum_{k=0}^{\infty} \frac{\left(T_\alpha^{t_0} f\right)^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!}, \quad t_0 < t < t_0 + R^{\frac{1}{\alpha}}, \quad R > 0.$$

Here $\left(T_\alpha^{t_0} f\right)^{(k)}(t_0)$ means the application of the conformable fractional derivative k times.

3. Conformable differential transform method

in this section, at first, we review the basic definitions and operations of RDTM which was introduced in [12].
Consider a function of two variables $u(x,t)$ and suppose that it can be represented as product of two single-variable functions, i.e., $u(x,t) = f(x)g(t)$. Based on the properties of differential transform [10], function $u(x,t)$ can be represented as

$$u(x,t) = \sum_{k=0}^{\infty} F_i x^i \sum_{j=0}^{\infty} G_j t^j = \sum_{k=0}^{\infty} U_k(x) t^k, \tag{3.1}$$

where $U_k(x)$ is called t-dimensional spectrom function of $u(x,t)$.

Definition 3.1. If function $u(x,t)$ is analytic and differentiated continuosly with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \tag{3.2}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function which is called T-function in brief. The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{3.3}$$

Combining (3.2) and (3.3) gives the solution as

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k. \tag{3.4}$$

In real applications, by taking to account n-terms of the series (3.4), the function $u(x,t)$ can be written by

$$u(x,t) = \lim_{n\to+\infty} u_n(x,t) = \lim_{n\to+\infty}\sum_{k=0}^{n} U_k(x)t^k. \tag{3.5}$$

The fundamental operations of reduced differential transform that can be deduced from Eqs. (3.2) and (3.3) are listed in Table 1.
Now, considering Theorem 2.3 and the fractional power series expansion, we can construct CRDTM, which is based on the conformable fractional derivative.

Definition 3.2. Suppose that $u(x,t)$ be an analytic function that satisfies the conditions of Theorem 2.3 with respect to variable t at $t_0 = 0$. Define

$$U_k(x) = \frac{1}{\alpha^k k!}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=0}, \tag{3.6}$$

where α is order of conformable derivative and $U_k(x)$ is the transformed function.

TABLE 1. Some basic reduced differential transformations.

Function Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$u(x,t) = v(x,t) + w(x,t)$	$U_k(x) = V_k(x) + W_k(x)$
$u(x,t) = cv(x,t)$	$U_k(x) = cV_k(x)$ (c *is a constant*)
$u(x,t) = v(x,t)w(x,t)$	$U_k(x) = \sum_{k_1=0}^{k} V_{k_1}(x)W_{k-k_1}(x)$
$u(x,t) = t^n v(x,t)$	$U_k(x) = V_{k-n}(x)$
$u(x,t) = x^m t^n$	$U_k(x) = x^m\delta(k-n) = \begin{cases} x^m & k=n \\ 0 & k\neq n \end{cases}$
$u(x,t) = \frac{\partial}{\partial t}v(x,t)$	$U_k(x) = (k+1)V_{k+1}(x)$
$u(x,t) = \frac{\partial^2}{\partial t^2}v(x,t)$	$U_k(x) = (k+1)(k+2)V_{k+2}(x)$
$u(x,t) = \frac{\partial^m}{\partial x^m}v(x,t)$	$U_k(x) = \frac{\partial^m}{\partial x^m}\mathrm{V}_k(x)$

Definition 3.3. Let $U_k(x)$ be the transform of $u(x,t)$ with respect to t, the differential inverse transform of $U_k(x)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = \sum_{k=0}^{\infty}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=0} t^{k\alpha}. \tag{3.7}$$

All properties of the CRDTM are similar to those in Table 1. For instance, the following theorem is proved.

Theorem 3.4. *If* $u(x,t) = \dfrac{\partial^\alpha v(x,t)}{\partial t^\alpha}$, *then* $U_k(x) = \alpha(k+1)V_{k+1}(x)$

Proof. From Definition 3.2 we have

$$\begin{aligned} U_k(x) &= \frac{1}{\alpha^k k!}[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)]_{t=0} = \frac{1}{\alpha^k k!}[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\frac{\partial^\alpha}{\partial t^\alpha}v(x,t)\right)]_{t=0} \\ &= \frac{1}{\alpha^k k!}[\frac{\partial^{(k+1)\alpha}}{\partial t^{(k+1)\alpha}}v(x,t)]_{t=0} = \alpha(k+1)\frac{1}{\alpha^{k+1}(k+1)!}[\frac{\partial^{(k+1)\alpha}}{\partial t^{(k+1)\alpha}}v(x,t)]_{t=0} \\ &= \alpha(k+1)V_{k+1}(x) \end{aligned}$$

□

Now, a weighted method according to the CRDTM will be presented to solve the problem (1.1)-(1.4). It is done in two step. At the first step, considering Theorem 3.4 and Table 2, the transformation of Eq.(1.1)

Table 2. Some basic properties of CRDTM.

Function Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{\alpha^k k!}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=0}$
$u(x,t) = v(x,t) + w(x,t)$	$U_k(x) = V_k(x) + W_k(x)$
$u(x,t) = cv(x,t)$	$U_k(x) = cV_k(x)$ (c is a constant)
$u(x,t) = v(x,t)w(x,t)$	$U_k(x) = \sum_{k_1=0}^{k} V_{k_1}(x)W_{k-k_1}(x) = \sum_{k_1=0}^{k} W_{k_1}(x)V_{k-k_1}(x)$
$u(x,t) = x^m t^n$	$U_k(x) = x^m\delta(k\alpha - n) \; = \begin{cases} x^m & k\alpha=n \\ 0 & k\alpha\neq n \end{cases}$
$u(x,t) = \frac{\partial^\alpha}{\partial t^\alpha}v(x,t)$	$U_k(x) = \alpha(k+1)V_{k+1}(x)$

and initial condition(1.2) with respect to t we have

(3.8)

$$\alpha(k+1)U_{k+1}(x)+a_0(x)U_k(x)+a_1(x)\frac{\partial}{\partial x}U_k(x)+a_2(x)\frac{\partial^2}{\partial x^2}U_k(x)-F_k(x) = 0,$$

where $F_k(x)$ is transformation of $f(x,t)$. By substituting of the $U_0(x) = \phi(x)$ as transformation of (1.2) into (3.8), the approximate solution

$$\hat{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^{k\alpha}. \tag{3.9}$$

will be obtained.

At the second step, we seek the series solution of the Eq.(1.1) according to boundary conditions (1.3) and (1.4). Again considering Theorem 3.4 and Table 1, take differential transform of Eq.(1.1) with respect to x and get

(3.10)

$$\frac{\partial^\alpha}{\partial t^\alpha}U_k(t) + \sum_{k_1=0}^{k} U_{k_1}(t)A_{0(k-k_1)}(x) + \sum_{k_1=0}^{k} k_1(k_1+1)U_{k_1+1}(t)A_{1(k-k_1)}(x) + \sum_{k_1=0}^{k} (k_1+1)(k_1+2)U_{k_1+2}(t)A_{2(k-k_1)}(x) - F_k(t) = 0$$

Where $U_i(t), A_i(x)$, and $F_i(t)$ are transformation of $u(x,t), a_i(x), i = 0,1,2$ and $f(x,t)$ with respect to x. From the boundary conditions (1.3) and (1.4), we have

$$U_0(t) = g(t), \tag{3.11}$$

and

$$U_1(t) = h(t). \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10), we can get the successive values of $U_r(t)$. In result, the series solution

$$\check{u}_n(x,t) = \sum_{r=0}^{n} U_r(t)x^r. \tag{3.13}$$

will be obtained. An approximate solution to problem (1.1)-(1.4) is considered as the following weighted combination.

$$u_n(x,t) = \lambda \hat{u}_n(x,t) + (1-\lambda)\check{u}_n(x,t), \tag{3.14}$$

where λ is a constant on the interval $[0,1]$. For determining the best value of λ for each n, we use the idea presented in [15].

Theorem 3.5. *Suppose that $\phi(x) \in L^2[(0,L)]$, $g(t), h(t) \in L^2[(0,T)]$ and $\|.\|$ denotes the L^2-norm. Let*

$$\lambda_1 = \|\hat{u}_n(0,t) - g(t)\|,$$
$$\lambda_2 = \|\frac{\partial \hat{u}_n}{\partial x}(0,t) - h(t)\|,$$
$$\lambda_3 = \|\check{u}_n(x,0) - \phi(x)\|.$$

Then the best value for λ in (3.14) is

$$\lambda = \frac{\lambda_3^2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}, \ n \geq 0.$$

Proof. refer to [15]. □

4. Illustrative examples

To show the applicability of the CRDTM, some examples will be presented. We use n terms in evaluating the approximate solution $u_n(x,t)$.

Example 4.1. As the first example consider

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial u(x,t)}{\partial x} - x\frac{\partial^2 u(x,t)}{\partial x^2} + 4x - 1 = 0, \ (x,t) \in [0,1]\times[0,1], \tag{4.1}$$

with the conditions

$$u(x,0) = x^2, \qquad 0 \leq x \leq 1, \tag{4.2}$$

$$u(0,t) = u_x(0,t) = \frac{1}{\alpha}t^\alpha, \qquad 0 \leq t \leq 1. \tag{4.3}$$

Using Theorem 3.4 and Table 2, transformation of the Eq.(4.1) with respect to t becomes

$$\alpha(k+1)U_{k+1}(x) - \frac{\partial}{\partial x}U_k(x) - x\frac{\partial^2}{\partial x^2}U_k(x) + (4x-1)\delta(k\alpha) = 0. \tag{4.4}$$

Substituting $U_0(x) = x^2$ as transformation of the initial condition (4.2) into recurrence relation (4.1) gives the next $U_k(x), k \geq 1$ as

$$U_1(x) = \frac{1}{\alpha},\ U_k(x) = 0,\ k = 2, 3, \cdots$$

From (3.9), the inverse differential transform of $U_k(x)$ gives:

$$\hat{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^{k\alpha} = x^2 + \frac{1}{\alpha}t^{\alpha}$$

Now, use the basic properties of the reduced differential transform of Table 1 with respect to x. Transformation of the Eq.(4.1) and the boundary conditions (4.3) to x becomes

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}U_k(t) - (k+1)U_{k+1}(t) - k(k+1)U_{k+1}(t) + 4\delta(k-1) - \delta(k) = 0 \tag{4.5}$$

and

$$U_0(t) = U_1(t) = \frac{1}{\alpha}t^{\alpha}. \tag{4.6}$$

To take the next $U_k(x), k \geq 2$, replace (4.6) into recurrence relation (4.5) and give

$$U_2(t) = \frac{5}{2}\alpha,\ U_k(t) = 0,\ k = 3, 4, \cdots.$$

From (3.13) the inverse differential transform of $U_k(x)$ gives:

$$\breve{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)x^k = \frac{1}{\alpha}t^{\alpha} + \frac{1}{\alpha}t^{\alpha}x + \frac{5}{2}\alpha x^2.$$

As $\lambda = 1$ for $n \geq 8$, the Eqs. (3.14) and (3.5) give

$$\lim_{n\to+\infty}\big(u_n(x,t) = \hat{u}_n(x,t)\big) = x^2 + \frac{1}{\alpha}t^{\alpha},$$

which is the exact solution of the problem.

Example 4.2. Consider the following problem

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} - \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} - u(x,t) + e^{\frac{1}{\alpha}t^{\alpha}} = 0,\ (x,t) \in [0,1]\times[0,1], \tag{4.7}$$

with the conditions

$$u(x,0)=x, \qquad 0\le x\le 1, \tag{4.8}$$

$$u(0,t)=0, u_x(0,t)=e^{\frac{1}{\alpha}t^{\alpha}}, \qquad 0\le t\le 1. \tag{4.9}$$

Being in a similar way with the first example, we apply the Theorem 3.4 and Table 2 to Eq.(4.7) and achieve the following relation.

$$\alpha(k+1)U_{k+1}(x)=\frac{\partial^2}{\partial x^2}U_k(x)+\frac{\partial}{\partial x}U_k(x)+U_k(x)-\frac{1}{\alpha^k k!}. \tag{4.10}$$

Substituting the initial condition (4.8), *i.e.* $U_0(x)=x$ into relation (4.10) we have

$$U_1(x)=\frac{1}{\alpha}x,\ U_2(x)=\frac{1}{2\alpha^2}x,\ U_3(x)=\frac{1}{6\alpha^3}x,\ \cdots,\ U_k(x)=\frac{1}{k!\alpha^k}x.$$

Therefore, we obtain the approximate solution

$$\hat{u}_n(x,t)=\sum_{k=0}^{n}U_k(x)t^{k\alpha}=x+\frac{1}{\alpha}xt^{\alpha}+\frac{1}{2\alpha^2}xt^{2\alpha}+\frac{1}{6\alpha^3}xt^{3\alpha}+\cdots+\frac{1}{n!\alpha^n}xt^{n\alpha}.$$

On the other side, using the basic properties of the reduced differential transform of Table 1 with respect to x, for the Eq.(4.7) and the boundary conditions (4.9) we take the relation

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}U_k(t)-(k+1)(k+2)U_{k+2}(t)-(k+1)U_{k+1}(t)-U_k(t)+e^{\frac{1}{\alpha}t^{\alpha}}\delta(k)=0,$$

or

$$(k+1)(k+2)U_{k+2}(t)=(k+1)U_{k+1}(t)+U_k(t)-\frac{\partial^{\alpha}}{\partial t^{\alpha}}U_k(t)-e^{\frac{1}{\alpha}t^{\alpha}}\delta(k) \tag{4.11}$$

and

$$U_0(t)=0,\ U_1(t)=e^{\frac{1}{\alpha}t^{\alpha}}. \tag{4.12}$$

Replace (4.12) into (4.11) and obtain $U_k(t)=0,\ k\ge 2$. So, we take the solution of the problem (4.7) with boundary conditions (4.9) as

$$\check{u}_n(x,t)=\sum_{k=0}^{n}U_kx^k=0+e^{\frac{1}{\alpha}t^{\alpha}}x+0=xe^{\frac{1}{\alpha}t^{\alpha}}.$$

Here, $\lambda=0$ for $n\ge 5$. Hence, by Eqs. (3.14) and (3.5),

$$\lim_{n\to+\infty}\big(u_n(x,t)=\check{u}_n(x,t)\big)=xe^{\frac{1}{\alpha}t^{\alpha}}.$$

which is the exact solution of the problem.

Example 4.3. The function $u(x,t)=\sin(x+\frac{1}{\alpha}t^{\alpha})$ is the exact solution of the problem

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+x\,(\,\frac{\partial^{2}u(x,t)}{\partial x^{2}}+\frac{\partial u(x,t)}{\partial x}\,)-\cos(\,x+\frac{1}{\alpha}t^{\alpha}\,)=0, \tag{4.13}$$

with the initial and boundary conditions

$$u(x,0)=\sin x, \qquad 0\le x\le \frac{1}{2}, \tag{4.14}$$

$$u(0,t)=\sin\,(\,\frac{1}{\alpha}t^{\alpha}\,)\,,\ u_x(0,t)=\cos\,(\,\frac{1}{\alpha}t^{\alpha}\,) \qquad 0\le t\le\frac{1}{2}. \tag{4.15}$$

For this problem, we compute an approximate solution and compare it with the exact solution.
The transformed form of the Eqs. (4.13) and (4.14) with respect to t becomes

$$(k+1)U_{k+1}(x)+x\frac{\partial^{2}}{\partial x^{2}}U_k(x)+xU_k(x)-\frac{1}{\alpha^{k}k!}\cos(\frac{k\pi}{2}+\frac{1}{\alpha}t^{\alpha})=0. \tag{4.16}$$

Substituting $U_0(x)=\sin x$, the transformed form of the initial condition (4.14), into the relation (4.16) gives
(4.17)

$$\begin{aligned}
&U_1(x)=\tfrac{1}{\alpha}\cos(x)\,,\,U_3(x)=-\tfrac{1}{6\alpha^{3}}\cos(x)\,,\,U_5(x)=\tfrac{1}{120\alpha^{5}}\cos(x)\,,\,\cdots\,,\\
&U_{2k+1}(x)=\tfrac{(-1)^{k}}{(2k+1)!\alpha^{2k+1}}\cos(x)\\
&U_2(x)=-\tfrac{1}{2\alpha^{2}}\sin(x)\,,\,U_4(x)=\tfrac{1}{24\alpha^{4}}\sin(x)\,,\,U_6(x)=-\tfrac{1}{720\alpha^{6}}\sin(x)\,,\,\cdots\,,\\
&U_{2k}(x)=\tfrac{(-1)^{k}}{(2k)!\alpha^{2k}}\sin(x)
\end{aligned}$$

Now, we take the differential transformation of the Eq.(4.13) with respect to x. Again we apply the properties of differential transformation in Table 1 and obtain

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}U_k(t)+k(k+1)U_{k+1}(t)+U_{k-1}(t)-\frac{1}{k!}\cos(\frac{k\pi}{2}+\frac{1}{\alpha}t^{\alpha}\,)=0,$$

or
(4.18)

$$U_{k+1}(t)=\frac{1}{k(k+1)}\left(-\frac{\partial^{\alpha}}{\partial t^{\alpha}}U_k(t)-U_{k-1}(t)+U_k(t)+\frac{1}{k!}\cos(\frac{k\pi}{2}+\frac{1}{\alpha}t^{\alpha}\,)\right).$$

Substituting $U_0(t)=\sin\,(\,\frac{1}{\alpha}t^{\alpha}\,)$ and $U_1(t)=\cos\,(\,\frac{1}{\alpha}t^{\alpha}\,)$, the transformed form of the boundary conditions (4.15), into relation (4.18) gives the next term of $U_k(t), k \geq 2$ as

(4.19)

$$U_2(x) = -\frac{1}{2}\sin(\frac{1}{\alpha}t^\alpha)\,,\ U_4(x) = \frac{5}{144}\sin(\frac{1}{\alpha}t^\alpha)\,,\ U_6(x) = -\frac{41}{43200}\sin(\frac{1}{\alpha}t^\alpha)\,,\ \cdots$$
$$U_3(x) = -\frac{1}{12}\cos(\frac{1}{\alpha}t^\alpha)\,,\ U_5(x) = \frac{1}{720}\cos(\frac{1}{\alpha}t^\alpha)\,,\ U_7(x) = \frac{1}{56700}\cos(\frac{1}{\alpha}t^\alpha)\,,\ \cdots .$$

Therefore, the Eqs. (4.17) and (4.19), give the approximate solution by (3.9), (3.13) and (3.14).
According to the Theorem 3.5, we get $\lambda = 0.449, 0.594, 0.663$ For $n = 5, 10, 15$ respectively. The relative errors of the approximation, have been given in Table 3, Table 4 and Table 5. Also, Figure 1 indicates the function error for $\{(x,t)|\, 0 \leq x \leq \frac{1}{2},\, 0 \leq t \leq \frac{1}{2}\}$.

Table 3. The relative error of the computed approximate solution of the Example 3 by $n = 5$.

x	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$	$t = 0.5$
0.1	$6.6286E-5$	$1.1920E-4$	$3.1249E-4$	$7.2514E-4$	$1.4639E-3$
0.2	$4.1012E-4$	$3.6751E-4$	$5.5493E-4$	$1.0451E-3$	$1.9685E-3$
0.3	$1.2332E-3$	$9.4466E-4$	$1.0219E-3$	$1.5056E-3$	$2.5452E-3$
0.4	$2.6828E-3$	$1.9772E-3$	$1.8118E-3$	$2.1743E-3$	$3.2292E-3$
0.5	$4.8871E-3$	$3.5857E-3$	$3.0238E-3$	$3.1206E-3$	$4.0549E-3$

Table 4. The relative error of the computed approximate solution of the Example 3 by $n = 10$.

x	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$	$t = 0.5$
0.1	$2.4524E-5$	$1.5120E-5$	$9.9538E-6$	$6.3766E-6$	$3.8580E-6$
0.2	$1.7586E-4$	$1.1260E-4$	$7.5169E-5$	$4.7064E-5$	$2.3597E-5$
0.3	$5.4093E-4$	$3.5778E-4$	$2.4264E-4$	$1.5235E-4$	$7.3519E-5$
0.4	$1.1849E-3$	$8.0621E-4$	$5.5490E-4$	$3.4989E-4$	$1.6451E-4$
0.5	$2.1641E-3$	$1.5105E-3$	$1.0541E-3$	$0.6740E-4$	$3.0572E-4$

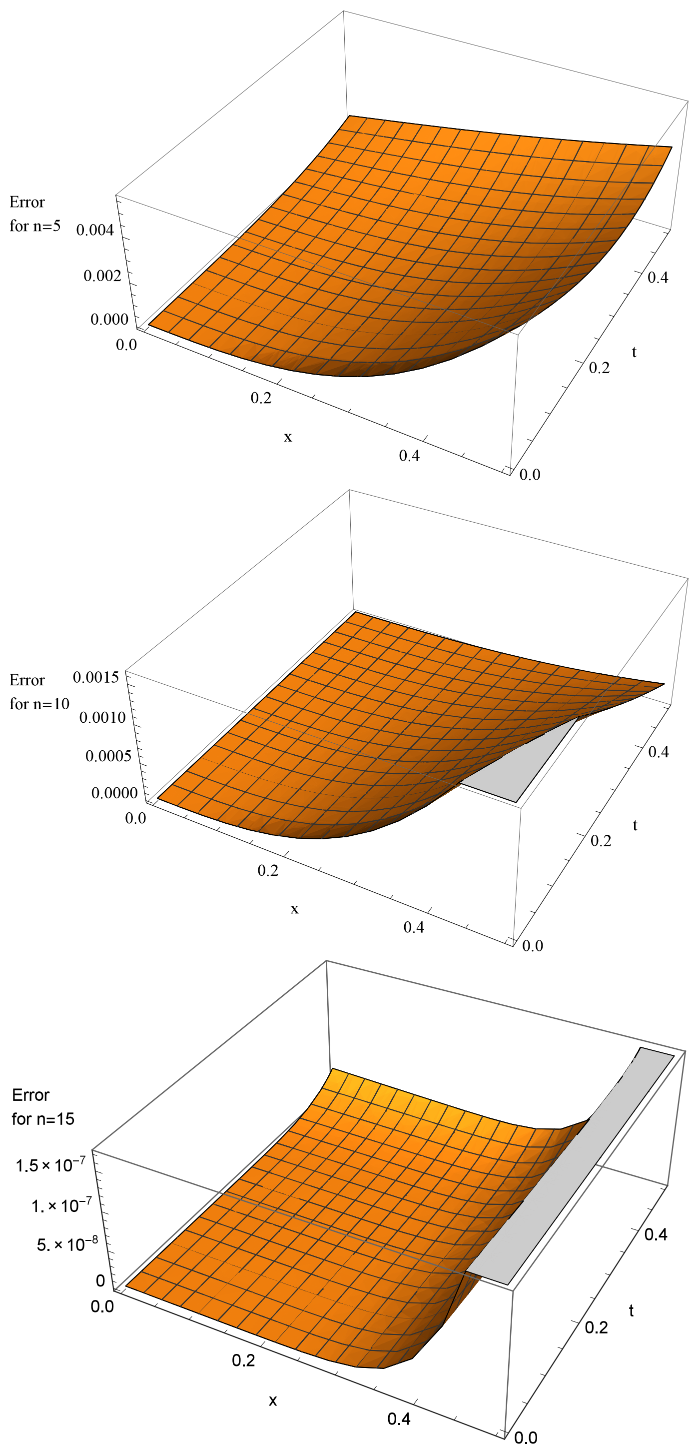

FIGURE 1. The error function of the approximate solution of the example 3 for $n = 5, 10, 15$.

TABLE 5. The relative error of the computed approximate solution of the Example 3 by $n = 15$.

x	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$	$t = 0.5$
0.1	$5.2963E-8$	$4.8109E-8$	$6.1769E-9$	$2.0888E-10$	$3.3379E-10$
0.2	$4.7110E-8$	$4.2793E-8$	$5.4943E-8$	$1.8580E-10$	$2.9323E-10$
0.3	$4.1394E-7$	$3.7599E-8$	$4.8276E-8$	$1.6325E-9$	$2.5846E-9$
0.4	$3.5883E-7$	$3.2594E-7$	$4.1849E-7$	$1.4151E-8$	$2.2515E-9$
0.5	$3.0648E-7$	$2.7839E-7$	$3.5744E-7$	$1.2087E-8$	$1.9105E-8$

5. Conclusion

In this study, the time fractional reaction-diffusion-convection problem with varriable coefficients has been solved. The time derivative has been considered the conformable fractional derivative. Reduced differential transform method has been adapted to conformable fractional derivative, then has been applied to obtain two approximate solution. One of them with respect to time variable t by initial condition and another, with respect to space variable x by boundary conditions. A convex combination of two solution has been introduced as the approximate solution of the problem. The given examples, have shown that the proposed method yield good results.

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