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SOME CHARACTERIZATIONS OF SURJECTIVE OPERATORS ON BANACH LATTICES

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Abstract. The concepts of compact and weakly compact operators on Banach spaces are considered and investigated in several papers. In this paper, taking idea from this notations, we consider the concept surjective compact and weakly compact operators on Banach lattices. In particular, we characterize Banach lattices on which operators must be surjective.

Key Words: Banach lattice, Semi-compact operator, Dunford-Pettis operator, L- and M-weakly compact operator.
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1. Introduction

An operator T from a Banach space X into a Banach space Y is compact (resp. weakly compact) if $\overline{T(B_X)}$ is compact (resp. weakly compact) where B_X is the closed unit ball of X. There exists compact operator T from a Banach space X into a Banach space Y which is not surjective. Indeed the operator $T : \ell^1 \to \ell^\infty$ defined by $T(\alpha_1, \alpha_2, ...) = (\Sigma_{n=1}^{\infty}\alpha_n, \Sigma_{n=1}^{\infty}\alpha_n, ...)$ is a compact operator which is not surjective. An operator T from a Banach space X into a Banach space Y is weakly compact if X or Y is reflexive but the converse is false in general. In fact each continuous operator from ℓ^∞ into c_0 is weakly compact but ℓ^∞ and c_0 are not reflexive. Let E be an infinite-dimensional AL-space, F a Banach lattice. If each operator $T : E \to F$ is weakly compact, then by [3, Corollary 2.6], F is reflexive. We show that, if an operator T

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from a Banach space X into a Banach space Y is compact (resp. weakly compact) and surjective, then Y must be finite-dimensional (resp. Y must be reflexive). We also show that if E and F are Banach lattices such that F is not reflexive and If a positive Dunford-Pettis operator $T : E \to F$ is surjective, then the norm of E' is not order continuous [Theorem 2.16]. In the final subsection we investigate surjective L- and M-weakly compact operators. Let E and F be Banach lattices such that the norm of E' is order continuous and F be infinite-dimensional. If $T : E \to F$ is a regular surjective L-weakly compact operator, then E' is not discrete [Proposition 2.22]. Finally we prove that if F is an infinite-dimensional discrete Banach lattice with an order continuous norm and if T is a regular M-weakly compact operator from a Banach lattice E (which is arbitrary) into F, then T is not surjective [Corollary 2.24].

1.1. **Some basic definitions.** A vector lattice E is an order vector space in which $\sup(x, y)$ exists for every $x, y \in E$. Let E be a vector lattice, for each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded (resp. b-order bounded) if it is included in some order interval (resp. order bonded in second order dual of E). A subset F of a vector lattice E is said to be sublattice if for every pair of elements $a, b \in F$ the supremum of a and b taken in E belongs to F . A subset B of a vector lattice E is said to be solid if it follows from $|y| \leq |x|$ whit $x \in B$ and $y \in E$ that $y \in B$. An order ideal of E is a solid subspace . The solid hull of a subset W is the smallest solid set including W and is exactly the set $\mathrm{Sol}(W) := \{x \in E : \exists y \in W \text{ with } |x| \leq |y|\}$. A sequence $(x_n)_n$ in a vector lattice is said to be disjoint whenever $|x_n| \wedge |x_m| = 0$ holds for $n \neq m$. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property : for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm and dual order is also a Banach lattice . A norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized sequence $(x_\alpha)_\alpha$ such that $x_\alpha \downarrow 0$ in E , the generalized sequence $(x_\alpha)_\alpha$ convergs to 0 for the norm $\|.\|$ where the notation $x_\alpha \downarrow 0$ means that $(x_\alpha)_\alpha$ is decreasing, its infimum exists and $\inf_\alpha(x_\alpha) = 0$. A Banach lattice E is is said to be an AM-space if for each $x, y \in E$ such that $|x| \wedge |y| = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. The Banach lattice E is an AL-space if its topological dual E' is an AM-space . We will use the term operator $T : E \to F$ between two

Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E . It is well known that each positive linear mapping on a Banach lattice is continuous . The operator T is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators . For terminology concerning Banach lattice theory and positive operators we refer the reader to [1].

2. Main Results

Theorem 2.1. *If an operator T from a Banach space X into a Banach space Y is compact and $T(X)$ is closed, then $T(X)$ is finite-dimensional.*

Proof. Let $T : X \to Y$ be a compact operator between Banach spaces. Since $\overline{T(X)} = T(X)$, $T(X)$ is a Banach space. If U, denotes the open ball of X then $T(U)$ is an open set in $T(X)$. On the other hand $\overline{T(U)}$ is compact. So, $T(X)$ is locally compact and then $T(X)$ is finite-dimensional. □

Corollary 2.2. *If $T : X \to Y$ is a surjective compact operator between Banach spaces, then Y is finite-dimensional.*

Corollary 2.3. *If X is an infinite-dimensional Banach space, then there is no surjective compact operator on X.*

Theorem 2.4. *Let $T : X \to Y$ be a weakly compact operator between Banach spaces. If $T(X)$ is closed, then $T(X)$ is reflexive.*

Proof. Let $T : X \to Y$ be a weakly compact operator between Banach spaces. Since $T(X)$ is closed, $T(X)$ is a Banach space and from equality $T(X) = \bigcup_{n\in\mathbb{N}} nT(B_X)$, we see that $T(B_X)$ contains a closed ball of $T(X)$. On the other hand, $\overline{T(B_X)}$ is weakly compact. So, that closed ball is weakly compact, then $T(X)$ is reflexive. □

Corollary 2.5. *If $T : X \to Y$ is a surjective weakly compact operator between Banach spaces, then Y is reflexive.*

Corollary 2.6. *If X is a non-reflexive Banach space, then there is no surjective weakly compact operator on X.*

An operator T from a Banach space E into a Banach lattice F is said to be semi-compact if for each $\epsilon > 0$, there exists some $u \in F^+$ such that, $T(B_E) \subset [-u, u] + \epsilon B_F$ where $F^+ = \{x \in F : x \geq 0\}$.

Example 2.7. The identity operator $i : \ell^\infty \to \ell^\infty$ is a surjective semi-compact operator, but the operator $T : \ell^\infty \to \ell^\infty$ defined by $T(\alpha_1, \alpha_2, ...) = (\alpha_1, \frac{\alpha_2}{2}, \frac{\alpha_3}{3}, ...)$ is a semi-compact operator which is not surjective.

Proposition 2.8. *Let E, F be Banach lattices such that F has an order continuous norm and $T : E \to F$ be a positive surjective semi-compact operator. Then F is reflexive.*

Proof. Let $T : E \to F$ be a positive surjectuve semi-compact operator between Banach lattices E and F such that F has an order continuous norm. By [2, Theorem 2.2], T is a weakly compact operator. Since T is surjective, by Corollary 2.5, F is reflexive. □

Corollary 2.9. *Let E be a non-reflexive Banach lattice with an order continuous norm. Then there is no positive surjective semi-compact operator on E.*

Recall that a nonzero element x of a Banach lattice E is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice E is discrete if it admits a complete disjoint system of discrete elements. For example the Banach lattice ℓ^2 is discrete but $L^1[0,1]$ is not.

Theorem 2.10. *Let E, F be Banach lattices and $T : E \to F$ be a positive injective semi-compact operator such that its range is closed. If one of the following statements is valid, then E is finite-dimensional .*

(1) *F is discrete and its norm is order continuous.*
(2) *E' is discrete of order continuous norm and F has an order continuous norm.*
(3) *The norms of E, E' and F are order continuous and E has the Dunford-pettis property (i.e. each weakly compact operator from E into an arbitrary Banach space F is Dunford-pettis).*

Proof. Let E, F be Banach lattices. Assume that $T : E \to F$ is a positive injective semi-compact operator such that its range is closed. We first show that T' is surjective . Since $T(E)$ is closed, $T(E)$ is a Banach space, so, $T_1 : E \to T(E)$ is a bijective operator between Banach spaces. Then $T_1' : T(E)' \to E'$ is bijective. Consequently $T' : F' \to E'$ is surjective. Now, by [2, Theorem 2.6], if one of the above conditions is valid, then T is compact. Therefore T' is compact. So, T' is a surjective compact operator. Then by Corollary 2.2, E' is finite dimensional. Hence E is finite-dimensional. This completes the proof. □

An operator T from a Banach space E into another F is said to be Dunford-Pettis, if it carries each weakly compact subset of E onto a compact subset of F (i.e. whenever $x_n \xrightarrow{w} 0$ implies $Tx_n \xrightarrow{\|\cdot\|} 0$). It is clear that any compact operator is Dunford-Pettis, while a Dunford-Pettis operator is not necessarily compact. Indeed the identity operator $i : \ell^1 \to \ell^1$ is Dunnford-Pettis but is not compact. The operator $T : \ell^\infty \to \ell^\infty$ defined by $T(\alpha_1, \alpha_2, ...) = (\alpha_1, \frac{\alpha_2}{2}, \frac{\alpha_3}{3}, ...)$ is a Dunford-Pettis operator which is not surjective.

Theorem 2.11. *Let E, F be Banach lattices and $T : E \to F$ be a positive surjective Dunford-Pettis operator. If one of the following conditions is valid, then F is finite-dimensional.*

(1) *The norm of E and E' are order continuous.*
(2) *E' is discrete and its norm is order continuous.*
(3) *The norm of E' is order continuous, F is discrete and its norm is order continuous.*

Proof. Let E and F be Banach lattices and $T : E \to F$ be a positive surjective Dunford-Pettis operator. By [5, Theorem 2.2], T is a compact operator. Applying Corollary 2.2, F is finite dimensional. $\square$

An operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact, if it maps each subset of E which is b-order bounded (i.e. order bounded in the topological bidual E'') into a relatively weakly compact subset of X.

Example 2.12. The identity operator $i : \ell^1 \to \ell^1$ is a surjective b-weakly compact operator, but the operator $T : \ell^2 \to \ell^2$ defined by $T(\alpha_1, \alpha_2, ...) = (\frac{\alpha_1}{2}, \frac{\alpha_2}{2^2}, \frac{\alpha_3}{2^3}, ...)$ is a b-weakly compact operator which is not surjective.

Theorem 2.13. *Let E and F be Banach lattices and $T : E \to F$ be an interval preserving surjective b-weakly compact operator. Then the norm of F is order continuous.*

Proof. Let E and F be Banach lattices and $T : E \to F$ be a positive surjective b-weakly compact operator. Assume that $\{y_n\}_n \subseteq F$ with $y_n \downarrow 0$. Since T is surjective, there is an element $x_1 \in E$ such that $Tx_1 = y_1$. It is clear that $\{y_n\}_n \subseteq [0, y_1] = T([0, x_1])$. Since $T([0, x_1])$ is relatively weakly compact, there is a subsequence $\{y_{n_j}\}_j$ from $\{y_n\}_n$ such that $y_{n_j} \xrightarrow{w} y_0 \in F$. Since $\{y_{n_j}\}_j$ is a decreasing sequence, by [1, Theorem 3.52], $y_{n_j} \xrightarrow{\|\cdot\|} y_0 \in F$. Since $y_n \downarrow 0$, $y_0 = 0$. Then $\|y_n\| \to 0$. Thus F has order continuous norm. $\square$

Recall that if E is a Banach lattice and if $0 \leqslant x'' \in E''$, then the principal ideal $I_{x''}$ generated by $x'' \in E''$ under the norm $\|.\|_\infty$ defined by

$$\|y''\|_\infty = \inf\{\lambda > 0 : \ |y''| \leq \lambda x''\}, \ y'' \in I_{x''},$$

is an AM-space with unit x'', whose closed unit ball is order interval $[-x'', x'']$.

Lemma 2.14. *Let E be a Banach lattice. Then every b-order bounded disjoint sequence in E is weakly convergent to zero.*

Proof. Let $\{x_n\}_n$ be a disjoint sequence in E such that $\{x_n\}_n \subseteq [-x'', x'']$ for some $x'' \in E''$. Let $Y = I_{x''} \cap E$ and equip Y with the order unit norm $\|.\|_\infty$ generated by x''. The space $(Y, \|.\|_\infty)$ is an AM-space, so, Y' is an AL-space and then its norm is order continuous. Now, by [8, Theorem 2.4.14], we see that $x_n \xrightarrow{w} 0$. □

Theorem 2.15. *Every Dunford-Pettis operator from a Banach lattice E into a Banach space X is b-weakly compact.*

Proof. Let T be a Dunford-Pettis operator from a Banach lattice E into a Banach space X. By [6, Proposition 1], it suffices to show that $\{Tx_n\}_n$ is norm convergent to zero for each b-order bounded disjoint sequence $\{x_n\}_n$ in E^+. Let $\{x_n\}_n$ be a b-order bounded disjoint sequence in E^+. As the canonical embedding of E into E'' is a lattice homomorphism, $\{x_n\}_n$ is an order bounded disjoint sequence in E''. Thus by preceding lemma, $\{x_n\}_n$ is $\sigma(E, E')$ convergent to zero in E. Now, since T is Dunford-Pettis, $\{Tx_n\}_n$ is norm convergent to zero. This completes the proof. □

Theorem 2.16. *Let E and F be Banach lattices such that F is not reflexive. If a positive Dunford-Pettis operator $T : E \to F$ is surjective, then the norm of E' is not order continuous.*

Proof. Assume that the norm of E' is order continuous. Let B be the band generated by E in its topological bidual E''. Since T is Dunford-Pettis, by Theorem 2.15, T is b-weakly compact. Now, [6, Proposition 2] may apply to yield $T''(B) \subset F$ where T'' is the second adjoint of T. On the other hand, since the norm of E' is order continuous, it follows from [8, Theorem 2.4.14] that $B = E''$. Thus $T''(E'') \subset F$ and hence from [1, Theorem 5.23], T is weakly compact. Since T is also surjective, by Corollary 2.5, F is reflexive which is a contradiction. □

As a consequence of Theorems 2.13 and 2.15, we obtain:

Corollary 2.17. *Let E and F be Banach lattices and $T : E \to F$ be an interval preserving surjective Dunford-Pettis operator. Then the norm of F is order continuous.*

2.1. **Surjective L- and M-weakly compact operators.** Recall that if E is a Banach lattice then E^a is the maximal order ideal in E on which the norm is order continuous. A positive element in E is discrete if its linear span is an order ideal in E. E is termed discrete if the band generated by the discrete elements is the whole space. For instance c, c_0, $l_p(1 \leqslant p \leqslant \infty)$ are discrete Banach lattices but the spaces $L^1[0,1]$ and $C[0,1]$ are not discrete. It turns out that the class of Banach lattices E that we need are those such that E^a are discrete. Although a sublattice of a discrete vector lattice need not be discrete, an ideal must be, so that if E is discrete then so is E^a. By [7, Corollary 2.3], we see that if E has order continuous norm then E is discrete, if and only if E has weakly sequentially continuous lattice operations. It is clear that if F is a closed sublattice of a Banach lattice with weakly sequentialy continuous lattice operations then the same is true for F. This applies, in particular, when $F = E^a$ so that if E has weakly sequentially continuous lattice operations, then E^a is discrete. By [1, Theorem 4.31], we see that if E is an AM-space then E has weakly sequentially continuous lattice operations, so that E^a is discrete.
An operator T from a Banach lattice E into a Banach space X is M-weakly compact if $lim_n\|T(x_n)\| = 0$ holds for every norm bounded disjoint sequence $(x_n)_n$ in E. An operator T from a Banach space X into a Banach lattice E is called L-weakly compact if $lim_n\|y_n\| = 0$ holds for every disjoint sequence $(y_n)_n$ in the solid hull of $T(B_X)$ where B_X is the closed unit ball of X. Let E and F be Banach lattices. If $T : E \to F$ is a surjective L-weakly (resp. M-weakly) compact operator then F is reflexive [1, Theorem 5.61]. A compact (and hence a weakly compact) operator need not be L- or M-weakly compact and L-weakly compact operator need not be compact operator:

Example 2.18. Consider the operator $T : \ell^1 \to \ell^\infty$ defined by

$$T(\alpha_1, \alpha_2, ...) = (\Sigma_{n=1}^{\infty}\alpha_n, \Sigma_{n=1}^{\infty}\alpha_n, ...).$$

Since T has rank one, is a compact operator. The sequence $\{e_n\}$ of the standard unit vectors is a norm bounded disjoint sequence of ℓ^1

satisfying $Te_n = (1, 1, 1, ...)$ for each n. This shows that T is not M-weakly compact. On the other hand, if U is the closed unit ball of ℓ^1, then it is easy to see that $\{e_n\}$ is also a disjoint sequence in the solid hull of $T(U)$. From $\|e_n\|_\infty \nrightarrow 0$, we see that T fails to be L-weakly compact.

Example 2.19. (1) The inclusion map $i : L^2[0, 1] \to L^1[0, 1]$ is M- and L-weakly compact operator which is not compact [8, Proposition 3.6.20].

(2) Let $(X, \sum, \mu)$ be a measure space such that μ is an atomless measure and let A_i be finite measurable pairwise disjoint sets. Then there exists a weakly null sequence $(f_n)_n$ in $L^1(\mu)$ which is not norm null. Hence, the operator $T : L^1(\mu) \to L^1(\mu)$ defined by $T(f) = \sum_{n=1}^{\infty}(\int_{A_i} f d\mu)f_n$ is M-weakly and L-weakly compact operator which is not compact. Note that T is not surjective.

Example 2.20. Let $E = \ell^2$ and $F = L^1[0, 1]$. By [8, Corollary 2.7.7], we see that F contains a closed subspace H which is isomorphic to ℓ^2, so that the isomorphism $T : E \to H \subset F$ is weakly compact and hence as F is an AL-space, surjective L-weakly compact operator.

Proposition 2.21. *Let E be a Banach lattice such that E^a is discrete and X be a Banach space. If $T : X \to E$ is a surjective L-weakly compact operator, then E is finite-dimensional.*

Proof. Let E be a Banach lattice such that E^a is discrete and X be a Banach space. Let $T : X \to E$ is a surjective L-weakly compact operator. Applying [4, Theorem 3.1], T is compact. Since T is also surjective, by Corollary 2.2, E is finite-dimensional. □

Proposition 2.22. *Let E and F be Banach lattices such that the norm of E' is order continuous and F be infinite-dimensional. If $T : E \to F$ is a regular surjective L-weakly compact operator, then E' is not discrete.*

Proof. Assume that E' is discrete. Clearly the operator $T' : F' \to E'$ is a regular M-weakly compact operator. Since the norm of E' is order continuous, by [8, Corollary 3.6.14], T' is a regular L-weakly compact operator and hence is compact by [4, Theorem 3.1]. Thus T is also compact and hence by Corollary 2.2, F is finite-dimensional which is a contradiction. □

Proposition 2.23. *Let E and F be Banach lattices such that F is infinite-dimensional and its norm is not order continuous. If $T : E \to F$ is a surjective M-weakly compact operator, then $(E')^a$ is not discrete.*

Proof. If $(E')^a$ is discrete, then by [4, Theorem 4.2], T is a surjective compact operator. Then F is finite-dimensional which is a contradiction. □

Finally, as a consequence of [4, Theorem 4.5], we obtain:

Corollary 2.24. *Let F be an infinite-dimensional discrete Banach lattice with an order continuous norm. If T is a regular M-weakly compact operator from a Banach lattice E (which is arbitrary) into F, then T is not surjective.*

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