Journal of Hyperstructures 12 *(2) (2023), 257-275.*
ISSN: 2322-1666 print/2251-8436 online
Research Paper

ZAGREB INDICES OF SOME CHEMICAL STRUCTURES USING NEW PRODUCTS OF GRAPHS

LIJU ALEX AND G. INDULAL

Abstract. Zagreb indices are one of the most extensively studied degree-based structural descriptors for analyzing various physico-chemical properties of chemical compounds. In this paper, we define four new products of graphs based on adjacency relations and compute their Zagreb indices. Using these expressions we compute the Zagreb indices of various chemical compounds such as linear polyacene, a class of nanotubes $N\mathbb{A}_m^{2n}$, toroidal fullerene $N\mathbb{C}_{2m}^{2n}(\mathbb{H}_{2m}^{2n})$ and hexagonal lattice.

Key Words: first Zagreb index ($M_1(G)$), second Zagreb index ($M_2(G)$), Fullerenes
2010 Mathematics Subject Classification: Primary: 05C90 Secondary: 92E10,94C15 .

1. Introduction

Topological indices, since their inception in 1947 by H. Wiener [29] has been subjected to an extensive study in analysing the structure-property relationship of compounds. In 1972, Gutman and Trinajstić defined the first and second Zagreb indices as an easier approximation in the computation of $\pi-$ electron energy of hydrocarbons [22]. Let G be any connected graph with vertex set $V(G)$ and edge set $E(G)$. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ [21] are

Received: 13 June 2023, Accepted: 29 December 2023. Communicated by Yuming Feng;
∗Address correspondence to L. Alex; E-mail:lijualex0@gmail.com.

defined as

$$M_1(G) = \sum_{u \in V(G)} d(u)^2$$

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v)$$

where $d(u)$ denotes the degree of the vertex u in G. Through the years, several mathematical properties and structure activity relationships of Zagreb indices have been extensively studied. For a detailed literature on Zagreb indices and other topological indices, see [1, 5, 13, 16, 20, 23, 24].

The determination of topological indices of chemical graphs is an important research problem for the past several years [9, 10]. The computation of topological indices of complex chemical structures is a challenging problem which requires polynomial time. Graph operations generalize various classes of graphs thus making the computation of topological indices easier for larger classes of graphs. Graovac and Pisanski were the first ones to study the topological indices of graph operations. They computed the Wiener index of the Cartesian product of graphs [19]. Klavžar [26] determined the closed expressions for the Szeged index of the Cartesian product of graphs. In [25], Khalifeh *et al.* determined the exact expressions for Zagreb indices of Cartesian product and some chemical structures. In 2009, Eliasi and Taeri defined $F-$ sums, a new set of operations on graphs and computed the Wiener index of the sums [17]. In [28], Metsidik *et al.* determined the hyper Wiener index and reverse Wiener index of $F-$ sums. In 2016, Deng *et al.* gave the explicit expressions for the Zagreb indices of $F-$ sums of graphs [14]. Akhtar and Imran computed the Forgotten index of $F-$ sums [2]. Basavanagoud *et al.* introduced sixty new operations related to $F-$ sums and computed Zagreb indices and Forgotten index of the operations [11]. In 2019, Liu *et al.* introduced the generalized form of subdivisions and F_k- sums. They also computed the Zagreb indices of the F_k- sums [27]. Awais *et al.* determined the exact expression for generalized F_k- sums of Forgotten index [8]. Numerous graph operations have been defined, and in-depth research has been done on computing various topological indices on various graph operations [3, 4, 7, 6, 12]. Although there are lots of graph operations which produce larger classes of graphs, most of them does not include large class of chemical structures. In this paper, we define a new graph operation called adjacency product or A-product

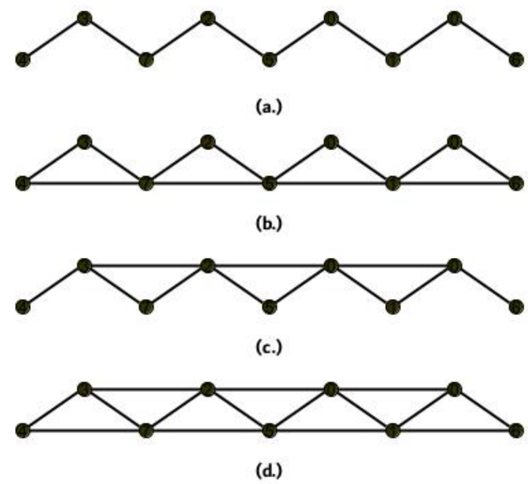

FIGURE 1. (a.) $S(P_5)$, (b.) $R(P_5)$,(c.) $Q(P_5)$, (d.) $T(P_5)$

which generalizes the structure of some chemical compounds and compute the Zagreb indices of adjacency products.

2. Four new adjacency based graph products

Let G be a connected graph, then the four subdivison graphs $S(G)$,$R(G)$, $Q(G)$ and $T(G)$ associated with G are [17]

- Subdivision graph $S(G)$ of a graph G is obtained from G by replacing each of its edges by a path of length 2, or equivalently, by inserting an additional vertex into each edge of G.
- $R(G)$ of a graph G is obtained from G by inserting an additional vertex into each edge of G and keeping every edge of G.
- $Q(G)$ of a graph G is obtained from G by inserting an additional vertex into each edge of G, then joining every pair of new vertices whose corresponding edges are adjacent in G.
- Total graph $T(G)$ of a graph G is obtained from G by inserting an additional vertex into each edge of G and keeping every edge of G and joining every pair of new vertices whose corresponding edges are adjacent in G.

For example, the subdvision graphs of path P_5 is plotted in Figure 1.

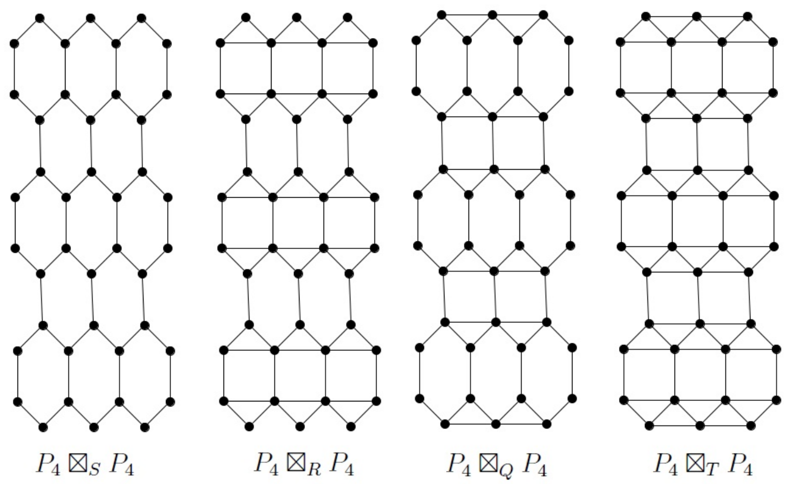

FIGURE 2. The Adjacency Products of two P_4's.

For convenience, in $F(G)$ where $F = \{S, R, Q, T\}$ we call newly introduced vertices as white vertices and the vertices of G as black vertices. Let $V_1 = \{u_1, u_2, \cdots, u_n\}$, $V_2 = \{v_1, v_2, \cdots, v_m\}$ denotes the vertex sets and $E_1 = \{e_1, e_2, \cdots, e_t\}$, $E_2 = \{f_1, f_2, \cdots, f_s\}$ denotes the edge sets of the graphs G_1 and G_2 respectively. Associated with each edge $f_j = v_p v_q \in G_2$, define a set $V_{f_j} = \{v_{pj}, v_{qj}\}$ of vertices and their union be $A(G_2) = \cup_{j=1}^{s} V_{f_j}$. Every vertex of $A(G_2)$ is of the form v_{ij} where the vertex v_{ij} denote the copy of the vertex v_i in G_2 corresponding to the edge f_j.

Definition 2.1. Let F be one among the four symbols S, R, Q, T, we define the adjacency product or A- product of G_1 and G_2 denoted by $G_1 \boxtimes_F G_2$ is the graph with vertex set $V(G_1 \boxtimes_F G_2) = (V_1 \cup E_1) \times (A(G_2))$ and edge set $E(G_1 \boxtimes_F G_2)$ consist of edges $(u_i, v_{jk})(u_p, v_{qr})$ if and only if either $u_i u_p \in E(F(G))$ and $v_{jk} = v_{qr}$ or $u_i = u_p$ with $v_j v_q \in E_2$ and $k = r$ or $u_i = u_p$ with $f_k f_r \in L(G_2)$ and $j = q$ where $f_k, f_r \in E_2$.

In other words, corresponding to each vertex $v \in G_2$ we take $d(v)$ copies of $F(G_1)$ and pairwise join the corresponding black vertices of copies $F(G_1)$ whenever the corresponding vertices are adjacent in G_2 and the corresponding white vertices of each pair will be adjacent to another pair whenever the corresponding edges are adjacent in G_2. Throughout this paper we consider generalized Zagreb index as $M_\alpha(G) = \sum_{u \in V(G)} d(u)^\alpha$,

$\alpha \geq 3$ is a natural number. When $\alpha = 3$ it is called Forgotten index [18]. Figure 2 is an example of A - product with $G_1, G_2 = P_4$. Based on these subdivisions we have the following preliminary result from [14].

Lemma 2.2. [14] *Let G_1 be a simple connected graph with vertex set V_1 and edges set E_1 and $F(G_1)$ be the subdivison graph of G_1 with $F = Q$ or T. If $u, x \in E_1$ with $u = u_iu_j$ and $x = u_ju_k$ where $u_i, u_j, u_k \in V_1$. Then*
(a).

$$\sum_{ux \in E(F(G_1)), u,x \in E_1} (d_{F(G_1)}(u) + d_{F(G_1)}(x)) = M_3(G_1) + 2M_2(G_1) - 2M_1(G_1)$$

(b.)

$$\sum_{ux \in E(F(G_1)), u,x \in E_1} d_{F(G_1)}(u) d_{F(G_1)}(x) = \frac{1}{2} M_4(G_1) - \frac{1}{2} M_3(G_1)$$

$$+ \sum_{u_i, u_j \in V_1} r_{ij} d_{G_1}(u_i) d_{G_1}(u_j) + \sum_{u_j \in V_1} d_{G_1}(u_j)^2 \sum_{u_i \in V_1, u_iu_j \in E_1} d_{G_1}(u_i) - 2M_2(G_1)$$

r_{ij} denotes the number of neighbouring vertices common to both u_i, u_j.

Fullerene is an all carbon skeleton of a molecule in which the atoms are arranged by means of pentagons and hexagons. Michel Deza [15] extended this fullerene structure onto other closed surfaces such as sphere, torus, Klein bottle and projective plane. Let L be a regular hexagonal lattice and P_n^m be an mn quadrilateral section cut from the regular hexagonal lattice. When $n = 1$, the structure is known as a linear hexagonal chain. When $n \geq 2$, if we identify the two lateral end sections of the hexagonal lattice and then identify the top and bottom sides of the lattice P_n^m, the resulting structure is known as toroidal fullerene with mn hexagons [10].

3. Main Results

In this section we obtain the expression for the first and second Zagreb indices of the adjacency product or A - product in terms of the constituent graphs.

Theorem 3.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple connected graphs. Then*

a. $M_1(G_1 \boxtimes_S G_2) = 2|E_2|(M_1(G_1) + 2|E_1|M_1(G_2) + |E_1|M_3(G_2) + 10|E_1||E_2| + 2|E_2||V_1|$

b. $M_2(G_1 \boxtimes_S G_2) = \frac{1}{2}|E_1|(M_4(G_2) + M_3(G_2)) + M_1(G_1)M_1(G_2) + 3|E_2|M_1(G_1) + \frac{3}{2}|E_1|M_1(G_2) + 7|E_1||E_2| + |E_2||V_1|$

Proof. From the definition of first Zagreb index, we have

$$\begin{aligned}
M_1(G_1 \boxtimes_S G_2) &= \sum_{(u,v)\in V(G_1\boxtimes_S G_2)} d^2_{(G_1\boxtimes_S G_2)}(u,v) \\
&= \sum_{(u_i,v_{jk})(u_p,v_{qr})\in E(G_1\boxtimes_S G_2)} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_S G_2)}(u_p,v_{qr})\big) \\
&= \sum_{u_i\in V_1}\sum_{v_jv_q\in E_2,k=r} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_j) + d_{(G_1\boxtimes_S G_2)}(u_i,v_q)\big) \\
&+ \sum_{u_i\in E_1}\sum_{f_kf_r\in E(L(G_2)),j=q} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_S G_2)}(u_i,v_{qr})\big) \\
&+ \sum_{v_{jk}\in V_2}\sum_{u_iu_p\in E(S(G_1))} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_S G_2)}(u_p,v_{jk})\big) \\
&= A_1 + B_1 + C_1
\end{aligned}$$

Now we separately find the values of each parts of the sum.

$$\begin{aligned}
A_1 &= \sum_{u_i\in V_1}\sum_{v_jv_q\in E_2,k=r} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_j) + d_{(G_1\boxtimes_S G_2)}(u_i,v_q)\big) \\
&= \sum_{u_i\in V_1}\sum_{v_jv_q\in E_2,k=r} [d_{G_1}(u_i) + 1 + d_{G_1}(u_i) + 1] \\
&= 2|E_2|(2|E_1| + |V_1|)
\end{aligned}$$

The second part is,

$$\begin{aligned}
B_1 &= \sum_{u_i\in E_1}\sum_{f_kf_r\in E(L(G_2)),j=q} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_S G_2)}(u_i,v_{qr})\big) \\
&= \sum_{u_i\in E_1}\sum_{f_kf_r\in E(L(G_2)),j=q} [d_{G_2}(v_j) + 1 + d_{G_2}(v_j) + 1] \\
&= \sum_{u_i\in E_1}\sum_{v_j\in V_2,d(v_j)\neq 1} d(v_j)(d(v_j)^2 - 1) \\
&= |E_1|(M_3(G_2) - 2|E_2|)
\end{aligned}$$

The third part is,

$$
\begin{aligned}
C_1 &= \sum_{v_{jk}\in V_2} \sum_{u_iu_p\in E(S(G_1))} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_S G_2)}(u_p,v_{jk})\big)\\
&= \sum_{v_{jk}\in V_2} \sum_{u_iu_p\in E(S(G_1))} (d_{G_1}(u_i) + 1 + d_{G_2}(v_j) + 1)\\
&= \sum_{v_{jk}\in V_2} (M_1(G_1) + 4|E_1| + 2|E_1|d_{G_2}(v_j))\\
&= 2|E_2|(M_1(G_1) + 2|E_1|M_1(G_2) + 8|E_1||E_2|
\end{aligned}
$$

From the expressions, we obtain

$$
\begin{aligned}
M_1(G_1\boxtimes_S G_2) = {}& 2|E_2|(M_1(G_1) + 2|E_1|M_1(G_2) + |E_1|M_3(G_2)\\
&+ 10|E_1||E_2| + 2|E_2||V_1|
\end{aligned}
$$

Next Consider

$$
\begin{aligned}
M_2(G_1\boxtimes_S G_2) = {}& \sum_{(u_i,v_{jk})(u_p,v_{qr})\in E(G_1\boxtimes_S G_2)} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_S G_2)}(u_p,v_{qr})\big)\\
&= \sum_{u_i\in V_1} \sum_{\substack{v_jv_q\in E_2\\ k=r}} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_j)d_{(G_1\boxtimes_S G_2)}(u_i,v_q)\big)\\
&+ \sum_{u_i\in E_1} \sum_{\substack{f_kf_r\in E(L(G_2))\\ j=q}} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_S G_2)}(u_i,v_{qr})\big)\\
&+ \sum_{v_{jk}\in V_2} \sum_{u_iu_p\in E(S(G_1))} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_S G_2)}(u_p,v_{jk})\big)\\
&= A_2 + B_2 + C_2
\end{aligned}
$$

First part of the sum is

$$
\begin{aligned}
A_2 &= \sum_{u_i\in V_1} \sum_{v_jv_q\in E_2, k=r} \big(d_{(G_1\boxtimes_S G_2)}(u_i,v_j)d_{(G_1\boxtimes_S G_2)}(u_i,v_q)\big)\\
&= \sum_{u_i\in V_1} \sum_{v_jv_q\in E_2, k=r} (d_{G_1}(u_i) + 1)(d_{G_1}(u_i) + 1)\\
&= \sum_{u_i\in V_1} \sum_{v_jv_q\in E_2, k=r} (d_{G_1}(u_i)^2 + 2(d_{G_1}(u_i)) + 1)\\
&= |E_2|(M_1(G_1) + 4|E_1| + |V_1|)
\end{aligned}
$$

The second part is,

$$\begin{aligned}
B_2 &= \sum_{u_i \in E_1} \sum_{f_k f_r \in E(L(G_2)), j=q} \big(d_{(G_1 \boxtimes_S G_2)}(u_i, v_{jk}) d_{(G_1 \boxtimes_S G_2)}(u_i, v_{qr})\big) \\
&= \sum_{u_i \in E_1} \sum_{f_k f_r \in E(L(G_2)), j=q} (d_{G_2}(v_j)+1)(d_{G_2}(v_j)+1) \\
&= \frac{1}{2} \sum_{u_i \in E_1} \sum_{v_j \in V_2, d(v_j) \neq 1} d(v_j)(d(v_j)^2 - 1)(d(v_j)+1) \\
&= \frac{1}{2} \sum_{u_i \in E_1} \sum_{v_j \in V_2} (d(v_j)^4 + d(v_j)^3 - d(v_j)^2 - d(v_j)) \\
&= \frac{1}{2}|E_1|(M_4(G_2) + M_3(G_2) - M_1(G_2) - 2|E_2|)
\end{aligned}$$

The third part is,

$$\begin{aligned}
C_2 &= \sum_{v_{jk} \in V_2} \sum_{u_i u_p \in E(S(G_1))} \big(d_{(G_1 \boxtimes_S G_2)}(u_i, v_{jk}) d_{(G_1 \boxtimes_S G_2)}(u_p, v_{jk})\big) \\
&= \sum_{v_{jk} \in V_2} \sum_{u_i u_p \in E(S(G_1))} (d_{G_1}(u_i)+1)(d_{G_2}(v_j)+1) \\
&= \sum_{v_{jk} \in V_2} \sum_{u_i u_p \in E(S(G_1))} (d_{G_1}(u_i) d_{G_2}(v_j) + d_{G_1}(u_i) + d_{G_2}(v_j) + 1) \\
&= M_1(G_1)M_1(G_2) + 2|E_2|M_1(G_1) + 2|E_1|M_1(G_2) + 4|E_1||E_2|
\end{aligned}$$

Thus we obtain, $M_2(G_1 \boxtimes_S G_2) = A_2 + B_2 + C_2$,

$$\begin{aligned}
M_2(G_1 \boxtimes_S G_2) = {} & \frac{1}{2}|E_1|(M_4(G_2) + M_3(G_2)) + M_1(G_1)M_1(G_2) \\
& + 3|E_2|M_1(G_1) + \frac{3}{2}|E_1|M_1(G_2) + 7|E_1||E_2| + |E_2||V_1|
\end{aligned}$$

□

Theorem 3.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple connected graphs. Then*

a. $M_1(G_1 \boxtimes_R G_2) = |E_1|M_3(G_2) + 8|E_2|M_1(G_1) + 2|E_1|M_1(G_2) + 18|E_1||E_2| + 2|E_2||V_1|$

b. $M_2(G_1 \boxtimes_R G_2) = \frac{1}{2}|E_1|(M_4(G_2) + M_3(G_2)) + 2M_1(G_1)M_1(G_2) + 12|E_2|M_1(G_1) + \frac{3}{2}|E_1|M_1(G_2) + 8|E_2|M_2(G_1) + 13|E_1||E_2| + |E_2||V_1|$

Proof. We have

$$\begin{aligned}
M_1(G_1 \boxtimes_R G_2) &= \sum_{(u,v)\in V(G_1\boxtimes_R G_2)} d^2_{(G_1\boxtimes_R G_2)}(u,v)\\
&= \sum_{(u_i,v_{jk})(u_p,v_{qr})\in E(G_1\boxtimes_R G_2)} \big(d_{(G_1\boxtimes_R G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_R G_2)}(u_p,v_{qr})\big)\\
&= \sum_{u_i\in V_1}\sum_{v_jv_q\in E_2,k=r} \big(d_{(G_1\boxtimes_R G_2)}(u_i,v_j) + d_{(G_1\boxtimes_R G_2)}(u_i,v_q)\big)\\
&+ \sum_{u_i\in E_1}\sum_{f_kf_r\in E(L(G_2)),j=q} \big(d_{(G_1\boxtimes_R G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_R G_2)}(u_i,v_{qr})\big)\\
&+ \sum_{v_{jk}\in V_2}\sum_{u_iu_p\in E(R(G_1))} \big(d_{(G_1\boxtimes_R G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_R G_2)}(u_p,v_{jk})\big)\\
&= A_1 + B_1 + C_1
\end{aligned}$$

Now we separately find the values of each parts of the sum

$$\begin{aligned}
A_1 &= \sum_{u_i\in V_1}\sum_{v_jv_q\in E_2,k=r} \big(d_{(G_1\boxtimes_R G_2)}(u_i,v_j) + d_{(G_1\boxtimes_R G_2)}(u_i,v_q)\big)\\
&= \sum_{u_i\in V_1}\sum_{v_jv_q\in E_2,k=r} [2d_{G_1}(u_i) + 1 + 2d_{G_1}(u_i) + 1]\\
&= 2|E_2|(4|E_1| + |V_1|)
\end{aligned}$$

The second part of the sum is same as the second part of $M_1(G_1 \boxtimes_S G_2)$.
That is $B_1 = |E_1|(M_3(G_2) - 2|E_2|)$.
The third part is,

$$\begin{aligned}
C_1 &= \sum_{v_{jk}\in V_2}\sum_{u_iu_p\in E(R(G_1))} \big(d_{(G_1\boxtimes_R G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_R G_2)}(u_p,v_{jk})\big)\\
&= \sum_{v_{jk}\in V_2}\sum_{u_iu_p\in E(R(G_1)),u_p\in E_1} (2d_{G_1}(u_i) + 1 + d_{G_2}(v_j) + 1)\\
&+ \sum_{v_{jk}\in V_2}\sum_{\substack{u_iu_p\in E(R(G_1))\\ u_i,u_p\in V_1}} (2d_{G_1}(u_i) + 1 + 2d_{G_1}(u_p) + 1)\\
&= \sum_{v_{jk}\in V_2} (2M_1(G_1) + 4|E_1| + 2|E_1|d_{G_2}(v_j)) + \sum_{v_{jk}\in V_2} 2M_1(G_1) + 2|E_1|\\
&= 8|E_2|M_1(G_1) + 2|E_1|M_1(G_2) + 12|E_1||E_2|
\end{aligned}$$

Thus,

$$\begin{aligned}M_1(G_1 \boxtimes_R G_2) &= |E_1|M_3(G_2) + 8|E_2|M_1(G_1) + 2|E_1|M_1(G_2) \\ &\quad + 18|E_1||E_2| + 2|E_2||V_1|\end{aligned}$$

Similarly,

$$\begin{aligned}M_2(G_1 \boxtimes_R G_2) &= \sum_{(u_i,v_{jk})(u_p,v_{qr})\in E(G_1\boxtimes_R G_2)} \big(d_{(G_1\boxtimes_R G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_R G_2)}(u_p,v_{qr})\big) \\ &= \sum_{u_i\in V_1}\sum_{\substack{v_jv_q\in E_2\\ k=r}} \big(d_{(G_1\boxtimes_R G_2)}(u_i,v_j)d_{(G_1\boxtimes_R G_2)}(u_i,v_q)\big) \\ &+ \sum_{u_i\in E_1}\sum_{\substack{f_kf_r\in E(L(G_2))\\ j=q}} \big(d_{(G_1\boxtimes_R G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_R G_2)}(u_i,v_{qr})\big) \\ &+ \sum_{v_{jk}\in V_2}\sum_{u_iu_p\in E(R(G_1))} \big(d_{(G_1\boxtimes_R G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_R G_2)}(u_p,v_{jk})\big) \\ &= A_2 + B_2 + C_2\end{aligned}$$

First part of the sum is

$$\begin{aligned}A_2 &= \sum_{u_i\in V_1}\sum_{\substack{v_jv_q\in E_2\\ k=r}} \big(d_{(G_1\boxtimes_R G_2)}(u_i,v_j)d_{(G_1\boxtimes_R G_2)}(u_i,v_q)\big) \\ &= \sum_{u_i\in V_1}\sum_{v_jv_q\in E_2,k=r} (2d_{G_1}(u_i)+1)(2d_{G_1}(u_i)+1) \\ &= \sum_{u_i\in V_1}\sum_{v_jv_q\in E_2,k=r} (4d_{G_1}(u_i)^2 + 4(d_{G_1}(u_i)) + 1) \\ &= |E_2|(4M_1(G_1) + 8|E_1| + |V_1|)\end{aligned}$$

The second part of the sum in both cases are same. i.e,

$$B_2 = \frac{1}{2}|E_1|(M_4(G_2) + M_3(G_2) - M_1(G_2) - 2|E_2|)$$

The third part is,

$$C_2 = \sum_{v_{jk}\in V_2}\sum_{u_iu_p\in E(R(G_1))}\left(d_{(G_1\boxtimes_R G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_R G_2)}(u_p,v_{jk})\right)$$

$$= \sum_{v_{jk}\in V_2}\sum_{\substack{u_iu_p\in E(R(G_1))\\ u_p\in E_1}}(2d_{G_1}(u_i)+1)(d_{G_2}(v_j)+1)$$

$$+ \sum_{v_{jk}\in V_2}\sum_{\substack{u_iu_p\in E(R(G_1))\\ u_i,u_p\in V_1}}(2d_{G_1}(u_i)+1)(2d_{G_2}(u_p)+1)$$

$$= \sum_{v_{jk}\in V_2}\sum_{u_iu_p\in E(R(G_1))}(2d_{G_1}(u_i)d_{G_2}(v_j)+2d_{G_1}(u_i)+d_{G_2}(v_j)+1)$$

$$+ \sum_{v_{jk}\in V_2}\sum_{\substack{u_iu_p\in E(R(G_1))\\ u_i,u_p\in V_1}}(4d_{G_1}(u_i)d_{G_2}(u_p)+2(d_{G_1}(u_i)+d_{G_2}(u_p))+1)$$

$$=2M_1(G_1)M_1(G_2)+4|E_2|M_1(G_1)+2|E_1|M_1(G_2)$$

$$+4|E_1||E_2|+8|E_2|M_2(G_1)+4|E_2|M_1(G_1)+2|E_1||E_2|$$

Thus we obtain, $M_2(G_1\boxtimes_R G_2) = A_2 + B_2 + C_2$,

$$M_2(G_1\boxtimes_R G_2) = \frac{1}{2}|E_1|(M_4(G_2)+M_3(G_2))+2M_1(G_1)M_1(G_2)$$

$$+12|E_2|M_1(G_1)+\frac{3}{2}|E_1|M_1(G_2)$$

$$+8|E_2|M_2(G_1)+13|E_1||E_2|+|E_2||V_1|$$

□

Theorem 3.3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple connected graphs, then*

a. $M_1(G_1\boxtimes_Q G_2) = 2|E_2|M_3(G_1)+|E_1|M_3(G_2)+2M_1(G_1)M_1(G_2)+4|E_2|M_2(G_1)-2|E_2|M_1(G_1)-2|E_1|M_1(G_2)+10|E_1||E_2|+2|E_2||V_1|$

b. $M_2(G_1\boxtimes_Q G_2) = |E_2|M_4(G_1)+\frac{1}{2}|E_1|(M_4(G_2)+\frac{3}{2}M_3(G_1)M_1(G_2)+\frac{3}{2}M_3(G_2)M_1(G_1)-4|E_2|M_3(G_1)-\frac{5}{2}|E_1|M_3(G_2))+3M_1(G_2)M_2(G_1)-5M_1(G_1)M_1(G_2)+8|E_2|M_1(G_1)-10|E_2|M_2(G_1)+\frac{7}{2}|E_1|M_1(G_2)-|E_1||E_2|+|E_2||V_1|+2|E_2|\left(\sum_{x,y\in V_1}r_{xy}d_{G_1}(x)d_{G_1}(y)+\sum_{y\in V_1}d_{G_1}(y)^2\sum_{x\in V_1,xy\in E_1}d_{G_1}(x)\right)$

where r_{xy} denote the number of common neighbours to the vertices x and y in G_1

Proof. Using the definition,

$$\begin{aligned}
M_1(G_1 \boxtimes_Q G_2) &= \sum_{(u,v)\in V(G_1\boxtimes_Q G_2)} d^2_{(G_1\boxtimes_Q G_2)}(u,v) \\
&= \sum_{(u_i,v_{jk})(u_p,v_{qr})\in E(G_1\boxtimes_Q G_2)} \Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_Q G_2)}(u_p,v_{qr})\Big) \\
&= \sum_{u_i\in V_1}\sum_{v_jv_q\in E_2,k=r} \Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_j) + d_{(G_1\boxtimes_Q G_2)}(u_i,v_q)\Big) \\
&+ \sum_{u_i\in E_1}\sum_{f_kf_r\in E(L(G_2)),j=q} \Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_Q G_2)}(u_i,v_{qr})\Big) \\
&+ \sum_{v_{jk}\in V_2}\sum_{u_iu_p\in E(Q(G_1))} \Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_Q G_2)}(u_p,v_{jk})\Big) \\
&= A_1 + B_1 + C_1
\end{aligned}$$

Now we separately find the values of the each parts of the sum. The first part of the sum is same as in Theorem 3.1, so $A_1 = 2|E_2|(2|E_1| + |V_1|)$. The second part is,

$$\begin{aligned}
B_1 &= \sum_{u_i\in E_1}\sum_{f_kf_r\in E(L(G_2)),j=q} \Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_{jk}) + d_{(G_1\boxtimes_Q G_2)}(u_i,v_{qr})\Big) \\
&= 2\sum_{u_i\in E_1}\sum_{f_kf_r\in E(L(G_2)),j=q} \Big[d_{Q(G_1)}(u_i) + d_{G_2}(v_j) - 1\Big] \\
&= \sum_{u_i\in E_1}\sum_{v_j\in E_2,d(v_j)\neq 1} d(v_j)(d(v_j)-1)(d_{Q(G_1)}(u_i)) \\
&+ \sum_{u_i\in E_1}\sum_{v_j\in E_2,d(v_j)\neq 1} d(v_j)(d(v_j)-1)^2 \\
&= \sum_{u_i\in E_1}(M_1(G_2) - 2|E_2|)(d_{Q(G_1)}(u_i)) + \sum_{u_i\in E_1}(M_3(G_2) - 2M_1(G_2) + 2|E_2|)
\end{aligned}$$

Also, $d_{Q(G_1)}(u_i) = d_{G_1}(x) + d_{G_1}(y)$ where u_i is the vertex inserted corresponding to the edge $xy \in E_1$. So the sum becomes

$$B_1 = M_1(G_1)(M_1(G_2) - 2|E_2|) + |E_1|(M_3(G_2) - 2M_1(G_2) + 2|E_2|)$$

The third part of the sum is,

$$
\begin{aligned}
C_1 = &\sum_{v_{jk}\in V_2}\sum_{u_iu_p\in E(Q(G_1))}\Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_{jk})+d_{(G_1\boxtimes_Q G_2)}(u_p,v_{jk})\Big)\\
=&\sum_{v_{jk}\in V_2}\sum_{u_iu_p\in E(Q(G_1)),u_p\in E_1}(d_{G_1}(u_i)+1+d_{Q(G_1)}(u_p)+d_{G_2}(v_j)-1)\\
&+\sum_{v_{jk}\in V_2}\sum_{\substack{u_iu_p\in E(Q(G_1))\\ u_i,u_p\in E_1}}(d_{Q(G_1)}(u_i)+d_{G_2}(v_j)-1+d_{Q(G_1)}(u_p)+d_{G_2}(v_j)-1)\\
&=C_{11}+C_{12}
\end{aligned}
$$

Also if u_i is the vertex corresponding to edge xy and u_p is the vertex corresponding to the edge yz, then

$$
\sum_{u_iu_p\in E(Q(G_1)),u_i\in V_1,u_p\in E_1} d_{Q(G_1)}(u_p) = 2\sum_{yz\in E_1}(d(y)+d(z)) = 2M_1(G_1)
$$

By using Lemma 2.2,

$$
\begin{aligned}
C_{11} =&\sum_{v_{jk}\in V_2}(M_1(G_1)+2M_1(G_1)+2|E_1|d_{G_2}(v_j))\\
=&6|E_2|M_1(G_1)+2|E_1|M_1(G_2)\\
C_{12} =&\sum_{v_{jk}\in V_2}\sum_{\substack{u_iu_p\in E(Q(G_1))\\ u_i,u_p\in E_1}}(d_{Q(G_1)}(u_i)+d_{G_2}(v_j)-1+d_{Q(G_1)}(u_p)+d_{G_2}(v_j)-1)\\
=&2|E_2|(M_3(G_1)+2M_2(G_1)-2M_1(G_1))\\
&+\sum_{v_{jk}\in V_2}(M_1(G_1)-2|E_1|)(d_{G_2}(v_j)-1)\\
=&2|E_2|(M_3(G_1)+2M_2(G_1)-2M_1(G_1))+M_1(G_1)M_1(G_2)\\
&-2|E_1|M_1(G_2)-2|E_2|M_1(G_2)+4|E_1||E_2|
\end{aligned}
$$

Thus we obtain,

$$
\begin{aligned}
M_1(G_1\boxtimes_Q G_2) =&2|E_2|M_3(G_1)+|E_1|M_3(G_2)+2M_1(G_1)M_1(G_2)\\
&+4|E_2|M_2(G_1)-2|E_2|M_1(G_1)-2|E_1|M_1(G_2)\\
&+10|E_1||E_2|+2|E_2||V_1|
\end{aligned}
$$

Next Consider

$$\begin{aligned}
M_2(G_1 \boxtimes_Q G_2) &= \sum_{(u_i,v_{jk})(u_p,v_{qr})\in E(G_1\boxtimes_Q G_2)} \Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_Q G_2)}(u_p,v_{qr})\Big)\\
&= \sum_{u_i\in V_1}\sum_{v_jv_q\in E_2,k=r}\Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_j)d_{(G_1\boxtimes_Q G_2)}(u_i,v_q)\Big)\\
&+ \sum_{u_i\in E_1}\sum_{f_kf_r\in E(L(G_2)),j=q}\Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_Q G_2)}(u_i,v_{qr})\Big)\\
&+ \sum_{v_{jk}\in V_2}\sum_{u_iu_p\in E(Q(G_1))}\Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_Q G_2)}(u_p,v_{jk})\Big)\\
&= A_2 + B_2 + C_2
\end{aligned}$$

First part of the sum is the same as in the proof of Theorem 3.1, $A_2 = |E_2|(M_1(G_1) + 4|E_1| + |V_1|)$.
The second part is,

$$\begin{aligned}
B_2 &= \sum_{u_i\in E_1}\sum_{f_kf_r\in E(L(G_2)),j=q}\Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_Q G_2)}(u_i,v_{qr})\Big)\\
&= \sum_{u_i\in E_1}\sum_{f_kf_r\in E(L(G_2)),j=q}\big(d_{Q(G_1)}(u_i) + d_{G_2}(v_j) - 1\big)^2\\
&= \frac{1}{2}\sum_{u_i\in E_1}\sum_{\substack{v_j\in V_2\\ d(v_j)\neq 1}} d(v_j)(d(v_j)-1)\big(d_{Q(G_1)}(u_i)^2 + 2d_{Q(G_1)}(u_i)(d(v_j)-1) + (d(v_j)-1)^2\big)\\
&= \frac{1}{2}\sum_{u_i\in E_1}\big(M_1(G_2) - 2|E_2|\big)d_{Q(G_1)}(u_i)^2 + \sum_{u_i\in E_1}\big(M_3(G_2) - 2M_1(G_2) + 2|E_2|\big)d_{Q(G_1)}(u_i)\\
&+ \frac{1}{2}\sum_{u_i\in E_1}\sum_{v_j\in V_2,d(v_j)\neq 1}\big(d(v_j)^4 - 3d(v_j)^3 + 3d(v_j)^2 - d(v_j)\big)\\
&= \frac{1}{2}\big(M_1(G_2) - 2|E_2|\big)\big(M_3(G_1) + 2M_2(G_1)\big) + \big(M_3(G_2) - 2M_1(G_2) + 2|E_2|\big)M_1(G_1)\\
&+ \frac{|E_1|}{2}\big(M_4(G_2) - 3M_3(G_2) + 3M_1(G_2) - 2|E_2|\big)
\end{aligned}$$

The third part is,

$$\begin{aligned}
C_2 =& \sum_{v_{jk}\in V_2} \sum_{\substack{u_iu_p\in E(Q(G_1))\\ u_i,u_p\in E_1}} \Big(d_{(G_1\boxtimes_Q G_2)}(u_i,v_{jk})d_{(G_1\boxtimes_Q G_2)}(u_p,v_{jk})\Big)\\
=& \sum_{v_{jk}\in V_2} \sum_{\substack{u_iu_p\in E(Q(G_1))\\ u_i,u_p\in E_1}} (d_{Q(G_1)}(u_i)+d_{G_2}(v_j)-1)(d_{Q(G_1)}(u_p)+d_{G_2}(v_j)-1)\\
=& \sum_{v_{jk}\in V_2} \sum_{\substack{u_iu_p\in E(Q(G_1))\\ u_i,u_p\in E_1}} d_{Q(G_1)}(u_i)d_{Q(G_1)}(u_p)\\
&+ \sum_{v_{jk}\in V_2} \sum_{\substack{u_iu_p\in E(Q(G_1))\\ u_i,u_p\in E_1}} \big(d_{Q(G_1)}(u_i)+d_{Q(G_1)}(u_p)\big)(d_{G_2}(v_j)-1)\\
&+ \sum_{v_{jk}\in V_2} \sum_{\substack{u_iu_p\in E(Q(G_1))\\ u_i,u_p\in E_1}} (d_{G_2}(v_j)-1)^2
\end{aligned}$$

$$\begin{aligned}
=&2|E_2|\left(\frac{1}{2}M_4(G_1)-\frac{1}{2}M_3(G_1)+\sum_{x,y\in V_1} r_{xy}d_{G_1}(x)d_{G_1}(y)\right)\\
&+2|E_2|\left(\sum_{y\in V_1} d_{G_1}(y)^2 \sum_{x\in V_1,xy\in E_1} d_{G_1}(x)-2M_2(G_1)\right)\\
&+M_1(G_2)\left(M_3(G_1)+2M_2(G_1)-2M_1(G_1)\right)-2|E_2|\left(M_3(G_1)+2M_2(G_1)-2M_1(G_1)\right)\\
&+\frac{1}{2}(M_1(G_1)-2|E_1|)\left(M_3(G_2)-2M_1(G_2)+2|E_2|\right)
\end{aligned}$$

Thus we obtain,

$$\begin{aligned}
M_2(G_1\boxtimes_Q G_2) =& |E_2|M_4(G_1)+\frac{1}{2}|E_1|(M_4(G_2)+\frac{3}{2}M_3(G_1)M_1(G_2)\\
&+\frac{3}{2}M_3(G_2)M_1(G_1)-4|E_2|M_3(G_1)-\frac{5}{2}|E_1|M_3(G_2))\\
&+3M_1(G_2)M_2(G_1)-5M_1(G_1)M_1(G_2)+8|E_2|M_1(G_1)\\
&-10|E_2|M_2(G_1)+\frac{7}{2}|E_1|M_1(G_2)-|E_1||E_2|+|E_2||V_1|\\
&+2|E_2|\left(\sum_{x,y\in V_1} r_{xy}d_{G_1}(x)d_{G_1}(y)+\sum_{y\in V_1} d_{G_1}(y)^2 \sum_{x\in V_1,xy\in E_1} d_{G_1}(x)\right)
\end{aligned}$$

where r_{xy} denote the number of common neighbours to the vertices x and y in G_1 □

Using Theorem 3.2, 3.3 along with the fact $d_{(G_1 \boxtimes_T G_2)}(x, y) = d_{(G_1 \boxtimes_Q G_2)}(x, y)$ for $x \in V_1$ and $y \in E_1$ or $x, y \in E_1$ and $d_{(G_1 \boxtimes_T G_2)}(x, y) = d_{(G_1 \boxtimes_R G_2)}(x, y)$ whenever $x, y \in V_1$. We can state the following theorem.

Theorem 3.4. *Let G_1 and G_2 be two simple connected graphs, then*

a. $M_1(G_1 \boxtimes_T G_2) = 2|E_2|M_3(G_1) + |E_1|M_3(G_2) + 2M_1(G_1)M_1(G_2) + 4|E_2|M_2(G_1) + 2|E_2|M_1(G_1) - 2|E_1|M_1(G_2) + 14|E_1||E_2| + 2|E_2||V_1|$

b. $M_2(G_1 \boxtimes_T G_2) = |E_2|M_4(G_1) + \frac{1}{2}|E_1|(M_4(G_2) + \frac{3}{2}M_3(G_1)M_1(G_2) + \frac{3}{2}M_3(G_2)M_1(G_1) - 4|E_2|M_3(G_1) - \frac{5}{2}|E_1|M_3(G_2)) + 3M_1(G_2)M_2(G_1) - 5M_1(G_1)M_1(G_2) + 12|E_2|M_1(G_1) - 2|E_2|M_2(G_1) + \frac{7}{2}|E_1|M_1(G_2) + |E_1||E_2| + |E_2||V_1| + 2|E_2|\left(\sum_{x,y \in V_1} r_{xy} d_{G_1}(x) d_{G_1}(y) + \sum_{y \in V_1} d_{G_1}(y)^2 \sum_{x \in V_1, xy \in E_1} d_{G_1}(x)\right)$

where r_{xy} denote the number of common neighbours to the vertices x and y in G_1

Corollary 3.5. *Let $n > 3$ be any postive integer, the Zagreb indices of linear hexagonal chain L_n*

a. $M_1(L_n) = 26n - 2$
b. $M_2(L_n) = 33n - 9$

Proof. Use the fact that $L_n = P_{n+1} \boxtimes_S P_2$ and by Theorem 3.1. □

Let P_n^m denote the hexagonal lattice, then using the results we can obtain

Corollary 3.6. *Let $n > 3, m > 2$ be postive integers, the Zagreb indices of linear hexagonal lattice P_{n+1}^{2m-1}*

a. $M_1(P_{n+1}^{2m-1}) = 36mn - 10n - 38m + 10$
b. $M_2(P_{n+1}^{2m-1}) = 54mn - 21n - 67m + 25$

Proof. Use the fact that $P_{n+1}^{2m-1} = P_n \boxtimes_S P_m$ and by Theorem 3.1. □

Using these results we can compute the Zagreb indices of certain fullerene nanotubes $N\mathbb{A}_m^{2n}$,$N\mathbb{C}_{2m}^{2n}(\mathbb{H}_{2m}^{2n})$ [9, 10].

Corollary 3.7. *Let $n, m > 3$ be postive integers, the Zagreb indices of the fullerene nanotubes $N\mathbb{A}_m^{2n}$ are*

a. $M_1(N\mathbb{A}_m^{2n}) = 36mn - 46m$
b. $M_2(N\mathbb{A}_m^{2n}) = 54mn - 75m$

Proof. Use $N\mathbb{A}_m^{2n} = C_m \boxtimes_S P_n$ and use Theorem 3.1. □

Example 3.8. When $G_1 = C_m$, $G_2 = C_n$, $n, m > 3$, using the Theorem 3.1

a. $M_1(C_m \boxtimes_S C_n) = 36mn$
b. $M_2(C_m \boxtimes_S C_n) = 54mn$
c. $M_1(C_m \boxtimes_R C_n) = 68mn$
d. $M_2(C_m \boxtimes_R C_n) = 144mn$
e. $M_1(C_m \boxtimes_Q C_n) = 68mn$
f. $M_2(C_m \boxtimes_Q C_n) = 144mn$
g. $M_1(C_m \boxtimes_T C_n) = 100mn$
h. $M_2(C_m \boxtimes_T C_n) = 250mn$

Also $N\mathbb{C}_{2m}^{2n} = C_m \boxtimes_S C_n$ which is computed in Example 3.8.

4. Summary and Conclusion

We have defined the Adjacency product or A - product in terms of the Cartesian product of graphs and computed some degree-based topological indices of A- products. These products generalize the structure of various chemical compounds such as graphene, linear polyacene, toroidal fullerene, and some fullerene nanotubes $N\mathbb{C}_{2m}^{2n}$ $(\mathbb{H}_{2m}^{2n})$,$N\mathbb{A}_m^{2n}$. These sums can be defined in terms of other graph products and computation of various other topological indices on A - products are also a problem for further research.

References

[1] Ayu Ameliatul Shahilah Ahmad Jamri, Roslan Hasni, and Sharifah Kartini Said Husain, *On the Zagreb indices of graphs with given Roman domination number*, Communications in Combinatorics and Optimization, **8(1)** (2023),141152.

[2] Shehnaz Akhter and Muhammad Imran, *Computing the forgotten topological index of four operations on graphs*, AKCE International Journal of Graphs and Combinatorics, **14(1)**,(2017),7079.

[3] Abdu Alameri, Mahmoud Al-Rumaima, and Mohammed Almazah, *Y-coindex of graph operations and its applications of molecular descriptors*, Journal of Molecular Structure, **1221:128754**, (2020).

[4] Liju Alex and Indulal Gopal, *On the Wiener index of FH sums of graphs*, Journal of Computer Science and Applied Mathematics,**3**,(2021),3757.

[5] Liju Alex and Indulal Gopalapillai, *On a conjecture on edge Mostar index of bicyclic graphs*, Iranian Journal of Mathematical Chemistry, **14(2)**,(2023),97108.
[6] Liju Alex and Indulal Gopalapillai, *Zagreb indices of a new sum of graphs*, Ural Mathematical Journal, **9(1)**,(2023),417.
[7] Liju Alex and G Indulal, *Some degree based topological indices of a generalised F sums of graphs*, Electronic Journal of Mathematical Analysis and Applications, **9(1)**,(2021),91111.
[8] HM Awais, M Javaid, and M Jamal, *Forgotten index of generalized F-sum graphs*, Journal of Prime Research in Mathematics, **15**,(2019),115128.
[9] Martin Bača, Jarmila Horváthová, Martina Mokrišová, Andrea Semaničová Feňovčíková, and Alžbeta Suhányiová. *On topological indices of a carbon nanotube network*. Canadian Journal of Chemistry, **93(10)**,(2015),11571160.
[10] Martin Bača, Jarmila Horváthová, Martina Mokrišová, and Alžbeta Suhányiová, *On topological indices of fullerenes*, Applied Mathematics and Computation, **251**,(2015),154161.
[11] B Basavanagoud, Wei Gao, Shreekant Patil, Veena R Desai, Keerthi G Mirajkar, and B Pooja, *Computing first Zagreb index and F-index of new C-products of graphs*, Applied Mathematics and Nonlinear Sciences, **2(1)**,(2017),285298.
[12] Muhammad Bilal, Muhammad Kamran Jamil, Muhammad Waheed, and Abdu Alameri, *Three topological indices of two new variants of graph products*, Mathematical Problems in Engineering, (2021),19.
[13] Roberto Cruz, Ivan Gutman, and Juan Rada, *Sombor index of chemical graphs*, Applied Mathematics and Computation, **399:126018**, (2021).
[14] Hanyuan Deng, D Sarala, SK Ayyaswamy, and Selvaraj Balachandran, *The Zagreb indices of four operations on graphs*, Applied Mathematics and Computation, **275**,(2016),422431.
[15] Michel Deza, Patrick W Fowler, A Rassat, and Kevin M Rogers, *Fullerenes as tilings of surfaces*, Journal of chemical information and computer sciences, **40(3)**,(2000),550558.
[16] Tomislav Došlić, Ivica Martinjak, Riste Skrekovski, Sanja Tipurić Spužević, and Ivana Zubac, *Mostar index*, Journal of Mathematical Chemistry, **56**,(2018),29953013.
[17] Mehdi Eliasi and Bijan Taeri, *Four new sums of graphs and their Wiener indices*, Discrete Applied Mathematics, **157(4)**,(2009),794803.
[18] Boris Furtula and Ivan Gutman, *A forgotten topological index*. Journal of mathematical chemistry, **53(4)**,(2015),11841190.
[19] Ante Graovac and Tomaž Pisanski, *On the Wiener index of a graph*, Journal of mathematical chemistry, **8(1)**,(1991),5362.
[20] Ivan Gutman, *On the origin of two degreebased topological indices*, Bulletin (Academie serbe des sciences et des arts. Classe des sciences mathématiques et naturelles. Sciences mathématiques), **39**,(2014),3952.
[21] Ivan Gutman and Kinkar Ch Das, *The first Zagreb index 30 years after*, MATCH Commun. Math. Comput. Chem, **50(1)**,(2004),8392.
[22] Ivan Gutman and Nenad Trinajstić, *Graph theory and molecular orbitals total ϕ-electron energy of alternant hydrocarbons*, Chemical physics letters, **17(4)**,(1972),535 538.

[23] Batmend Horoldagva and Kinkar Chandra Das, *On Zagreb indices of graphs*, MATCH Commun. Math. Comput. Chem, **85**,(2021),295301.
[24] Gopalapillai Indulal, Liju Alex, and Ivan Gutman, *On graphs preserving PI index upon edge removal*, Journal of Mathematical Chemistry, **59(7)**,(2021),16031609.
[25] MH Khalifeh, Hassan Yousefi-Azari, and Ali Reza Ashrafi, *The first and second Zagreb indices of some graph operations*, Discrete applied mathematics, **157(4)**,(2009),804811.
[26] Sandi Klavžar, Amal Rajapakse, and Ivan Gutman, *The Szeged and the Wiener index of graphs*, Applied Mathematics Letters, **9(5)**,(1996),4549.
[27] Jia-Bao Liu, Muhammad Javaid, and Hafiz Muhammad Awais, *Computing Zagreb indices of the subdivision-related generalized operations of graphs*, IEEE Access, **7**,(2019),105479105488.
[28] Metrose Metsidik, Weijuan Zhang, and Fang Duan, *Hyper-and reverse-Wiener indices of F-sums of graphs*, Discrete Applied Mathematics, **158(13)**,(2010),14331440.
[29] Harry Wiener, *Structural determination of paraffin boiling points*, Journal of the American chemical society, **69(1)**,(1947),1720.

Liju Alex
Department of Mathematics, Bishop Chulaprambil Memorial College(BCM), Kottayam - 686001 , India and Marthoma College, Thiruvalla , India
Email: `lijualex0@gmail.com`

G Indulal
Department of Mathematics, St.Aloysius College, Edathua, Alappuzha - 689573, India
Email: `indulalgopal@gmail.com`