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Research Paper

QUASI-PARTIAL BRANCIARI b-METRIC SPACES AND FIXED POINT RESULTS WITH AN APPLICATION

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Abstract. The main aim of this research paper is to introduce concept of quasi-partial Branciari b-metric space. Such spaces are an extension of quasi-partial metric spaces, quasi-partial b-metric spaces and quasi-partial Branciari metric spaces. In this article, firstly, Conditions for the existence and uniqueness of fixed points in underlying spaces are discussed and related theorems are proved. After that various consequences of these theorems are given and specific examples are presented. Final

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1. Introduction

In the year 1993, Czerwik [9] introduced the concept of a b-metric space as a generalization of the concept of metric space. Several researchers have determined on fixed point results for a metric space, a partial-metric space, a quasi-partial metric space and a partial b-metric space see e.g.[2-5], [8], [13], [16-17]. Karapnar et al. presented the concept of a quasi-partial-metric space [11].

In this sequel, S. Shukla [15] defined partial-b metric space, A. Gupta and P. Gautam [12] introduced the concept of quasi partial b-metric

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space and K. Sarkar established partial Branciari b-metric space [14] and they presented some fixed point theorems in these spaces.

Motivated by the concepts presented in [1], [7], [10], [18], we exhibit an innovative notion, called quasi-partial Branciari b-metric spaces. The fixed point results are proved in setting of such spaces and some innovative examples are given to verify the effectiveness of the main theorems. At the end, an application to the solutions of linear equations.

2. Preliminaries

In this section we give some basic definitions and related examples.

Definition 2.1. [13]. Let X be a non-empty set. A distance function $B_b : X \times X \to \mathbb{R}^+$ is called Branciari b-metric if the following conditions are satisfied :

(Bbm_1) $B_b(x,y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$;
(Bbm_2) $B_b(x,y) = B_b(y,x)$ for all $x, y \in X$;
(Bbm_3) $B_b(x,z) \leq s\Big[B_b(x,u)+B_b(u,v)+B_b(v,z)\Big]$ for some fixed $s \geq 1$ and for all distinct points $x, z, u, v \in X$.

In this case (X, B_b) is called a Branciari b-metric space and $s \geq 1$ is called coefficient of (X, B_b) .

This definition turns in to b-metric space [7] if (Bbm_3) is replaced by $B_b(x,z) \leq s\Big[B_b(x,u)+B_b(u,z)\Big]$ for some fixed $s \geq 1$ and for all distinct points $x, z, u \in X$ and Branciari metric space [13] if we set $s = 1$.

Definition 2.2. [1]. Let X be a non-empty set and $s \geq 1$ given real number. A distance function $d : X \times X \to \mathbb{R}^+$ is said to be a quasi Branciari b-metric if following conditions hold :

(qB_{b_1}) $qB_b(x,y) = 0 \Leftrightarrow x = y$;
(qB_{b_2}) $qB_b(x,y) \leq s\big[qB_b(x,z) + qB_b(z,w) + qB_b(w,y)\big]$.

In this case, the pair (X, qB_b) is called a quasi Branciari b-metric space. The number $s \geq 1$ is called the coefficient of (X, qB_b).

Definition 2.3. [14]. Let X be a non-empty set and $s \geq 1$ given real number. A distance function $d : X \times X \to \mathbb{R}^+$ is said to be a partial

Branciari b-metric if following conditions hold :

(pB_{b_1}) $x = y \Leftrightarrow pB_b(x,x) = pB_b(x,y) = pB_b(y,y)$;

(pB_{b_2}) $pB_b(x,x) \leq pB_b(x,y)$;

(pB_{b_3}) $pB_b(x,y) = pB_b(y,x)$;

(pB_{b_4}) $pB_b(x,y) \leq s\big[pB_b(x,z) + pB_b(z,w) + pB_b(w,y) - pB_b(z,z)\big]$
$$- pB_b(w,w) + \frac{1-s}{2}[pB_b(x,x) + pB_b(y,y)].$$

In this case, the pair (X, pB_b) is called a partial Branciari b-metric space. The number $s \geq 1$ is called the coefficient of (X, pB_b).

Note that every partial Branciari metric space is a partial Branciari b-metric space with coefficient $s = 1$ and every Branciari b-metric space is a partial Branciari b-metric space with the same coefficient and zero self-distance. However, the converse of this fact may not hold.

3. Quasi-Partial Branciari b-Metric Space

We start by introducing the notion of a quasi-partial Branciari b-metric space as follows:

Definition 3.1. Let X be a non-empty set and $s \geq 1$ given real number. A distance function $d : X \times X \to \mathbb{R}^+$ is said to be a quasi-partial Branciari b-metric if following conditions hold :

(qpB_{b_1}) $d(x,x) \leq d(x,y)$;

(qpB_{b_2}) $d(x,y) \leq s\big[d(x,z) + d(z,w) + d(w,y) - d(z,z) - d(w,w)\big]$
$$+ \frac{1-s}{2}[d(x,x) + d(y,y)].$$

In this case, the pair (X, d) is called a quasi-partial Branciari b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

This definition turns in to quasi-partial Branciari metric space for $s = 1$.

Note that every quasi-partial Branciari metric space is a quasi-partial Branciari b-metric space with coefficient $s = 1$. However, the converse of this fact need not hold.

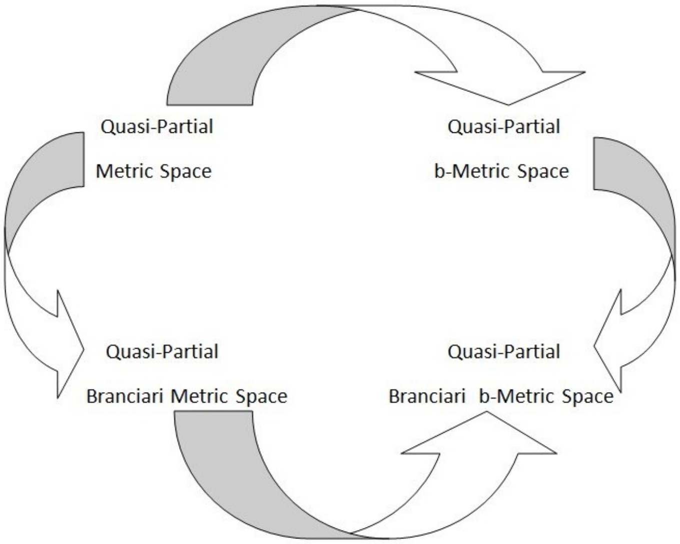

FIGURE 1. Illustration of different Metric Spaces

Definition 3.2. Let (X, d) be a quasi-partial Branciari b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be any sequence in X and $x \in X$. Then

(a) The sequence $\{x_n\}$ is said to be convergent and converges to x, if $\lim_{n\to\infty} d(x_n, x) = d(x, x)$ i.e. limit exists and is finite.
(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d) if $\lim_{n,m\to\infty} d(x_n, x_m)$ exists and finite.
(c) (X, d) is said to be complete quasi-partial Branciari b-metric space if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that

$$\lim_{n,m\to\infty} d\big(x_n, x_m\big) = \lim_{n\to\infty} d\big(x_n, x\big) = d(x, x).$$

The following example highlight the importance of quasi-partial Branciari b-metric space.

Example 3.3 Let $X = \{1, 2, 3, 4\}$ and $d : X \times X \to \mathbb{R}$ be defined by

$$d(x, y) = \begin{cases} |x-y|^2 + \max\{x, y\}, & \text{if } x \neq y \\ x, & \text{if } x = y \neq 1 \\ 0, & \text{if } x = y = 1. \end{cases}$$

Then (X, d) is a complete quasi-partial Branciari b-metric space with coefficient $s > 1$.
Therefore by given definition, we get

$$
\begin{aligned}
&d(1,4) = 3^2 + 4 = 9 + 4 = 13.\\
&d(1,3) = 2^2 + 3 = 4 + 3 = 7.\\
&d(1,2) = 1^2 + 2 = 1 + 2 = 3.\\
&d(2,4) = 2^2 + 4 = 4 + 4 = 8.\\
&d(2,3) = 1^2 + 3 = 1 + 3 = 4.\\
&d(3,4) = 1^2 + 4 = 1 + 4 = 5.\\
&d(1,1) = 0.\\
&d(2,2) = 2.\\
&d(3,3) = 3.\\
&d(4,4) = 4.
\end{aligned}
$$

We have the following observations:

$$
\begin{aligned}
(a) \qquad d(1,4) &\leq d(1,2) + d(2,4) - d(2,2).\\
13 &\leq 3 + 8 - 2\\
&= 9,
\end{aligned}
$$

which is a contradiction. Thus (X, d) is not a complete quasi-partial metric space.

$$
\begin{aligned}
(b)\ d(1,4) &\leq d(1,2) + d(2,3) + d(3,4) - d(2,2) - d(3,3).\\
13 &\leq 3 + 4 + 5 - 2 - 3\\
&= 7,
\end{aligned}
$$

which is a contradiction. Thus (X, d) is not a complete quasi-partial Branciari metric space.

$$
\begin{aligned}
(c)\ d(1,4) &\leq s\{d(1,2)+d(2,3)+d(3,4)-d(2,2)-d(3,3)\} \\
&\quad + \frac{1-s}{2}\{d(1,1)+d(4,4)\}. \\
13 &\leq s(3+4+5-2-3)+\frac{1-s}{2}(0+4) \\
&= 7s+\frac{1-s}{2}(4) \\
&= 7s+2-2s. \\
11 &\leq 5s.
\end{aligned}
$$

Therefore quasi-partial Branciari b-inequality (qpB_{b_2}) holds for $s = 2.2$. Hence (X, d) is a complete quasi-partial Branciari b-metric space with coefficient $s \geq 1$.

4. Main Results

In this section, first, we prove unique fixed point result in quasi-partial Branciari b-metric space.

Theorem 4.1. Let (X, d) be a complete quasi-partial Branciari b-metric space. Let T be a self maps on X and satisfying for any $x, y \in X$ such that

$$
d(Tx,Ty) \leq a \max\Big\{d(x,Tx), d(y,Ty), d(x,y)\Big\} \tag{4.1}
$$

where $a > 0$ such that $as \leq 1$ and $s \geq 1$. Then T has a unique fixed point $u \in X$ and $d(u, u) = 0$.

Proof: Let us first show that if fixed point of T exists, then it is unique. On the contrary suppose $u, v \in X$ be two distinct fixed point of T, then we have

$$
Tu = u \ \text{ and } \ Tv = v.
$$

It follows from inequality (4.1) that

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\le a \max\Big\{d(u,Tu), d(v,Tv), d(u,v)\Big\} \\ &\le a \max\Big\{d(u,u), d(v,v), d(u,v)\Big\} \\ &= a\, d(u,v). \\ d(u,v) &< d(u,v) \qquad\qquad \text{for } \ a<1, \end{aligned}$$

which is a contradiction. Therefore, we must have $d(u,v) = 0$, that is, $u = v$. Thus if fixed point of T exists then it is unique.
Further, if u is a fixed point of T and $d(u,u) > 0$ then from inequality (4.1)

$$\begin{aligned} d(Tu,Tu) &\le a \max\Big\{d(u,Tu), d(u,Tu), d(u,u)\Big\} \\ &= a \max\Big\{d(u,u), d(u,u), d(u,u)\Big\} \\ &= a\, d(u,u). \\ d(u,u) &< d(u,u), \qquad \text{a contradiction.} \end{aligned}$$

Therefore, $d(u,u) = 0$.

For the existence of fixed point, let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in x. Define the recursion

$$x_n = Tx_{n-1} = T^n x_0, \ \ n = 1,2,3,... \tag{4.2}$$

From inequality (4.1) and (4.2), we obtain

$$\begin{aligned} d\big(x_n, x_{n+1}\big) &= d\big(Tx_{n-1}, Tx_n\big) \\ &\le a \max\Big\{d\big(x_{n-1}, Tx_{n-1}\big), d\big(x_n, Tx_n\big), d\big(x_{n-1}, x_n\big)\Big\} \\ &= a \max\Big\{d\big(x_{n-1}, x_n\big), d\big(x_n, x_{n+1}\big), d\big(x_{n-1}, x_n\big)\Big\} \\ &\le a \max\Big\{d\big(x_{n-1}, x_n\big), d\big(x_n, x_{n+1}\big)\Big\}. \end{aligned}$$

Now two cases arises:

Case-(i): If $\max\Big\{d\big(x_{n-1},x_n\big),d\big(x_n,x_{n+1}\big)\Big\}= d\big(x_n,x_{n+1}\big)$.
Then we get

$$\begin{aligned} d\big(x_n,x_{n+1}\big) &\leq a\, d\big(x_n,x_{n+1}\big) \\ \big(1-a\big)d\big(x_n,x_{n+1}\big) &\leq 0. \end{aligned}$$

Letting $n\to\infty$, we obtain that $d\big(x_n,x_{n+1}\big)\to 0$.

Case-(ii): If $\max\Big\{d\big(x_{n-1},x_n\big),d\big(x_n,x_{n+1}\big)\Big\}= d\big(x_{n-1},x_n\big)$.
Then we have

$$d\big(x_n,x_{n+1}\big) \leq a\, d\big(x_{n-1},x_n\big).$$

Continuing the above process, we get

$$d\big(x_n,x_{n+1}\big) \leq a^n d\big(x_0,x_1\big). \tag{4.3}$$

Letting $n\to\infty$, we obtain that $d\big(x_n,x_{n+1}\big)\to 0$.

Now we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.

Let $m, n \in \mathbb{N}$ such that $m > n$, then by definition 3.1, we have

$$\begin{aligned}
d\big(x_n, x_m\big) &\leq s\Big\{d\big(x_n, x_{n+1}\big) + d\big(x_{n+1}, x_{n+2}\big) + d\big(x_{n+2}, x_m\big) \\
&- d\big(x_{n+1}, x_{n+1}\big) - d\big(x_{n+2}, x_{n+2}\big)\Big\} + \frac{1-s}{2}\big\{d\big(x_n, x_n\big) \\
&+ d\big(x_m, x_m\big)\big\} \\
&\leq s\Big\{d\big(x_n, x_{n+1}\big) + d\big(x_{n+1}, x_{n+2}\big) + d\big(x_{n+2}, x_m\big)\Big\} \\
&\leq s\, d\big(x_n, x_{n+1}\big) + s\, d\big(x_{n+1}, x_{n+2}\big) + s^2\Big\{d\big(x_{n+2}, x_{n+3}\big) \\
&+ d\big(x_{n+3}, x_{n+4}\big) + d\big(x_{n+4}, x_m\big) - d\big(x_{n+3}, x_{n+3}\big) \\
&- d\big(x_{n+4}, x_{n+4}\big)\Big\} + s\frac{1-s}{2}\big\{\, d\big(x_{n+2}, x_{n+2}\big) + d\big(x_{n+2}, x_{n+2}\big)\big\} \\
&\leq s\, d\big(x_n, x_{n+1}\big) + s\, d\big(x_{n+1}, x_{n+2}\big) + s^2\Big\{d\big(x_{n+2}, x_{n+3}\big) \\
&+ d\big(x_{n+3}, x_{n+4}\big) + d\big(x_{n+4}, x_m\big)\Big\} \\
&\leq \big\{sa^n + sa^{n+1} + s^2a^{n+2} + s^2a^{n+3} + s^3a^{n+4} + s^3a^{n+5} + ...\big\}d\big(x_0, x_1\big) \\
&\leq \big\{sa^n(1 + sa^2 + s^2a^4 + ...) + sa^{n+1}(1 + sa^2 + s^2a^4 + ...)\big\}d\big(x_0, x_1\big) \\
&\leq (sa^n + sa^{n+1})(1 + sa^2 + s^2a^4 + ...)d\big(x_0, x_1\big) \\
&\leq \frac{sa^n}{(1 - sa^2)}(1 + a)d\big(x_0, x_1\big).
\end{aligned}$$

Letting $n, m \to \infty$, we get $\lim_{n\to\infty} d\big(x_n, x_m\big) = 0$, since $a < 1$.

Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. By completeness of X there exists $u \in X$ such that

$$\lim_{n\to\infty} d\big(x_n, u\big) = \lim_{n,m\to\infty} d\big(x_n, x_m\big) = d(u, u) = 0. \tag{4.4}$$

Now we show that u is a fixed point of T.

$$\begin{aligned} d(u,Tu) &\leq s\Big\{d\big(u,x_{n+1}\big)+d\big(x_{n+1},x_{n+2}\big)+d\big(x_{n+2},Tu\big) \\ &\quad - d\big(x_{n+1},x_{n+1}\big)-d\big(x_{n+2},x_{n+2}\big)\Big\}+\frac{1-s}{2}\big\{d\big(x_{n+1},x_{n+1}\big) \\ &\quad + d\big(x_{n+2},x_{n+2}\big) \\ &\quad \leq s\,d\big(u,x_{n+1}\big)+sa^{n+1}d\big(x_0,x_1\big)+s\,d\big(Tx_{n+1},Tu\big) \\ &\quad \leq s\,d\big(u,x_{n+1}\big)+sa^{n+1}d\big(x_0,x_1\big)+sa\max\Big\{d(x_{n+1},Tx_{n+1}), \\ &\quad d(u,Tu),d(x_{n+1},u)\Big\}. \end{aligned}$$

Case-(i):
If $\max\Big\{d(x_{n+1},Tx_{n+1}),d(u,Tu),d(x_{n+1},u)\Big\}=d(x_{n+1},Tx_{n+1})$.
Then we get

$$\begin{aligned} d(u,Tu) &\leq s\,d\big(u,x_{n+1}\big)+sa^{n+1}d\big(x_0,x_1\big)+sa\,d(x_{n+1},Tx_{n+1}) \\ &\leq s\,d\big(u,x_{n+1}\big)+sa^{n+1}d\big(x_0,x_1\big)+sa\,d(x_{n+1},x_{n+2}) \\ &\leq s\,d\big(u,x_{n+1}\big)+sa^{n+1}d\big(x_0,x_1\big)+sa^{n+2}d\big(x_0,x_1\big). \end{aligned}$$

Letting $n\to\infty$, we get

$$\lim_{n\to\infty} d(u,Tu)=0 \Rightarrow u=Tu.$$

Therefore u is a fixed point of T.

Case-(ii):
If $\max\Big\{d(x_{n+1},Tx_{n+1}),d(u,Tu),d(x_{n+1},u)\Big\}=d(u,Tu)$.
Then we get

$$\begin{aligned} d(u,Tu) &\leq s\,d\big(u,x_{n+1}\big)+sa^{n+1}d\big(x_0,x_1\big)+sa\,d(u,Tu). \\ (1-sa)d(u,Tu) &\leq s\,d\big(u,x_{n+1}\big)+sa^{n+1}d\big(x_0,x_1\big). \\ (1-sa)d(u,Tu) &\leq \frac{s}{1-sa}\,d\big(u,x_{n+1}\big)+\frac{sa^{n+1}}{1-sa}d\big(x_0,x_1\big). \end{aligned}$$

Letting $n\to\infty$, we get

$$\lim_{n\to\infty} d(u,Tu)=0.$$
$$\Rightarrow u=Tu.$$

Therefore u is a fixed point of T.

Case-(iii):
If $\max\Big\{d(x_{n+1}, Tx_{n+1}), d(u, Tu), d(x_{n+1}, u)\Big\} = d(x_{n+1}, u)$.
Then we get

$$\begin{aligned} d(u, Tu) &\leq s\, d\big(u, x_{n+1}\big) + sa^{n+1} d\big(x_0, x_1\big) + sa\, d(x_{n+1}, u) \\ &\leq s(1+a)\, d\big(u, x_{n+1}\big) + sa^{n+1} d\big(x_0, x_1\big). \end{aligned}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} d(u, Tu) = 0 \Rightarrow u = Tu.$$

Therefore u is the fixed point of T.

Corollary 4.2. Let (X, d) be a complete quasi-partial Braciari b-metric space, T be a self map on X and satisfying for any $x, y \in X$ such that

$$d(Tx, Ty) \leq a\, d(x, y) \tag{4.4}$$

where $a \in [0, 1/s]$ and $s \geq 1$. Then T has a unique fixed point $u \in X$ and $d(u, u) = 0$.

Now, we present common fixed point theorem for two distance functions in quasi-partial Branciari b-metric space.

Theorem 4.3. Let (X, d) be a complete quasi-partial Branciari b-metric space. Let f and g be two maps define onto X itself and satisfying follow inequality for any $x, y \in X$

$$d(fx, gy) \leq \alpha\{d(x, gx) + d(y, fy)\} \tag{4.5}$$

where $\alpha \in [0, \frac{1}{s})$ with $s \geq 1$ Then f and g have a common unique fixed point.

Proof Firstly we prove the existence of common fixed point of f and g.

Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}$ in X as $x_{n+1} = fx_n$ and $x_{n+2} = gx_{n+1}$ $\forall\ n \geq 1$.

If $x_n = x_{n+1}$ $\forall\ n \geq 1$. Then $gx_n = x_{n+1} = x_n = fx_n$. Thus f and g have a unique common fixed point x_n.

Now, suppose that $x_n \neq x_{n+1}\ \forall\ n \geq 1$.
Therefore from (4.5)

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(fx_{n-1}, gx_n) \\
&\leq \alpha\{d(x_{n-1}, gx_{n-1}) + d(x_n, fx_n)\} \\
&= \alpha\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}. \\
(1-\alpha)d(x_n, x_{n+1}) &\leq \alpha\, d(x_{n-1}, x_n). \\
d(x_n, x_{n+1}) &\leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n) \\
&\leq \eta\, d(x_{n-1}, x_n) (\eta = \frac{\alpha}{1-\alpha} < 1).
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \eta\, d(x_{n-1}, x_n) \\
&\leq \eta^2\, d(x_{n-2}, x_{n-1}) \leq \ldots \leq \eta^n\, d(x_0, x_1).
\end{aligned}$$

$$d(x_n, x_{n+1}) \leq \eta^n\, d(x_0, x_1). \tag{4.6}$$

Letting $n \to \infty$, we obtain that $d\big(x_n, x_{n+1}\big) \to 0$.

Now,

$$\begin{aligned}
d(x_n, x_{n+2}) &= d(fx_{n-1}, gx_{n+1}) \\
&\leq \alpha\{d(x_{n-1}, gx_{n-1}) + d(x_{n+1}, fx_{n+1})\} \\
&= \alpha\{d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})\}.
\end{aligned}$$

Therefore , we get

$$d(x_n, x_{n+2}) \leq \alpha\{\eta^{n-1} d(x_0, x_1) + \eta^{n+1} d(x_0, x_1)\}. \tag{4.7}$$

Letting $n \to \infty$, we obtain that $d\big(x_n, x_{n+2}\big) \to 0$.

Now we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.

Let $m, n \in \mathbb{N}$ such that $m > n$, then by definition 3.1, we have

$$
\begin{aligned}
d\big(x_n, x_m\big) \leq\; & s\Big\{d\big(x_n, x_{n+1}\big) + d\big(x_{n+1}, x_{n+2}\big) + d\big(x_{n+2}, x_m\big) \\
& - d\big(x_{n+1}, x_{n+1}\big) - d\big(x_{n+2}, x_{n+2}\big)\Big\} + \frac{1-s}{2}\big\{d\big(x_n, x_n\big) \\
& + d\big(x_m, x_m\big)\big\} \\
& \leq s\Big\{d\big(x_n, x_{n+1}\big) + d\big(x_{n+1}, x_{n+2}\big) + d\big(x_{n+2}, x_m\big)\Big\} \\
& \leq s\, d\big(x_n, x_{n+1}\big) + s\, d\big(x_{n+1}, x_{n+2}\big) + s^2\Big\{d\big(x_{n+2}, x_{n+3}\big) \\
& + d\big(x_{n+3}, x_{n+4}\big) + d\big(x_{n+4}, x_m\big) - d\big(x_{n+3}, x_{n+3}\big) \\
& - d\big(x_{n+4}, x_{n+4}\big)\Big\} + s\frac{1-s}{2}\big\{\, d\big(x_{n+2}, x_{n+2}\big) + d\big(x_{n+2}, x_{n+2}\big)\big\} \\
& \leq s\, d\big(x_n, x_{n+1}\big) + s\, d\big(x_{n+1}, x_{n+2}\big) + s^2\Big\{d\big(x_{n+2}, x_{n+3}\big) \\
& + d\big(x_{n+3}, x_{n+4}\big) + d\big(x_{n+4}, x_m\big)\Big\} \\
& \leq \big\{sa^n + sa^{n+1} + s^2a^{n+2} + s^2a^{n+3} + s^3a^{n+4} + s^3a^{n+5} + ...\big\}d\big(x_0, x_1\big) \\
& \leq \big\{sa^n(1 + sa^2 + s^2a^4 + ...) + sa^{n+1}(1 + sa^2 + s^2a^4 + ...)\big\}d\big(x_0, x_1\big) \\
& \leq (sa^n + sa^{n+1})(1 + sa^2 + s^2a^4 + ...)d\big(x_0, x_1\big) \\
& \leq \frac{sa^n}{(1 - sa^2)}(1 + a)d\big(x_0, x_1\big).
\end{aligned}
$$

Letting $n, m \to \infty$, we get $\lim_{n\to\infty} d\big(x_n, x_m\big) = 0$, since $a < 1$.

Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. By completeness of X there exists $u \in X$ such that

$$\lim_{n\to\infty} d\big(x_n, u\big) = \lim_{n,m\to\infty} d\big(x_n, x_m\big) = d(u, u) = 0.$$

Since (X, d) is a complete quasi-partial Branciari b-metric space then there exists $r \in X$ such that

$$\lim_{n\to\infty} x_n = r. \tag{4.8}$$

Now, we find out $fr = r$.

In quasi-partial Branciari b-metric space, we have

$$\begin{aligned} d(fr,r) &\leq s\big\{d(fr,x_n)+d(x_n,x_{n+1})+d(x_{n+1},r)-d(x_n,x_n) \\ &\quad -d(x_{n+1},x_{n+1})\big\}+\frac{1-s}{2}\big\{d(fr,fr)+d(r,r)\big\} \\ &\leq s\big\{d(fr,x_n)+d(x_n,x_{n+1})+d(x_{n+1},r)\big\} \\ &\leq s[d(fr,x_n)+\eta^n d(x_0,x_1)+d(x_{n+1},r)]. \end{aligned}$$

Letting $n\to\infty$ and using inequality (4.7), we find for $s>1$

$$d(fr,r)=0.$$

Therefore

$$fr=r.$$

Hence, r is a fixed point of f.

Now, we show that r is a fixed point of g. i.e. $gr=r$.
Suppose $gr\neq r$. Then by inequality (4.1)

$$\begin{aligned} d(r,gr) &= d(fr,gr) \\ &\leq \alpha\{d(r,gr)+d(r,fr)\} \\ &= \alpha\{d(r,gr)+d(r,r)\} \\ &\leq \alpha d(r,gr). \end{aligned}$$

Thus

$$(1-\alpha)d(r,gr)\leq 0.$$

but $1-\alpha\neq 0$, therefore $d(r,gr)=0$. i.e. $gr=r$.
Hence,

$$fr=gr=r.$$

Therefore f and g have a unique common fixed point of X .

Uniqueness of Fixed Point

On the contrary suppose p and q are two common fixed points of f and g. Then clearly $p=fp=gp$ and $q=fq=gq$.

$$\begin{aligned} d(p,q) &= d(fp,gq) \\ &\leq \alpha\{d(p,gp)+d(q,fq)\} \\ &\leq \alpha\{d(p,p)+d(q,q)\} \\ &\leq 0. \end{aligned}$$

Thus

$$d(p,q) = 0 \Rightarrow p = q.$$

Hence our result is proved.

Corollary 4.4. Let (X,d) be a complete quasi-partial Branciari b-metric space. Let f and g be two maps define onto X itself and satisfying follow inequality for any $x, y \in X$

$$d(fx, gy) \leq \alpha d(x,y) \tag{4.9}$$

where $\alpha \in [0, \frac{1}{s})$ with $s \geq 1$ Then f and g have a common unique fixed point.

Now, we demonstrate an example to verify our first fixed point result (4.1).

Example 4.5. Let $X = \{1,2,3,4\}$ and $d : X \times X \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} |x-y|^2 + \max\{x,y\}, & \text{if } x \neq y \\ x, & \text{if } x = y \neq 1 \\ 0, & \text{if } x = y = 1. \end{cases}$$

Then (X,d) is a complete quasi-partial Branciari b-metric space with coefficient $s \geq 1$.

Define $T : X \to X$ by $T_1 = 2, T_2 = 2, T_3 = 1, T_4 = 1$.
Then T satisfies all the conditions of theorem 4.1 with $a \in [0, 1/s]$ and has a unique fixed point $2 \in X$ is the unique fixed point of T.

Now,

$$\begin{aligned} d(4,2) &\leq s\{d(4,3) + d(3,1) + d(1,4) - d(3,3) - d(1,1)\} \\ &\quad + \frac{1-s}{2}\{d(4,4) + d(2,2)\} \\ 8 &\leq s(5 + 7 + 13 - 3 - 0) + \frac{1-s}{2}(4+2) \\ &= 22s + \frac{1-s}{2}(6) \\ &= 22s + 3 - 3s \\ 5 &\leq 19s. \end{aligned}$$

Therefore quasi-partial Branciari b-inequality holds for $s \geq 1$. Thus theorems 4.1 is applicable here.

Let $x = 1, y = 2$

$$\begin{aligned} d(T_1, T_2) &\leq a \max\Big\{d(1, T_1), d(2, T_2), d(1, 2)\Big\}. \\ d(2, 2) &\leq a \max\Big\{d(1, 2), d(2, 2), d(1, 2)\Big\}. \\ 2 &\leq a \max\{3, 2, 3\}. \\ 2 &\leq 3a. \\ Therefore\ \frac{2}{3} &\leq a < 1. \end{aligned} \tag{4.10}$$

Now take $x = 1$ and $y = 3$

$$\begin{aligned} d(T_1, T_3) &\leq a \max\Big\{d(1, T_1), d(3, T_3), d(1, 3)\Big\}. \\ d(2, 1) &\leq a \max\Big\{d(1, 2), d(3, 1), d(1, 3)\Big\}. \\ 3 &\leq a \max\{3, 7, 7\}. \\ 3 &\leq 7a. \\ Therefore\ \frac{3}{7} &\leq a < 1 \end{aligned} \tag{4.11}$$

Now take $x = 1$ and $y = 4$

$$\begin{aligned} d(T_1, T_4) &\leq a \max\Big\{d(1, T_1), d(4, T_4), d(1, 4)\Big\}. \\ d(2, 1) &\leq a \max\Big\{d(1, 2), d(3, 1), d(1, 4)\Big\}. \\ 3 &\leq a \max\{3, 7, 13\}. \\ 3 &\leq 13a. \\ Therefore\ \frac{3}{13} &\leq a < 1 \end{aligned} \tag{4.12}$$

Now take $x = 2$ and $y = 3$

$$d(T_2, T_3) \leq a \max\Big\{d(2, T_2), d(3, T_3), d(2, 3)\Big\}.$$

$$d(2, 1) \leq a \max\Big\{d(2, 2), d(3, 1), d(2, 3)\Big\}.$$

$$3 \leq a \max\{2, 7, 4\}$$

$$\leq 7a.$$

$$\text{(4.13)} \qquad Therefore\ \frac{3}{7} \leq a < 1$$

Now take $x = 2$ and $y = 4$

$$d(T_2, T_4) \leq a \max\Big\{d(2, T_2), d(4, T_4), d(2, 4)\Big\}.$$

$$d(2, 1) \leq a \max\Big\{d(2, 2), d(4, 1), d(2, 4)\Big\}.$$

$$3 \leq a \max\{2, 13, 8\}.$$

$$3 \leq 13a.$$

$$\text{(4.14)} \qquad Therefore\ \frac{3}{13} \leq a < 1$$

Now take $x = 3$ and $y = 4$

$$d(T_3, T_4) \leq a \max\Big\{d(3, T_3), d(4, T_4), d(3, 4)\Big\}.$$

$$d(1, 1) \leq a \max\Big\{d(3, 1), d(4, 1), d(3, 4)\Big\}.$$

$$0 \leq a \max\{4, 13, 7\}.$$

$$0 \leq 13a.$$

$$\text{(4.15)} \qquad Therefore\ 0 \leq a < 1$$

Hence the condition (4.1) of theorem 4.1, is satisfied for $as \leq 1$.

5. An Application to Linear Equation

In this section, we give an application to unique solution of linear equations using corollary 4.2.

Theorem 4.1. Let $X = \mathbb{R}^n$ be complete quasi-partial Branciari b-metric space define by

$$\text{(4.16)} \qquad d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

where $x, y \in X$. If $\sum_{j=1}^{n} |a_{ij}| \leq a < 1$ for all $i = 1, 2, ...n$, then the linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ . \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

of n linear equations in n unknowns has a unique solution.

Proof. we need to prove that the mapping $T : X \to X$ defined by

$$T(x) = Ax + b \tag{4.17}$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \; : \; b = (b_1, b_2, ..., b_n) \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is a contraction.
Since

$$\begin{aligned} d(Tx, Ty) &= \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} a_{ij}(x_j - y_j) \right| \\ &\leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}||(x_j - y_j)| \\ &\leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \Big(\max_{1 \leq j \leq n} |(x_j - y_j)| \Big) \\ &= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| d(x, y) \\ &\leq a \; d(x, y). \end{aligned}$$

We conclude that T is a contraction mapping. This result obviously proves.

4.2. Conclusion

In this article, we proposed some fixed point theorems in quasi-partial Branciari b-metric space which generalize results of Gupta and Gautam [12]. We have introduced some very interesting contractions in such spaces and obtained results are validated by appropriate examples. Applications to the solutions of linear equations are also entrusted to manifest the viability of the obtained results.

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