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Research Paper

ON THE GENERALIZATION OF THE TORSION FUNCTOR AND P-SEMIPRIME MODULES OVER NONCOMMUTATIVE RINGS

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Abstract. Let R be an associative Noetherian unital noncommutative ring R. We introduce the functor $P\Gamma_P$ over the category of R-modules and use it to characterize P-semiprime. P-semisecond R-modules also characterized by the functor $P\Lambda_P$. We also show that the Greenless-May type Duality (GM) and Matlis Greenless-May Equality(MGM) hold over the full subcategory of R-Mod consisting of P-semiprime and P-semisecond modules. Finally, we generate a one-sided right ideal $P\Gamma_P(R)$, which gives an equivalent formulation to solve Köthe conjecture positively or negatively.

Key Words: P-semiprime, P-semisecond, torsion functor, adic completion and Köthe conjecture.

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1. Introduction

Let R be a unital Noetherian ring which is not necessarily commutative. R is semiprime if for all ideals P of R, $P^2 = 0$ implies that $P = 0$. R is reduced if for all $a \in R$, $a^2 = 0$ implies that $a = 0$. If R is commutative, the two notions coincides. An ideal P of a ring R is semiprime (resp. completely semiprime) if the quotient ring R/I is a semiprime (resp. reduced) ring. Any reduced ring is semiprime. However, the ring

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$M_2(\mathbb{Z})$ is a semiprime ring which is not reduced. The notion of reduced, semiprime, semisecond and coreduced modules has been widely studied see for instance [11, 15, 16, 18, 21]. When R is commutative, the locally nilradical functor $a\Gamma_a(-)$ over a category of R-modules, where a is the ring element has been studied as a measure of how far a module from being reduced in [9]. Moreover, in [19], it has been studied about P-reduced and P-coreduced modules in relation with Matlis-Greenlees-May Equivalence and Greenlees-May duality. The Köthe conjecture states that if a ring R has no nonzero nil ideals then R has no nonzero nil one-sided ideals and it has existed since 1930. Even if the problem is still open lots of equivalent formulations has been made. The sum of two right nil ideals in any ring R is nil is also an equivalent formulation for the conjecture, [1, 8, 14]. This paper is organized as follows: In Section 2 we introduce the functor $P\Gamma_P$ and show that it is a 1) radical over a category of R-modules; 2) left exact over an abelian full subcategory of R-Mod consisting of flat modules; 3) we also use it to characterize the P-semiprime modules; 4) we characterize P-semisecond modules using the functor $P\Lambda_P$. In Section 3 we study applications of P-semiprime and P-semisecond modules and show that the Greenless-May duality and Matlis-Greenless-May Equality holds. In Section 4, we generate a right nil ideal by considering $P\Gamma_P$ over rings. We also provide a gadget that produces one sided nil ideals for any noncommutative ring for a given right ideal (Proposition 4.2). This is useful in the possible counter examples to answer the Köthe conjecture in the negative. Finally, we pose questions which are equivalent to solve the Köthe conjecture in the negative (Question 4.8) and positive (Question 4.9) using the ideal $P\Gamma_P$.

2. P-SEMIPRIME AND P-SEMISECOND MODULES

In this section unless otherwise mentioned R-Mod represents the category of left R-modules. Let P be an ideal of a ring R. A submodule N of an R-module M is P-semiprime if for all $m \in M$, $P^2m \subseteq N$ implies that $Pm \subseteq N$. A submodule N of an R-module M is semiprime if it is P-semiprime for all ideals P of R. An R-module M/N is semiprime (resp. P-semiprime) if N is a semiprime (resp. P-semiprime) submodule of M.

Remark 2.1. A ring R is semiprime if and only if the R-module R is semiprime.

Let R-Mod be a category of left R-modules. A functor $\gamma(-) : R\text{-Mod} \to R\text{-Mod}$ is a *preradical* if $\gamma(M)$ is a submodule of M and for every R-homomorphism $f : M \to N$, $f(\gamma(M)) \subseteq \gamma(N)$. γ is a *radical* if it is a preradical and for all $M \in R\text{-Mod}$, $\gamma(M/\gamma(M)) = 0$. A radical γ is left exact if for every submodule N of a module $M \in R\text{-Mod}$, $\gamma(N) = N \cap \gamma(M)$. Equivalently, if for any exact sequence $0 \to N \to M \to K$ of R-modules, the sequence $0 \to \gamma(N) \to \gamma(M) \to \gamma(K)$ is also exact.

Definition 2.2. Let R be a Noetherian ring and P a right ideal of R. $\Gamma_P(-) : R\text{-Mod} \to R\text{-Mod}$ $M \mapsto \Gamma_P(M) := \{m \in M : P^k m = 0, \text{for some } k \in \mathbb{Z}^+\}$.

Proposition 2.3. *The functor $\Gamma_P(-) : R$-Mod $\to R$-Mod is a left exact radical.*

Proof.

1. Consider the R-module homomorphism $f : M \to N$. Let $y \in f(\Gamma_P(M))$, $y = f(m) \in N$ for some $m \in \Gamma_P(M)$, i.e., $P^k m = 0$ for some positive integer k. Now, $P^k y = P^k f(m) = f(P^k m) = f(0) = 0$ which implies $y \in \Gamma_P(N)$.
2. To show it is radical, it is enough to show that $\Gamma_P(M/\Gamma_P(M)) = 0$. Let $y \in \Gamma_P(M/\Gamma_P(M))$ such that $P^k m \in \Gamma_P(M)$, where $y = m + \Gamma_P(M)$ for some $m \in M$. Then there exists a positive integer k_1 such that $P^{k_1}(P^k m) = 0$ which implies $P^{k_1+k} m = 0$. It follows that $m \in \Gamma_P(M)$ and thus $y = 0$.
3. It is similar to the proof of [3, Lemma 1.16]

□

By multiplying the torsion functor Γ_P by P from the left we define $P\Gamma_P$ as follows:

Definition 2.4. Let P be an ideal of a ring R. A functor

$$P\Gamma_P(-) : R\text{-Mod} \to R\text{-Mod is defined by}$$

$$M \mapsto P\Gamma_P(M) := \left\{ \sum_{i=1}^{n} r_i m_i : r_i \in P \text{ and } m_i \in \Gamma_P(M) \right\}.$$

Proposition 2.5. *Let $M \in R$-Mod and P an ideal of R. The following are equivalent:*

1. *M is P-semiprime.*
2. *$(0 : m)$ is an P-semiprime left ideal of R for all $0 \neq m \in M$.*
3. *$(0 :_M P) = (0 :_M P^2)$.*
4. *$Hom_R(R/P, M) = Hom_R(R/P^2, M)$.*

5. $\Gamma_P(M) \cong Hom(R/P, M)$.
6. $P\Gamma_P(M) = 0$.

Proof. 1 $\Rightarrow$ 2 For any left ideal P of R, let $P^2 \subseteq (0 : m)$. Then this implies that $P^2m = 0$ for all nonzero $m \in M$, since M is P-semiprime R-module it follows that $Pm = 0$ and hence $P \subseteq (0 : m)$.

2 $\Rightarrow$ 1 For any ideal P of R and $0 \neq m \in M$, let $P^2m = 0$ implies $P^2 \in (0 : m)$ then by hypothesis $P \in (0 : m)$ which implies $Pm = 0$, thus P-semiprime.

2 $\Rightarrow$ (3) Since $(0 : m)$ is P-semiprime ideal of R, it follows that $(0 :_M P^2) \subseteq (0 :_M P)$, the other inclusion is obvious.

3 $\Rightarrow$ 4 Since $(0 :_M P)$ is a left R-module, then it coincides with $\mathrm{Hom}_R(R/P, M)$ then the result follows.

(4) $\Rightarrow$ (5) since $\Gamma_P(M) \cong \varinjlim_k \mathrm{Hom}_R(R/P^k, M)$. then by (4) we have
$\mathrm{Hom}_R(R/P, M) = \mathrm{Hom}_R(R/P^k, M)$ for all $k \in \mathbb{Z}^+$. So, $\Gamma_P(M) \cong \mathrm{Hom}_R(R/P, M)$.

(5) $\Rightarrow$ (6) $P\Gamma_P(M) \cong P(\mathrm{Hom}(R/P, M)) = 0$.

1 $\Rightarrow$ 6 Let M be P-semiprime module, $m \in P\Gamma_P(M)$ and $m = \sum_{i=1}^n a_im_i$, $a_i \in P$ and $m_i \in \Gamma_P(M)$, i.e., $P^{k_i}m_i = 0$ for some positive integers k_i. By hypothesis $Pm_i = 0$ then for each $a_i \in P$ we have $a_im_i = 0$ then $m = 0$. So, $P\Gamma_P(M) = 0$.

6 $\Rightarrow$ 1 Suppose $P\Gamma_P(M) = 0$. Let $m \in M$ and $P^2m = 0$ which implies $m \in \Gamma_P(M)$ then $Pm \subseteq P\Gamma_P(M) = 0$, so M is P-semiprime R-module.

□

So, the functor $P\Gamma_P(-)$ on R-Mod is a measure of how far a module is from being P-semiprime.

A left R-module F is *flat* if the functor $- \otimes F$ is exact. A short exact sequence $0 \to L \to M \to N \to 0$ of left R-modules is called *pure* if $K \otimes L \to K \otimes M$ is a monomorphism for every right R-module K.

In general, $P\Gamma_P$ is not left exact. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{12}$ and the submodule $N = \{0, 6\}$ and take $I = \langle 2\rangle$. Now, $\langle 2\rangle\Gamma_{\langle 2\rangle}(M) = \{0, 3, 6, 9\}$, $\langle 2\rangle\Gamma_{\langle 2\rangle}(N) = \{0\}$ and $\langle 2\rangle\Gamma_{\langle 2\rangle}(M)\cap N = N$, but $\langle 2\rangle\Gamma_{\langle 2\rangle}(N) = \{0\}$. Let $\mathrm{Fl}(R)$ denote an abelian full subcategory of R-Mod consisting of all flat R-modules. This abelian category was studied in [6, 7, 20]. Note that since any free R-module is flat, $R \in \mathrm{Fl}(R)$.

Proposition 2.6. Consider the functor

$$P\Gamma_P(-) : R\text{-Mod} \to R\text{-Mod}, M \mapsto P\Gamma_P(M).$$

Then,

1 $P\Gamma_P(-)$ is a radical and it is left exact over $\mathrm{Fl}(R)$.
2. $P\Gamma_P(M) \cong P\otimes_R\Gamma_P(M)$, i.e., a composition of $\Gamma_P(-)$ and $P\otimes -$.
3. If in addition $\mathrm{Fl}(R)$ has enough injectives, then $P\Gamma_P$ is exact.

Proof. 1. To show $P\Gamma_P$ is a preradical, consider the module homomorphism $f : M \to N$ for any left R-modules M and N. Let $y \in f(P\Gamma_P(M))$, $y = f(m) \in N, m = \sum_{i=1}^{l} r_i m_i$ and $r_i \in P, m_i \in \Gamma_P(M)$, i.e., $P^k m_i = 0$ for some positive integer k. Now, $P^k(y) = P^k(f(m)) = f(P^k \sum_{i=1}^{l} r_i m_i) = f(0) = 0$ which implies $y \in \Gamma_P(N)$. To show it is radical, it is enough to show that $M/P\Gamma_P(M)$ is P-semiprime. Let $\bar{m} \in M/P\Gamma_P(M)$ and $P^2\bar{m} = \bar{0}$ which implies $P^2(m+P\Gamma_P(M)) = P\Gamma_P(M))$, then $P^2 m \in P\Gamma_P(M)$ and hence $Pm \in \Gamma_P(M)$ such that $P^k m = 0$ for some positive integer k which implies $m \in \Gamma_P(M)$ and so, $Pm \subseteq P\Gamma_P(M)$ and thus, $P\bar{m} = \bar{0}$. Then by Proposition 2.5 the functor is radical.

$P\Gamma_P$ is left exact if and only if for all submodules N of M $P\Gamma_P(N) = P\Gamma_P(M) \cap N$, [12]. Since, $P\Gamma_P(N)$ is submodule of both $P\Gamma_P(M)$ and N it is easy to show that $P\Gamma_P(N) \subseteq P\Gamma_P(M) \cap N$. However, by hypothesis every submodule is pure, so $P\Gamma_P(N) = PM \cap \Gamma_P(N)$, [5, Proposition 8.1]. Then it follows that $PM \cap \Gamma_P(N) \subseteq P\Gamma_P(M) \cap N$. To show the reverse inclusion let $y \in P\Gamma_P(M) \cap N$. Then, $y = \sum_{i=1}^{n} r_i m_i$ such that $P^{k_i} r_i = 0$ for some positive integers k_i, $1 \le i \le n$. Now, $P^k y = \sum_{i=1}^{n} P^k r_i m_i = 0$, where $k = k_1 + \cdots + k_n$, which shows that $y \in PM \cap \Gamma_P(N)$. Hence, $P\Gamma_P(N) = P\Gamma_P(M) \cap N$.

2. Since $M \in \mathrm{Fl}(R)$ and $\mathrm{Fl}(R)$ is an abelian subcategory, $\Gamma_P(M) \in \mathrm{Fl}(R)$. It follows that $P\Gamma_P(M) \cong P \otimes_R \Gamma_P(M)$.
3. Since $P\Gamma_P$ is left exact by part 1 and $\mathrm{Fl}(R)$ has enough injectives, we can compute the right derived functor of $P\Gamma_P$. By [17, Theorem 10.47] $\mathbf{R}^i(P\Gamma_P(M)) \cong \mathbf{R}^i(P \otimes_R \Gamma_P(M)) \cong \mathbf{R}^i(P \otimes_R \mathbf{R}^i(\Gamma_P(M)) \cong 0$. It is zero because $P \otimes_R -$ is exact.

□

In [4] examples for which $\mathrm{Fl}(R)$ has enough injectives were given. This happens when R_P is quasi-Frobenius for all $P \in ASS(R)$, the assassinator of R. However, the only two examples of rings which were given namely; commutative Noetherian domain and $R = k[x, y]/\langle xy\rangle$ are both reduced and flat modules over reduced rings in which case, our functor $P\Gamma_P(-) : \mathrm{Fl}(R) \to \mathrm{Fl}(R)$ will be trivial. see [4, Theorem 3].

Example 2.7. If $R = k[t]/\langle t^2\rangle$. Then R is a flat R-Mod which is not semiprime and also $\langle t\rangle\Gamma_{\langle t\rangle}(R) \neq 0$.

Definition 2.8. Let P be an ideal of R. Define the *P-adic completion* functor $\Lambda_P(-) : R\text{-Mod} \to R\text{-Mod}$ by $M \mapsto \Lambda_P(M) := \varprojlim_k M/P^kM$.

Definition 2.9. Let P be an ideal of R. A left R-module M is said to be *P-semisecond* if $P^2M = PM$.

A left R-module M is said to be semisecond if M is P-semisecond module for every ideal P of R, [2].

Proposition 2.10. For any ideal P of R and R-module M the following are equivalent:

1. M is P-semisecond.
2. $R/P \otimes_R M \cong R/P^2 \otimes_R M$,
3. $R/P \otimes M \cong \Lambda_P(M)$.
4. $P\Lambda_P(M) = 0$.

Proof. (1) $\Rightarrow$ (2) $R/P \otimes_R M \cong M/PM$ since M is semisecond $R/P \otimes_R M \cong M/P^2M \cong R/P^2 \otimes_R M$.
(2) $\Rightarrow$ (3) $\Lambda_P(M) = \varprojlim_k (M/P^kM) \cong \varprojlim_k (M/P^k \otimes_R M) \cong \varprojlim_k (M/P \otimes_R M) = R/P \otimes_R M$.
(3) $\Rightarrow$ (4) $P\Lambda_P(M) \cong P(R/P \otimes_R M) = P(M/PM) = 0$
(1) $\Rightarrow$ (4) Since $P^2M = PM$, it follows that $P^kM = PM$ for each positive integer k, then $P\Lambda_P(M) = 0$
(4) $\Rightarrow$ (1) Since $P\Lambda_P(M) = 0$ which implies $P(\varprojlim_k M/P^kM) = 0$, then $PM = P^kM$ for all $k \in \mathbb{Z}^+$ which implies $PM = P^2M$. □

3. APPLICATIONS OF P-SEMIPRIME AND P-SEMISECOND MODULES

If R is commutative ring and P is an ideal of R, then P-reduced and P-semiprime coincide and similarly, P-coreduced also coincide with P-semisecond modules. So, Matlis-Greenlees-May Equivalence and Greenlees-May type duality holds, see [19]. However, when R is noncommutative the above notions are different. Hence in this section we prove that Matlis-Greenlees-May Equivalence and Greenlees-May type duality holds in the settings of P-semiprime and P-semisecond modules. In this section the modules under considerations are bimodules and R-Mod:=R-R-Mod.

We denote by $(R\text{-Mod})_{P\text{-ss}}$(resp. $(R\text{-Mod})_{P\text{-sp}}$) the subcategory of R-Mod consisting of P-semisecond (resp. P-semiprime) R-modules. A left R-module is said to be P-*torsion* (resp. P-*complete*) if and only if $\Gamma_P(M) = M$ (resp. $\Lambda_P(M) = M$).

Proposition 3.1. If M is P-semisecond module and N an R-module, then $\mathrm{Hom}_R(M, N)$ is P-semiprime.

Proof. Suppose that $P^2M = PM$, then $M/P^2M = M/PM$. So, Hom_R $(R/P^2, \mathrm{Hom}_R(M, N)) \cong \mathrm{Hom}_R(R/P^2 \otimes M, N) \cong \mathrm{Hom}_R(R/P \otimes M, N) \cong \mathrm{Hom}_R(R/P, \mathrm{Hom}_R(M, N))$, then by Proposition 2.5 $\mathrm{Hom}_R(M, N)$ is P-semiprime. For the converse assume that $\mathrm{Hom}_R(M, N)$ is P-semiprime module. Then, $\mathrm{Hom}_R(R/P^2, \mathrm{Hom}_R(M, N)) \cong \mathrm{Hom}_R(R/P, \mathrm{Hom}(M, N))$.

$\mathrm{Hom}_R(R/P^2 \otimes M, N) \cong \mathrm{Hom}_R(R/P \otimes M, N)$ if and only if $\mathrm{Hom}_R(M/P^2M \otimes N) \cong \mathrm{Hom}_R(M/PM, N)$. Since N reflects isomorphism, $M/P^2M \cong M/PM$. So, M is P-semisecond. □

Proposition 3.2. For any ideal P of a ring R we have:

1. $R/P \otimes -$ and $\mathrm{Hom}_R(R/P, -)$ are idempotent functors from R-Mod to
$(R\text{-Mod})_{P\text{-ss}} \cap (R\text{-Mod})_{P\text{-sp}}$.
2. For any R-module M, $R/P \otimes \mathrm{Hom}_R(R/P, M) \cong \mathrm{Hom}_R(R/P, M)$ and
$\mathrm{Hom}_R(R/P, R/P \otimes M) \cong R/P \otimes M$.
3. For any R-module M, the R-modules $\mathrm{Hom}_R(R/P, M)$ and $R/P \otimes M$ are
both P-torsion and P-complete.

Proof.

1. $R/P \otimes (R/P \otimes M) \cong (R/P \otimes R/P) \otimes M \cong R/P \otimes M$ and $\mathrm{Hom}_R(R/P,$ $\mathrm{Hom}_R(R/P, M)) \cong \mathrm{Hom}_R(R/P \otimes R/P, M) \cong \mathrm{Hom}_R(R/P, M)$. For any

R-module M , $R/P \otimes M \cong M/PM$ and $\mathrm{Hom}_R(R/P, M) \cong (0 :_M P)$. Also, $P(M/PM) = 0$ and $P(0 :_M P) = 0$. So, the R-modules M/PM and $(0 :_M P)$ are P-semisecond. It is also easy to see that $(0 :_{(0:_M P)} P) = (0 :_{(0:_M P)} P^2) = (0 :_M P)$ and $(\bar{0} :_{M/PM} P) = (\bar{0} :_{M/PM} P^2) = M/PM$. Thus, the R-modules M/PM and $(0 :_M P)$ are P-semiprime.

2. $R/P \otimes \mathrm{Hom}_R(R/P, M) \cong \frac{\mathrm{Hom}_R(R/P,M)}{P\mathrm{Hom}_R(R/P,M)} = \mathrm{Hom}_R(R/P, M)$. $\mathrm{Hom}_R(R/P, R/P \otimes M) \cong \mathrm{Hom}_R(R/P, M/PM) = (\bar{0} :_{M/PM} P) = M/PM \cong R/P \otimes M$.
3. The following maps hold true:
   a. $\mathrm{Hom}_R(R/P, \mathrm{Hom}_R(R/P, M)) \hookrightarrow \Gamma_P(\mathrm{Hom}_R(R/P, M)) \hookrightarrow \mathrm{Hom}_R(R/P, M)$.
   b. $\mathrm{Hom}_R(R/P, R/P \otimes M) \hookrightarrow \Gamma_P(R/P \otimes M) \hookrightarrow R/P \otimes M$.
   c. $\mathrm{Hom}_R(R/P, M) \twoheadrightarrow \Lambda_P(\mathrm{Hom}_R(R/P, M)) \twoheadrightarrow R/P \otimes \mathrm{Hom}_R(R/P, M)$.
   d. $R/P \otimes M \twoheadrightarrow \Lambda_I(R/P \otimes M) \twoheadrightarrow R/P \otimes (R/P \otimes M)$,

   where $\hookrightarrow$ denotes a monomorphism and $\twoheadrightarrow$ denotes epimorphism. The first and the last maps are all isomorphisms since $\mathrm{Hom}_R(R/P, M)$ and $R/P \otimes M$ are idempotent. Moreover, $\mathrm{Hom}_R(R/P, M)$ and $R/P \otimes M$ are P-torsion and P-complete. Invariance of $R/P \otimes M$ and $\mathrm{Hom}_R(R/P, M)$ under the functor $\mathrm{Hom}_R(R/P, -)$ and $R/P \otimes -$ respectively shows that the morphisms in (b) and (c) maps are all isomorphisms. This shows that $R/P \otimes M$ and $\mathrm{Hom}_R(R/P, M)$ are P-torsion and P-complete respectively.

□

3.1. **Greenless-May type Duality.** In general, functors Γ_P and Λ_P are not adjoint to each other. However, over commutative ring their derived functors $\mathbf{R}\Gamma_P$ and $\mathbf{L}\Lambda_P$ on the derived category of R-modules are adjoint this is what is called Greenless-May duality, see [13, Theorem 7.12]. In this subsection we show that the functors Γ_P and Λ_P are adjoint in the category of P-semiprime modules and P-semisecond modules, for an ideal P of R (noncommutative).

Lemma 3.3. For any ideal P of a ring R,

1. The functor $\Gamma_P(-) : (R\text{-Mod})_{P\text{-sp}} \to (R\text{-Mod})_{P\text{-ss}}$ is idempotent and for any $M \in (R\text{-Mod})_{P\text{-sp}}$, $\Gamma_P(M) \cong \mathrm{Hom}_R(R/P, M)$.
2. The functor $\Lambda_P : (R\text{-Mod})_{P\text{-ss}} \to (R\text{-Mod})_{P\text{-sp}}$ is idempotent and for any $M \in (R\text{-Mod})_{P\text{-ss}}$, $\Lambda_P(M) \cong R/P \otimes_R M$.

Proof.

1. It follows from Proposition 2.5 (5) and Proposition 3.2 (1).
2. Follows from Proposition 2.10 (3) and Proposition 3.2 (1).

□

Theorem 3.4 (GM type Duality in R-Mod)**.** For any ideal P of a ring R and any $N \in (R\text{-Mod})_{P\text{-sp}}$ and $M \in (R\text{-Mod})_{P\text{-ss}}$,

$$\mathrm{Hom}_R(\Lambda_P(M), N) \cong \mathrm{Hom}_R(M, \Gamma_P(N)).$$

Proof. Consider the functor $\Gamma_P(-) : (R\text{-Mod})_{P\text{-sp}} \to (R\text{-Mod})_{P\text{-ss}}$. For any module $M \in (R\text{-Mod})_{P\text{-sp}}, \Gamma_P(M) \cong \mathrm{Hom}_R(R/P, M)$, Lemma 3.3 (1). However, the functor $R/P \otimes -$ is left-adjoint to $\mathrm{Hom}_R(R/P, -)$. By uniqueness of adjoints, the functor $\Lambda_I(-) : (R\text{-Mod})_{P\text{-ss}} \to (R\text{-Mod})_{I\text{-sp}}$ which has the property that for all $M \in (R\text{-Mod})_{P\text{-ss}}$ $\Lambda_P(M), \cong R/P \otimes M$, Lemma 3.3 (2). Then, Λ_P is the left adjoint of Γ_P. □

3.2. **Matlis-Greenless-May Equality.** Let R be a commutative ring and $\mathbf{D}(R)$ denote the derived category of the abelian category R-Mod. The Matlis-Greenless-May Equivalence (MGM) duality on derived category is given as:

Theorem 3.5. [MGM Equivalence] [13, Theorem 7.11] Let R be a ring, and P be a weakly proregular ideal in it.

1. If $M \in \mathbf{D}(R)$, then $\mathbf{R}\Gamma_P(M) \in \mathbf{D}(R)_{P\text{-tor}}$ and $\mathbf{L}\Lambda_P(M) \in \mathbf{D}(R)_{P\text{-com}}$.
2. The functor $\mathbf{R}\Gamma_P(-) : \mathbf{D}(R)_{P\text{-com}} \to \mathbf{D}(R)_{P\text{-tor}}$ is an equivalence, with quasi-inverse $\mathbf{L}\Lambda_P$.

In this subsection we prove the MGM Equality in the setting of P-semiprime and P-semisecond modules.

Proposition 3.6. A left R-module M is P-torsion and P-semiprime if and only if M is P-complete and P-semisecond.

Proof. Suppose M be P-torsion and P-semiprime. $M = \Gamma_P(M) = \mathrm{Hom}_R(R/P, M)$
$= (0 :_M P)$, hence it follows that $PM = 0$ which implies $P^k M = 0$ for any $k \in \mathbb{Z}^+$ then $\Lambda_P(M) = M$ and $P^2M = PM$. Conversely, let M be an P-complete and P-semisecond. To show it is P-semiprime, let $P^k M = 0$ for some $k \in \mathbb{Z}^+$, but since M is P-complete the previous relation satisfied for all $k \in \mathbb{Z}^+$, thus $PM = 0$. Now by Proposition 2.6, $\Gamma_P(M) = \mathrm{Hom}_R(R/P, M) = (0 :_M P) = M$. □

Lemma 3.7. If P is an ideal of a ring R and M a P-semiprime (resp. P-semisecond) R-module, then $\Gamma_P(M)$ (resp. $\Lambda_P(M)$) is a P-complete (resp. P-torsion) R-module.

Proof. Suppose M is P-semiprime. Then by Proposition 3.2 $\Gamma_P(M) = \operatorname{Hom}_R(R/P,$
$M)$ is both an P-semiprime and P-semisecond R-module. To show that $\Gamma_P(M)$ is P-complete, $\Lambda_P(\Gamma_P(M)) \cong R/P \otimes \operatorname{Hom}_R(R/P, M) \cong \operatorname{Hom}_R(R/P, M) \cong \Gamma_P(M)$. Let M be P-semisecond, by Proposition 3.2, $\Lambda_P(M) \cong R/P \otimes M$ which is also both an P-semiprime and P-semisecond R-module. Now, $\operatorname{Hom}_R(R/P, \Lambda_P(M)) \cong \operatorname{Hom}_R(R/P, R/P\otimes M) \cong R/P \otimes M \cong \Lambda_P(M)$. This proves that $\Lambda_P(M)$ is P-torsion. □

Let $\mathcal{C} := (R\text{-Mod})_{P\text{-com}} \cap (R\text{-Mod})_{P\text{-ss}}$ and $\mathcal{T} := (R\text{-Mod})_{P\text{-tor}} \cap (R\text{-Mod})_{P\text{-sm}}$.

Theorem 3.8 (MGM Equality)**.** Let P be any ideal of a ring R,

1. If $M \in (R\text{-Mod})_{P\text{-sm}}$, then $\Gamma_P(M) \in \mathcal{C}$ and if $M \in (R\text{-Mod})_{P\text{-ss}}$, then $\Lambda_p(M) \in \mathcal{T}$.
2. The functor $\Gamma_P(-) : R\text{-Mod}_{P\text{-sm}} \to R\text{-Mod})_{P\text{-ss}}$ restricted to $\mathcal{T}$ is equality between $\mathcal{C}$ and $\mathcal{T}$ with quasi inverse Λ_P.

Proof.
1. Let $M \in (R\text{-Mod})_{P\text{-sm}}$ then by Theorem 3.4 it follows that $\Gamma_P(M) \in (R\text{-Mod})_{P\text{-ss}}$ and by Lemma 3.7, $\Gamma_P(M) \in (R\text{-Mod})_{P\text{-com}}$. Then $\Gamma_P(M) \in \mathcal{C}$. Similarly, applying Theorem 3.4 and Lemma 3.7 we get $\Lambda_P(M) \in \mathcal{T}$.
2. Proposition 3.6 and Lemma 3.7 assures that there is equality between the categories $\mathcal{C}$ and $\mathcal{T}$ which is the Matlis-Greenless-May Equality for R-modules holds.

□

4. THE FUNCTOR $P\Gamma_P$ OVER RINGS

In this section we study some properties of the ideal $P\Gamma_P(R)$ and we use it to formulate Köthe conjecture.

Lemma 4.1.
1. *If P is a left ideal of R, then $P\Gamma_P(R)$ is a two sided ideal of R.*
2. *If P is a right ideal of R, then $P\Gamma_P(R)$ is a right ideal of R.*

Proof.
1. Suppose P is a left ideal of R. Let $r \in R$ and $x \in P\Gamma_I(R)$, thus $x = \sum_{i=1}^{l} a_i r_i$ such that $P^{k_i} r_i = 0$, for some positive integers k_i, where $a_i \in P$ and $r_i \in R$. Now, $rx = \sum_{i=1}^{l} (ra_i) r_i$,

since P is left ideal $ra_i \in I$ and by hypothesis $P^{k_i}r_i = 0$. Then $x \in P\Gamma_P(R)$ and hence $P\Gamma_P(R)$ is left ideal. To show it is right ideal, $xr = \sum_{i=1}^{l} a_i(r_ir)$, multiplying $P^{k_i}r_i = 0$ from the right by r we get $P^{k_i}r_ir = 0$ which shows that $xr \in I\Gamma_P(R)$ and hence it is right ideal of R.

2. Suppose P is a right ideal of R. Let $r \in R$ and $x \in P\Gamma_P(R)$, thus $x = \sum_{i=1}^{l} a_ir_i$ such that $P^{k_i}r_i = 0$, for some positive integers k_i, where $a_i \in P$ and $r_i \in R$. To show $P\Gamma_P(R)$ is right ideal of R, $xr = \sum_{i=1}^{l} a_i(r_ir)$, multiplying $P^{k_i}r_i = 0$ from the right by r we get $P^{k_i}r_ir = 0$ which shows that $xr \in P\Gamma_P(R)$ and hence it is right ideal of R.

□

Proposition 4.2. For any right ideal P of a ring R, $P\Gamma_P(R)$ is a nil right ideal.

Proof. By Lemma 4.1 (2), $P\Gamma_P(R)$ is a right ideal, whenever P is a right ideal of R. Let $y \in P\Gamma_P(R)$. Then $y = \sum_{i=1}^{l} a_ir_i$ where $r_i \in R$ and $a_i \in P$ such that $P^{n_i}r_i = 0$ for each $1 \leq i \leq l$ and let $n = n_1 + \cdots + n_l$. Then, $y^n = (\sum_{i=1}^{l} a_ir_i)^n = (a_1r_1)^n + (a_1r_1)(a_2r_2)^{n-1} + \cdots + (a_1r_1)(a_ir_i)^{n-1} + \cdots$ (l^n terms) each of total degree n. Now, $(a_1r_1)(a_ir_i)^{n-1} = (a_1r_1)(a_ir_i)$ $(a_ir_i)\cdots(a_i)r_i$, i.e., this is a product of one a_1r_1 term, $n-2$ terms of a_ir_i, one term of a_i and r_i, then $(a_1r_1)(a_ir_i)^{n-1} = (a_1r_1)(a_ir_i)(a_ir_i)\cdots(a_i)r_i \subseteq PP^{n-2}Pr_i = P^nr_i = 0$. In a similar fashion every term in the expression y^n undergoes such steps and then $y^n = 0$ and hence $P\Gamma_P(R)$ is a nil right ideal. □

Corollary 4.3. Let R be a ring and $\mathcal{U}$ denotes the upper nilradical of R.

1. For any ideal P of R
   a. $P\Gamma_P(R)$ is nil.
   b. $\sum_{P \lhd R} P\Gamma_P(R) \subseteq \mathcal{U}(R)$.
2. If R is P-torsion for any ideal P of R, then every ideal P is nil and $\sum_{P \lhd R} P\Gamma_P(R) = \mathcal{U}(R)$

Proof. 1. a and b are direct consequences of Proposition 4.2.
2. By hypothesis $\Gamma_P(R) = R$ for all ideals I of R which implies $P\Gamma_P(R) = PR = P$. By 1a) P is nil, so $\sum_{P \lhd R} P\Gamma_P(R) = \sum_{P \lhd R} P = \mathcal{U}(R)$.

□

Corollary 4.4. *Let R be a right noetherian ring and P be a right ideal of R, then $P\Gamma_P(R)$ is nilpotent.*

Proof. By proposition 2.9, $P\Gamma_P(R)$ is a nil right ideal, for any right ideal P of R then by [10, Levitzki's Theorem] $P\Gamma_P(R)$ is nilpotent. □

Proposition 4.5. For a ring R, $P\Gamma_P(R[x]) = (P\Gamma_p(R))[x]$.

Proof. Let $f(x) \in P\Gamma_p(R)[x]$. Then $f(x) = \sum_{i=0}^{n} a_{ij}x^i$, where $a_{ij} \in P\Gamma_P(R)$ which implies for each i there exists S_{ij} and $k_i \in \mathbb{Z}^+$ such that $I^{k_j}s_{ij} = 0$ and $a_{ij} = \sum_{j=0}^{k} r_{ij}s_{ij}$. Now, $f(x) = \sum_{i=1}^{n} a_{ij}x^i = \sum_{i=1}^{n}(\sum_{j=0}^{k} r_{ij}s_{ij})x^i = \sum_{i=0}^{n}(r_{i0}s_{i0} + \cdots + r_{ik}s_{ik})x^i = \sum_{i=0}^{n} r_{i0}(s_{i0}x^i) + \cdots + r_{ik}(s_{ik}x^i)$. Since $I^{k_j}s_{ij} = 0$ it follows that $s_{ij}x^i \in \Gamma_P(R[x])$. Thus, $f(x) \in P\Gamma_P(R[x])$. Suppose $g(x) = \sum_{i=0}^{n} a_i f_i(x) \in P\Gamma_P(R[x])$, where $f_i(x) = (r_0 + r_1 x + \cdots + r_n x^n)_i$ and $P^{k_i}f_i(x) = 0$ which implies $P^{k_i}r_i = 0$ for each $i, 0 \leq i \leq n$ and thus $\sum_{i=0}^{n} a_i r_i \in P\Gamma_P(R)$. Therefore, $g(x) = \sum_{i=0}^{n} a_i f_i(x) = \sum_{i=0}^{n}(a_i r_i)x^i$, so $g(x) \in P\Gamma_P(R)[x]$.

□

Proposition 4.6. Let J_1 and J_2 be ideals of R and P be a right ideal of R. Then $P\Gamma_P(J_1) + I\Gamma_P(J_2)$ is a nil right ideal of R.

Proof. $P\Gamma_P(J_1) + P\Gamma_P(J_2) = P\Gamma_P(J_1 + J_2) \subseteq P\Gamma_P(R)$. From Proposition 4.2 $P\Gamma_P(R)$ is a right nil ideal of R, then it follows that $P\Gamma_P(J_1) + P\Gamma_P(J_2)$ is a right nil ideal of R. □

Corollary 4.7. If R is a ring such that all its right sided nil ideals are of the form $P\Gamma_P(J)$ for some ideal J of R, then R satisfies the Köthe conjecture

question 4.8. Let P_1 and P_2 be two right ideals of a ring R. Is the sum $P_1\Gamma_{P_1}(R) + P_2\Gamma_{P_2}(R)$ a nil right ideal? A negative answer to this question would answer the Köthe conjecture in the negative.

question 4.9. Is $\sum_{P \triangleleft_r R} P\Gamma_P(R) = \mathcal{U}(R)$? A positive answer to this question would solve the Köthe conjecture in the affirmative.

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