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ON CARTAN TORSION OF 4-DIMENSIONAL FINSLER MANIFOLDS

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Abstract. There are several non-Riemannian curvatures in Finsler geometry which show the complexity of Finsler geometry with respect to Riemannian geometry. Among these quantities, the Cartan and mean Cartan torsion have very important and brilliant positions. In this paper, we find the necessary and sufficient condition under which a 4-dimensional Finsler manifold is C-reducible. Also, we find the necessary and sufficient condition under which an arbitrary 4-dimensional Finsler manifold has vanishing $\bar{I}$-curvature.

Key Words: Cartan torsion, mean Cartan torsion, C-reducible metric.
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1. Introduction

There are several important non-Riemannian quantities in Finsler geometry which show the complexity of Finsler geometry with respect to Riemannian geometry. Among these quantities, the Cartan and mean Cartan torsion have very important and brilliant positions (see [1], [9], [11], [12], [13], [14] and [16]). For a Finsler manifold (M, F), the second and third order derivatives of $1/2F_x^2$ at $y \in T_xM_0$ are inner products $\mathbf{g}_y$ and symmetric trilinear forms $\mathbf{C}_y$ on T_xM, respectively. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. The

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Cartan torsion is one of the most important non-Riemannian quantity in Finsler geometry and it was first introduced by Finsler [5] and emphased by Cartan [2]. A Finsler metric reduces to a Riemannian metric if and only if it has vanishing Cartan torsion. Taking a trace of Cartan torsion yields the mean Cartan torsion $\mathbf{I}_y$. In [4], Deicke proves that a positive definite Finsler metric F is Riemannian if and only if the mean Cartan torsion vanishes.

However, the Cartan torsion of special Finsler metrics explains that this quantity needs more attention. For example, according to Matsumoto-Hōjō's conclusive theorem, a Finsler manifold of dimension $n \geq 3$ is of the Randers or Kropina-type if and only if its Cartan torsion satisfies in the C-reducibility condition. Namely, its Cartan torsion is given by

$$\mathbf{C}_y(u,v,w) = \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v,w) + \mathbf{I}_y(v)\mathbf{h}_y(u,w) + \mathbf{I}_y(w)\mathbf{h}_y(u,v)\Big\},$$

where

$$\mathbf{h}_y(u,v) = \mathbf{g}_y(u,v) - F^{-2}(y)\mathbf{g}_y(y,u)\mathbf{g}_y(y,v).$$

It is remarkable that $\mathbf{h}_y(u,v)$ is called the angular form in direction y. A Randers metric on a manifold M is a positive scalar function on TM defined by $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M (see [3]). The Kropina metrics $F = \alpha^2/\beta$ are closely related to physical theories. These metrics, was introduced by Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and was investigated by Kropina [7]. All of these metrics are special Finsler metrics so- called (α,β)-metrics. Also, every Finsler surface is C-reducible.

The 4-dimensional Finsler manifolds have an interesting history. In 1941, Randers published a paper concerned with an asymmetric metric in the four-space of general relativity. His metric is in the form $F = \alpha + \beta$, where α is gravitation field and β is the electromagnetic field. He regarded these metrics not as Finsler metrics but as “affinely connected Riemannian metrics”. This metric was first recognized as a kind of Finsler metric in 1957 by Ingarden [6], who first named them Randers metrics. It is interesting to find the necessary and sufficient condition under which a 4-dimensional Finsler manifold is C-reducible. In this paper, we consider the class of 4-dimensional Finsler manifolds and find the necessary and sufficient condition under which a 4-dimensional Finsler manifold is C-reducible. More precisely, we prove the following.

Theorem 1.1. *A 4-dimensional Finsler manifold (M,F) is C-reducible if and only if its main scalars satisfy*

$$\mathcal{A}=3\mathcal{B}=3\mathcal{C}, \tag{1.1}$$

$$\mathcal{D}=\mathcal{E}=\mathcal{F}=\mathcal{G}=\mathcal{H}=0, \tag{1.2}$$

where $\mathcal{A}=\mathcal{A}(x,y)$, $\mathcal{B}=\mathcal{B}(x,y)$, $\mathcal{C}=\mathcal{C}(x,y)$, $\mathcal{D}=\mathcal{D}(x,y)$, $\mathcal{E}=\mathcal{E}(x,y)$ and $\mathcal{F}=\mathcal{F}(x,y)$ are scalar functions on TM and called the main scalars of F.

In [15], Shen defined a new non-Riemannian quantity that is close to the mean Cartan torsion and mean Landsberg curvature. Indeed, by taking a horizontal derivation of mean Cartan torsion, one can find a new quantity $\bar{\mathbf{I}}$, namely,

$$\bar{\mathbf{I}}:=\nabla_l\mathbf{I},$$

where ∇_l denotes the horizontal derivation with respect to the Berwald connection of F. Every Riemannian metric has vanishing $\bar{\mathbf{I}}$-curvature. In this paper, we find the necessary and sufficient condition under which a 4-dimensional Finsler manifold has vanishing $\bar{I}$-curvature.

Theorem 1.2. *Let (M,F) be a 4-dimensional Finsler manifold. Then F satisfies $\bar{\mathbf{I}}=0$ if and only if its main scalars satisfy*

$$\mathcal{A}_{|i}+\mathcal{B}_{|i}+\mathcal{C}_{|i}=0, \quad h_i=0, \quad j_i=0. \tag{1.3}$$

where $h_i=h_i(x,y)$ and $j_i=j_i(x,y)$ are called the h-connection vectors.

2. PRELIMINARY

Let M be an n-dimensional C^∞ manifold, $TM=\bigcup_{x\in M}T_xM$ the tangent bundle and $TM_0:=TM-\{0\}$ the slit tangent bundle. A Finsler structure on M is a function $F:TM\to[0,\infty)$ with the following properties:
(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x,\lambda y)=\lambda F(x,y)$, $\forall\lambda>0$;
(iii) The following quadratic form $\mathbf{g}_y:T_xM\times T_xM\to\mathbb{R}$ is positively defined on TM_0

$$\mathbf{g}_y(u,v):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y+su+tv)\Big]_{s=t=0}, \quad u,v\in T_xM.$$

The pair (M,F) is called a Finsler manifold.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]_{t=0}, \quad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y \in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y$, $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where

$$I_i := g^{jk}C_{ijk}.$$

Let (M,F) be an n-dimensional Finsler manifold. For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u,v,w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\Big\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\Big\},$$

$h_{ij} := FF_{y^iy^j}$ is the angular metric.

Lemma 2.1. ([8]) A Finsler metric F on a manifold M of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

Let us define $\bar{\mathbf{I}}_y : T_xM \times T_xM \to \mathbb{R}$ by $\bar{\mathbf{I}}_y(u,v) = \bar{I}_{ij}u^iv^j$, where

$$\bar{I}_{ij} := I_{i|j}.$$

Here, "|" denotes the horizontal covariant differentiation with respect to the Berwald connection. Thus $\bar{\mathbf{I}}$-curvature is defined as the horizontal derivation of mean Cartan torsion.

The horizontal covariant derivatives of the Cartan torsion $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k$, where

$$L_{ijk} := C_{ijk|s}y^s.$$

The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature.

Throughout this paper, we use the Berwald connection ∇ on Finsler manifolds. Let (M,F) be an n-dimensional Finsler manifold. Let $\{e_j\}$ be a local frame for π^*TM, $\{\omega^i,\omega^{n+i}\}$ be the corresponding local coframe for $T*(TM_0)$ and $\{\omega^i_j\}$ be the set of local Berwald connection forms with respect to $\{e_j\}$. Then the connection forms are characterized by the structure equations as follows

- Torsion freeness:

$$d\omega^i = \omega^j \wedge \omega^i{}_j. \tag{2.1}$$

- Almost metric compatibility:

$$dg_{ij} - g_{kj}\omega^k{}_i - g_{ik}\omega^k{}_j = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}, \tag{2.2}$$

where $\omega^i := dx^i$ and $\omega^{n+k} := dy^k + y^j\omega^k{}_j$.

The horizontal and vertical covariant derivations with respect to the Berwald connection respectively are denoted by "||" and ", ". For more details, one can see [15].

3. Proof of Theorems

In this section, we are going to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1: Let (M,F) be a 4-dimensional Finsler manifold. Suppose that

$$\ell_i := F_{y^i} = \frac{\partial F}{\partial y^i}$$

is the unit vector along the element of support, m_i is the unit vector along mean Cartan torsion I_i, i.e.,

$$m_i := \frac{1}{||\mathbf{I}||} I_i,$$

where $||\mathbf{I}|| := \sqrt{g^{ij}I_iI_j}$, and n_i and p_i are unit vectors orthogonal to the vectors ℓ_i and m_i. Then the quadruple (ℓ_i, m_i, n_i, p_i) is called the Miron frame. In this frame, we have

$$g_{ij} = \ell_i\ell_j + m_im_j + n_in_j + p_ip_j, \tag{3.1}$$

$$g^{ij} = \ell^i\ell^j + m^im^j + n^in^j + p^ip^j. \tag{3.2}$$

Thus

$$h_{ij} = m_im_j + n_in_j + p_ip_j.$$

Taking a vertical derivative of (3.1) yields the Cartan torsion as follows

$$\begin{aligned} FC_{ijk} = {} & \mathcal{A}m_im_jm_k + \mathcal{B}(m_in_jn_k + n_im_jn_k + n_in_jm_k) + \mathcal{C}(m_ip_jp_k \\ & + p_im_jp_k + p_ip_jm_k) + \mathcal{D}(m_im_jn_k + m_in_jm_k + n_im_jm_k) \\ & + \mathcal{E}n_in_jn_k + \mathcal{F}(m_im_jp_k + m_ip_jm_k + p_im_jm_k) + \mathcal{G}(n_in_jp_k \\ & + n_ip_jn_k + p_in_jn_k) + \mathcal{H}(m_in_jp_k + m_ip_jn_k + n_im_jp_k + n_ip_jm_k \\ & + p_im_jn_k + p_in_jm_k) - (\mathcal{D} + \mathcal{E})(n_ip_jp_k + p_in_jp_k + p_ip_jn_k) \\ & - (\mathcal{F} + \mathcal{G})p_ip_jp_k, \end{aligned} \tag{3.3}$$

where $\mathcal{A} = \mathcal{A}(x,y)$, $\mathcal{B} = \mathcal{B}(x,y)$, $\mathcal{C} = \mathcal{C}(x,y)$, $\mathcal{D} = \mathcal{D}(x,y)$, $\mathcal{E} = \mathcal{E}(x,y)$ and $\mathcal{F} = \mathcal{F}(x,y)$ are scalar functions on TM and called the main scalars of F. By (3.3), we have

$$FI_k = (\mathcal{A} + \mathcal{B} + \mathcal{C})m_k. \tag{3.4}$$

Then, we get

$$\begin{aligned} h_{ij}I_k + h_{jk}I_i + h_{ki}I_j = (\mathcal{A} + \mathcal{B} + \mathcal{C})\Big\{ & 3m_im_jm_k + m_kn_in_j + m_kp_ip_j \\ & + m_in_jn_k + m_ip_jp_k + m_jn_in_k + m_jp_ip_k\Big\}. \end{aligned} \tag{3.5}$$

By (3.3) and (3.5), we get

$$\begin{aligned} & (\mathcal{A} + \mathcal{B} + \mathcal{C})\Big\{3m_im_jm_k + m_kn_in_j + m_kp_ip_j + m_in_jn_k + m_ip_jp_k \\ & + m_jn_in_k + m_jp_ip_k\Big\} - 4\mathcal{A}m_im_jm_k - 4\mathcal{B}\big(m_in_jn_k + n_im_jn_k \\ & + n_in_jm_k\big) - 4\mathcal{C}\big(m_ip_jp_k + p_im_jp_k + p_ip_jm_k\big) - 4\mathcal{D}\big(m_im_jn_k \\ & + m_in_jm_k + n_im_jm_k\big) - 4\mathcal{E}n_in_jn_k - 4\mathcal{F}\big(m_im_jp_k + m_ip_jm_k \\ & + p_im_jm_k\big) - 4\mathcal{G}\big(n_in_jp_k + n_ip_jn_k + p_in_jn_k\big) - 4\mathcal{H}\big(m_in_jp_k \\ & + m_ip_jn_k + n_im_jp_k + n_ip_jm_k + p_im_jn_k + p_in_jm_k\big) \\ & + 4\big(\mathcal{D} + \mathcal{E}\big)\big(n_ip_jp_k + p_in_jp_k + p_ip_jn_k\big) \\ & + 4\big(\mathcal{F} + \mathcal{G}\big)p_ip_jp_k = 0, \end{aligned} \tag{3.6}$$

which yields

$$\frac{3}{4}(\mathcal{A}+\mathcal{B}+\mathcal{C})=\mathcal{A}, \tag{3.7}$$

$$\frac{1}{4}(\mathcal{A}+\mathcal{B}+\mathcal{C})=\mathcal{B}, \tag{3.8}$$

$$\frac{1}{4}(\mathcal{A}+\mathcal{B}+\mathcal{C})=\mathcal{C}, \tag{3.9}$$

$$\mathcal{D}=\mathcal{E}=\mathcal{F}=\mathcal{G}=\mathcal{H}=0. \tag{3.10}$$

By (3.7), (3.8) and (3.9), we get (1.1). This completes the proof. □

Proof of Theorem 1.2: The horizontal derivation of Miron frame are given by following

$$\begin{aligned}
&\ell_{i|j}=0,\\
&m_{i|j}=h_j n_i+j_j p_i,\\
&n_{i|j}=k_j p_i-h_j m_i,\\
&p_{i|j}=-j_j m_i-k_j n_i,
\end{aligned}$$

where $h_s=h_s(x,y)$, $j_s=j_s(x,y)$ and $k_s=k_s(x,y)$ are called the h-connection vectors (for more details, see [10]). Taking a horizontal derivation of (3.4) implies

$$FI_{i|j}=(\mathcal{A}_{|j}+\mathcal{B}_{|j}+\mathcal{C}_{|j})m_i+(\mathcal{A}+\mathcal{B}+\mathcal{C})(h_j n_i+j_j p_i). \tag{3.11}$$

By assumption, (3.11) gives us

$$(\mathcal{A}_{|j}+\mathcal{B}_{|j}+\mathcal{C}_{|j})m_i+(\mathcal{A}+\mathcal{B}+\mathcal{C})(h_j n_i+j_j p_i)=0. \tag{3.12}$$

Contracting (3.12) with m^i and using $n_i m^i=p_i m^i=0$ implies that

$$\mathcal{A}_{|j}+\mathcal{B}_{|j}+\mathcal{C}_{|j}=0. \tag{3.13}$$

By multiplying (3.12) with n^i, and using $m_i n^i=p_i n^i=0$ yields

$$h_j=0. \tag{3.14}$$

Also, contracting (3.12) with p^i, and using $m_i p^i=n_i p^i=0$ and $p_i p^i=1$ gives us

$$j_j=0. \tag{3.15}$$

This completes the proof. □

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