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Research Paper

ON THE SEIDEL LAPLACIAN SPECTRUM OF THRESHOLD GRAPHS

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ABSTRACT

A graph which does not contain C_4, P_4 or $2K_2$ as its induced subgraphs, is called a threshold graph. In this paper, we consider Seidel Laplacian matrix of a connected threshold graph and determine Seidel Laplacian spectrum. Also the characterization of threshold graphs having atmost four distinct Seidel Laplacian eigenvalues have been done.

1. Introduction

For a connected graph $\mathcal{G}$ with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, the adjacency matrix of $\mathcal{G}$ is denoted by $\mathcal{A}_\mathcal{G}$ and defined as $\mathcal{A}_\mathcal{G} = (a_{ij})$ where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The energy of the graph $\mathcal{G}$ is denoted by$E_\mathcal{A}$ and defined as sum of the absolute values of the eigenvalues of $\mathcal{A}_\mathcal{G}$. The study of spectrum of adjacency matrix of a graph paved a way to the area of spectral graph theory. Recently, graph energies related to different graph matrices such as laplacian matrix, seidel matrix etc have been introduced and studied by Furtula B, Gutman I, Merris R, et al. [7, 8, 18, 19, 21] . We use $D_\mathcal{G}$ to denote

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the diagonal degree matrix. That is,

$$D_{\mathcal{G}} = \begin{bmatrix} deg(v_1) & 0 & \cdots & 0 \\ 0 & deg(v_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & deg(v_n) \end{bmatrix}.$$

The Laplacian matrix of $\mathcal{G}$, denoted by $\mathcal{L}_{\mathcal{G}}$ is defined as $\mathcal{L}_{\mathcal{G}} = D_{\mathcal{G}} - \mathcal{A}_{\mathcal{G}}$. Let $\mu_1, \cdots, \mu_n$ be eigenvalues of $\mathcal{L}_{\mathcal{G}}$. The Laplacian energy of $\mathcal{G}$ was defined as $E_{\mathcal{L}} = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$, where m denotes the number of edges in $\mathcal{G}$. Van Lint and Seidel [9] defined the Seidel matrix $\mathcal{S}_{\mathcal{G}}$ of $\mathcal{G}$ as $\mathcal{S}_{\mathcal{G}} = (s_{ij})$ where

$$s_{ij} = \begin{cases} -1 & \text{if } v_i \text{ is adjacent to } v_j \\ 1 & \text{if } v_i \text{ is not adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

.

Clearly $\mathcal{S}_{\mathcal{G}} = J_n - I_n - 2\mathcal{A}_{\mathcal{G}}$, where I_n is the identity matrix of order n and J_n is $n \times n$ matrix with all entries are 1. Let $\theta_1, \cdots, \theta_n$ be the eigenvalues of $\mathcal{S}_{\mathcal{G}}$. The Seidel energy of the graph $\mathcal{G}$ was defined as $E_{\mathcal{S}_{\mathcal{G}}} = \sum_{i=1}^{n} |\theta_i|$ [10]. The Seidel Laplacian matrix of graph was introduced and properties of their spectrum and Seidel Laplacian energy were studied in detail by Gutman I, Jummannaver R.B and Ramane H.S [11]. Let $\mathcal{D}_{\mathcal{S}_{\mathcal{G}}}$ denote $Diag(k_{ii})$ where $k_{ii} = n-1-2deg(v_i)$. The Seidel Laplacian matrix is given by $\mathcal{SL}_{\mathcal{G}} = \mathcal{D}_{\mathcal{S}_{\mathcal{G}}} - \mathcal{S}_{\mathcal{G}}$. Let $\sigma_1, \cdots, \sigma_n$ be the eigenvalues of $\mathcal{SL}_{\mathcal{G}}$. The multiset of Seidel Laplacian eigenvalues of the graph G is the Seidel Laplacian spectrum, denoted by $Spec_{\mathcal{SL}_{\mathcal{G}}} = \{\sigma_1^{m_1}, \cdots, \sigma_l^{m_l}\}$ where m_i is the multiplicity of σ_i. The Seidel Laplacian energy was given by $E_{\mathcal{SL}_{\mathcal{G}}} = \sum_{i=1}^{l} m_i|\sigma_i - \frac{n(n-1)-4m}{n}|$[11].

The concept of equitable partition acts as a strong tool in the study of spectral graph theory. An idea about equitable partitions of symmetric matrices are given as follows. Let $M = (m_{ij})$ be a symmetric real matrix of order n. Let $X = \{1, 2, \cdots, n\}$. Let $\Pi = \{X_1, X_2 \cdots, X_k\}$ be a partition of X. Then the matrix M can be written as

$$M = \begin{pmatrix} M_{1,1} & \cdots & M_{1,k} \\ \vdots & & \vdots \\ M_{k,1} & \cdots & M_{k,k} \end{pmatrix}$$

where $M_{i,j}$ is the submatrix of M defined by $M_{i,j} = (m_{rs})$ where $r \in X_r,\ s \in X_s; r, s = 1, 2, \cdots, k$. The characteristic matrix $P = (p_{ij})$ of Π is the $n \times k$ matrix such that

$$p_{ij} = \begin{cases} 1 & \text{if } i \in X_j \\ 0 & \text{otherwise} \end{cases}$$

and its j^{th} column is the characteristic vector of X_j for $1 \leq j \leq k$. If $q_{i,j}$ is the average row sum of $M_{i,j}$ then $Q_M = (q_{ij})$ is the quotient matrix of M. If the each block has constant row sum, then the partition Π is called equitable partition [4]. The following result is well known on an equitable parition of a matrix. "Let M be a real symmetric matrix and let Π be an equitable partition of M with quotient matrix Q_M. Then the characteristic polynomial of the quotient matrix Q_M divides the characteristic polynomial of M.[17]"

This paper concentrates on the Seidel Laplacian spectrum of threshold graphs. Threshold graphs are $\{C_4, P_4, 2K_2\}$-free graphs. In literature, threshold graphs are also defined based on its binary string representation. The repetitive process of constructing threshold graph on n vertices as follows, K_1 is threshold graph with one vertex. Assuming H is a threshold graph on $(n-1)$ vertices, a threshold graph on n vertices can be formed by adding a new vertex v such that either v is adjacent to all vertices in H (dominating vertex) or v is nonadjacent to all vertices in H (isolated vertex).

If $\mathcal{T}$ is a threshold graph on n vertices such that v_i is the added vertex in the i^{th} step of operations, then $\mathcal{T}$ can be obtained by the binary string $(t_1t_2\cdots t_n)$ such that $t_1 = 0$ and

$$t_i = \begin{cases} 0, \text{if } v_i \text{ is an isolated vertex} \\ 1, \text{if } v_i \text{ is a dominating vertex} \end{cases}$$

for $i \geq 2$.

Threshold graphs were introduced by Chvtal and Hammer [5] and Henderson and Zalcstein [6] in 1977 and these graphs having numerous applications in areas including computer science, psychology and so on [5]. The spectral properties of the adjacency matrix of threshold graphs have been studied by Sciriha, Farrugia in 2011 [23]. Bapat [12] derived formulas for the determinant, the inverse, when it exists, and inertia of the adjacency matrix of a threshold graph. D.P. Jacobs, Trevisan and Tura [13, 15] presented algorithms to locate eigenvalues and to compute characteristic polynomial of a threshold graph. They showed that all the eigenvalues of threshold graph other than 0 or -1, are simple [14]. Normalized spectrum of a threshold graph have been studied by Anirban Banerjee and Ranjit Mehatari in 2017 [16].

In this paper, the Seidel Laplacian spectrum of threshold graphs has been studied. Also threshold graphs with at most four distinct Seidel Laplacian eigenvalues have been characterized. In coming discussions, we use the term T_H- graphs instead of threshold graphs.

2. Seidel Laplacian Spectrum of T_H- graphs

2.1. **Seidel Laplacian matrix of T_H- graph.** Let $\mathcal{T}$ be a connected T_H graph by the binary string $(0^{a_1}, 1^{b_1}, \cdots, 0^{a_k}, 1^{b_k})$. We set $a = \sum_{i=1}^{k} a_i$, and $b = \sum_{j=1}^{k} b_j$, $a_i, b_j \geq 1$, for $1 \leq i, j \leq k$. By the binary sequence of $\mathcal{T}$, we have $V(\mathcal{T}) = U_1 \cup V_1 \cup U_2 \cup V_2 \cdots \cup U_k \cup V_k$, where U_1 contains a_1 vertices, V_1 contains b_1 vertices and so on. The Seidel matrix of $\mathcal{T}$, $\mathcal{S}_{\mathcal{T}}$ is a square matrix of order $a+b$ given by

$$\mathcal{S}_{\mathcal{T}} = \begin{bmatrix} (J_{a_1} - I_{a_1}) & -J_{a_1\times b_1} & J_{a_1\times a_2} & -J_{a_1\times b_2} & \cdots & J_{a_1\times a_k} & -J_{a_1\times b_k} \\ -J_{b_1\times a_1} & (I_{b_1} - J_{b_1}) & J_{b_1\times a_2} & -J_{b_1\times b_2} & \cdots & J_{b_1\times a_k} & -J_{b_1\times b_k} \\ J_{a_2\times a_1} & J_{a_2\times b_1} & (J_{a_2} - I_{a_2}) & -J_{a_2\times b_2} & \cdots & J_{a_2\times a_k} & -J_{a_2\times b_k} \\ -J_{b_2\times a_1} & -J_{b_2\times b_1} & -J_{b_2\times a_2} & (I_{b_2} - J_{b_2}) & \cdots & J_{b_2\times a_k} & -J_{b_2\times b_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ J_{a_k\times a_1} & J_{a_k\times b_1} & J_{a_k\times a_2} & J_{a_k\times b_2} & \cdots & (J_{a_k} - I_{a_k}) & -J_{a_k\times b_k} \\ -J_{b_k\times a_1} & -J_{b_k\times b_1} & -J_{b_k\times a_2} & -J_{b_k\times b_1} & \cdots & -J_{b_k\times a_k} & (I_{b_k} - J_{b_k}) \end{bmatrix}$$

By the construction of $\mathcal{T}$, for $u \in U_i, 1 \leq i \leq k$ we have $deg(u) = b - b_1 - b_2 - \cdots - b_{i-1}$. Then, $n - 1 - 2deg(u) = a - b - 1 + 2(b_1 + b_2 + \cdots + b_{i-1})$.
For $v \in V_j, 1 \leq j \leq k$ we have $deg(v) = b + a_1 + a_2 + \cdots + a_j - 1$. Then, $n - 1 - 2deg(v) = a - b + 1 - 2(a_1 + a_2 + \cdots + a_j)$. Then $\mathcal{D}_{\mathcal{S}_\mathcal{T}} = Diag(\mathcal{D}_{\mathcal{S}_{\mathcal{T}(U_1)}}, \mathcal{D}_{\mathcal{S}_{\mathcal{T}(V_1)}}, \cdots, \mathcal{D}_{\mathcal{S}_{\mathcal{T}(U_k)}}, \mathcal{D}_{\mathcal{S}_{\mathcal{T}(V_k)}})$ where $\mathcal{D}_{\mathcal{S}_{\mathcal{T}(U_i)}}$ and $\mathcal{D}_{\mathcal{S}_{\mathcal{T}(V_j)}}$ are diagonal matrices of order a_i and b_j respectively, for $1 \leq i, j \leq k$. The Seidel Laplacian matrix $\mathcal{SL}_\mathcal{T}$ is a square matrix of order $a + b$ given by

$$\mathcal{SL}_\mathcal{T} = \begin{bmatrix} \mathcal{SL}_{\mathcal{T}(U_1)_{a_1}} & J_{a_1 \times b_1} & -J_{a_1 \times a_2} & J_{a_1 \times b_2} & \cdots & J_{a_1 \times b_k} \\ J_{b_1 \times a_1} & \mathcal{SL}_{\mathcal{T}(V_1)_{b_1}} & -J_{b_1 \times a_2} & J_{b_1 \times b_2} & \cdots & J_{b_1 \times b_k} \\ -J_{a_2 \times a_1} & -J_{a_2 \times b_1} & \mathcal{SL}_{\mathcal{T}(U_2)_{a_2}} & J_{a_2 \times b_2} & \cdots & J_{a_2 \times b_k} \\ J_{b_2 \times a_1} & J_{b_2 \times b_1} & J_{b_2 \times a_2} & \mathcal{SL}_{\mathcal{T}(V_2)_{b_2}} & \cdots & J_{b_2 \times b_k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -J_{a_k \times a_1} & -J_{a_k \times b_1} & -J_{a_k \times a_2} & -J_{a_k \times b_2} & \cdots & J_{a_k \times b_k} \\ J_{b_k \times a_1} & J_{b_k \times b_1} & J_{b_k \times a_2} & J_{b_k \times b_2} & \cdots & \mathcal{SL}_{\mathcal{T}(V_1)_{b_k}} \end{bmatrix}$$

where $\mathcal{SL}_{\mathcal{T}(U_i)} = (a - b + 2(b_1 + b_2 + \cdots + b_{i-1}))I - J$ and $\mathcal{SL}_{\mathcal{T}(V_j)} = (a - b - 2(a_1 + a_2 + \cdots + a_j))I + J$ are block matrices of order a_i and b_j respectively.

2.2. **Spectrum of equitable quotient matrix of $\mathcal{SL}_\mathcal{T}$.** In this section, we determine eigenvalues of equitable quotient matrix of $\mathcal{SL}_\mathcal{T}$. Obviously, the partition $\Pi : V = U_1 \cup V_1 \cup \cdots \cup U_k \cup V_k$ is a equitable partition. Therefore the quotient matrix $Q_{\mathcal{SL}}$ of $\mathcal{SL}_\mathcal{T}$ is a square matrix of order $2k$ given by

$$Q_{\mathcal{SL}} = \begin{bmatrix} \rho_{11} & b_1 & -a_2 & b_2 & \cdots & -a_k & b_k \\ a_1 & \eta_{11} & -a_2 & b_2 & \cdots & -a_k & b_k \\ -a_1 & -b_1 & \rho_{22} & b_2 & \cdots & -a_k & b_k \\ a_1 & b_1 & a_2 & \eta_{22} & \cdots & -a_k & b_k \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & -b_1 & -a_2 & -b_2 & \cdots & \rho_{kk} & b_k \\ a_1 & b_1 & a_2 & b_2 & \cdots & a_k & \eta_{kk} \end{bmatrix}$$

where $\rho_{ii} = a - b + 2(b_1 + b_2 + \cdots + b_{i-1}) - a_i$ and $\eta_{jj} = a - b - 2(a_1 + a_2 + \cdots + a_j) + b_j$. Let $D = Diag(a_1, b_1, \cdots, a_k, b_k)$ be the diagonal matrix. We get $D^{1/2} Q_{\mathcal{SL}} D^{-1/2}$ is a symmetric matrix. Therefore $Q_{\mathcal{SL}}$ is similar to the symmetric matrix $D^{1/2} Q_{\mathcal{SL}} D^{-1/2}$, which implies it is similar to a diagonal matrix. So $Q_{\mathcal{SL}}$ is diagonalizable. Using the elementary operations of determinant, we get $|\lambda - Q_{\mathcal{SL}}| = \lambda(\lambda - (\eta_{11} - b_1))(\lambda - (\rho_{22} + a_2)) \cdots (\lambda - (\rho_{kk} + a_k))(\lambda - (\eta_{kk} - b_k))$. So the spectrum of $Q_{\mathcal{SL}}$ is given by $(0, a - b + 2(b_1 + b_2 + \cdots + b_{i-1}), a - b - 2(a_1 + a_2 + \cdots + a_j))$ where $2 \leq i \leq k$ and $1 \leq j \leq k$. Thus all the eigenvalues of $Q_{\mathcal{SL}}$ are simple.

Theorem 2.1. *Let $t = 0^{a_1} 1^{b_1} \cdots 0^{a_k} 1^{b_k}$ be the binary string of the T_H- graph $\mathcal{T}$. Then 0 is a simple eigenvalue of $Q_{\mathcal{SL}}$ if $a \neq b - 2(b_1 + b_2 + \cdots + b_{i-1})$ or $a \neq b + 2(a_1 + a_2 + \cdots + a_j)$ where $2 \leq i \leq k$ and $1 \leq j \leq k - 1$.*

Proof. 0 is an eigenvalue of $Q_{\mathcal{SL}}$. It is simple if $a \neq b - 2(b_1 + b_2 + \cdots + b_{i-1})$ or $a \neq b + 2(a_1 + a_2 + \cdots + a_j)$ where $2 \leq i \leq k$ and $1 \leq j \leq k-1$. Otherwise algebraic multiplicity of 0 is two. □

Theorem 2.2. *Let $t = 0^{a_1}1^{b_1} \cdots 0^{a_k}1^{b_k}$ be the binary string of the T_H- graph $\mathcal{T}$. The Seidel Laplacian spectrum of $\mathcal{T}$ consists the following*

- $0 \in Spec_{\mathcal{SL}_{\mathcal{T}}}$.
- $a - b$ *with multiplicity* $a_1 - 1$.
- $a - b + 2(b_1 + b_2 + \cdots + b_{i-1})$ *with multiplicity* $a_i; 2 \leq i \leq k$
- $a - b - 2(a_1 + a_2 + \cdots + a_j)$ *with multiplicity* $b_j; 1 \leq j \leq k-1$
- $-(a + b)$ *with multiplicity* b_k.

Proof. By the construction of $\mathcal{T}$, each U_i is an independent set with order a_i where $1 \leq i \leq k$. Every vertex in U_i is adjacent only to the dominating vertices after them. Then, we have $deg(u) = b - b_1 - b_2 - \cdots - b_{i-1}$, for $u \in U_i$. $\mathcal{SL}_{\mathcal{T}} - (n - 2deg(u))I$ has a_i identical rows and so $(\lambda - (n - 2deg(u)))^{a_i - 1}$ is the factor of characteristic polynomial of $\mathcal{SL}_{\mathcal{T}}$. For $1 \leq j \leq k$, the subset V_j is a clique of order b_j with $N(x)\backslash V_j = N(y)\backslash V_j$ for all $x, y \in V_j$. For $v \in V_j, 1 \leq j \leq k$, we have $deg(v) = b + a_1 + a_2 + \cdots + a_j - 1$. Similarly $\mathcal{SL}_{\mathcal{T}} - (n - 2deg(v) - 2)I$ has b_j identical rows and so $(\lambda - (n - 2deg(v) - 2))^{b_j - 1}$ is the factor of characteristic polynomial of $\mathcal{SL}_{\mathcal{T}}$. Remaining eigenvalues of $\mathcal{SL}_{\mathcal{T}}$ are eigenvalues of equitable quotient matrix $Q_{\mathcal{SL}}$.
For $v \in V_k$, $n - 2deg(v) - 2 = (a + b) - 2(b + a_1 + a_2 + \cdots + a_k - 1) - 2 = -(a + b)$. By Theorem 2.1, $0 \in Spec_{\mathcal{SL}_{\mathcal{T}}}$. Let m_0 be the algebraic multiplicity of the eigenvalue 0. For $2 \leq i \leq k$ and $1 \leq j \leq k-1$

$$m_0 = \begin{cases} a_1, & \text{if } a = b \\ a_i + 1, & \text{if } a = b - 2(b_1 + b_2 + \cdots + b_{i-1}) \\ b_j + 1, & \text{if } a = b + 2(a_1 + a_2 + \cdots + a_j) \\ 1, & \text{otherwise} \end{cases}$$

Finally Seidel Laplacian spectrum of $\mathcal{T}$ is given by $(0^{m_0}, (a-b)^{a_1-1}, (a - b + 2(b_1 + b_2 + \cdots + b_{i-1}))^{a_i}, (a - b - 2(a_1 + a_2 + \cdots + a_j))^{b_j}, -(a+b)^{b_k})$ where $2 \leq i \leq k$ and $1 \leq j \leq k-1$. □

Example 2.3. Let $\mathcal{T}$ be a T_H-graph with binary string $t = 0^2101$.

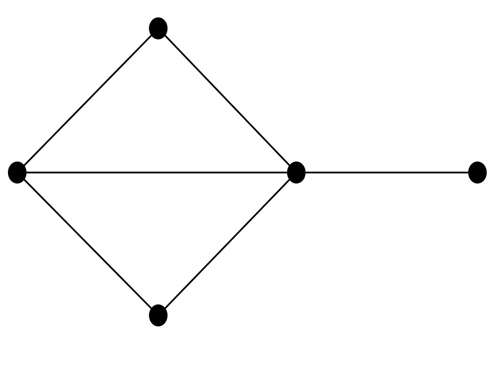

FIGURE 1. $\mathcal{T} : (0^2101)$

We have $\mathcal{SL}_{\mathcal{T}} = \begin{bmatrix} 0 & -1 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 & 1 \\ 1 & 1 & -2 & -1 & 1 \\ -1 & -1 & -1 & 2 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}$.

The eigenvalues of $\mathcal{SL}_{\mathcal{T}}$ are $-5, -3, 0, 1, 3$.

Corollary 2.4. *Let $\mathcal{T}$ be a T_H- graph with binary representation $t = 0^{a_1}1^{b_1}\cdots 0^{a_k}1^{b_k}$ where $a_i = b_j = r \geq 1$. Then the distinct Seidel Laplacian eigenvalues are given by $0, 2r, 4r, \cdots, 2(k-1)r, -2r, -4r, \cdots, -2kr$ with same multiplicity r.*

Proof. Here $a = b = kr$, 0 is the Seidel Laplacian eigenvalue with multiplicity $a_1 = r$. For $2 \leq i \leq k$, $a-b+2(b_1+b_2+\cdots+b_{i-1}) = 2(i-1)r$ and for $1 \leq j \leq k$, $a-b-2(a_1+a_2+\cdots+a_j) = -2jr$ are distinct Seidel Laplacian eigenvalues with same multiplicity r. ☐

3. T_H- graphs with a maximum of four distinct Seidel Laplacian eigenvalues

This section classify all such T_H- graphs which have at most four distinct Seidel Laplacian eigenvalues. We start with spectral properties of T_H- graphs where $k = 1$ and $k = 2$.

Case I: k = 1

Let $\mathcal{T}$ be a T_H- graph with binary string $t = 0^a1^b$. Then seidel laplacian spectrum of $\mathcal{T}$ consists of following

(1) $a - b$ with multiplicity $a - 1$
(2) $-(a + b)$ with multiplicity b
(3) 0 with
  (a) multiplicity 1 if $a \neq b$
  (b) multiplicity a if $a = b$

Remark 3.1. If $a = 1$, then $\mathcal{T}$ is the complete graph K_n while if $b = 1$, then $\mathcal{T}$ is the star graph S_n. In these two cases, the Seidel Laplacian eigenvalues are $0, (-n)^{n-1}$ and $0, (n-2)^{n-2}, -n$ respectively.

Remark 3.2. If $a = b \geq 1$ in case I, then the Seidel Laplacian eigenvalues are $0^a, -(2a)^a$.

Case II: k = 2

Let $\mathcal{T}$ be a T_H- graph with binary string $t = 0^{a_1}1^{b_1}0^{a_2}1^{b_2}$. Then distinct Seidel Laplacian spectrum of $\mathcal{T}$ consists of following

(1) $a - b$ with multiplicity $a_1 - 1$
(2) $a - b + 2b_1$ with multiplicity a_2
(3) $a - b - 2a_1$ with multiplicity b_1
(4) $-(a + b)$ with multiplicity b_2.
(5) 0 with
  (a) multiplicity 1 if $a \neq b$ or $a \neq b - 2b_1$ or $a \neq b + 2a_1$

(b) multiplicity a_1 if $a = b$
(c) multiplicity $a_2 + 1$ if $a = b - 2b_1$
(d) multiplicity $b_1 + 1$ if $a = b + 2a_1$

Pineapple Graph: A pineapple graph is obtained by joining pendant vertices to a vertex of a complete graph. Clearly a pineapple graph is always a connected T_H- graph. The binary representation of pineapple graph is $01^{b-1}0^{n-b-1}1$. When $b = 1$, we have a star graph of order n. When $n = 2b + 2$, Seidel Laplacian spectrum of a pineapple graph is given by $0^b, (n-2)^{n-b-1}, -n$ whereas $n \neq 2b+2$, Seidel Laplacian eigenvalues are $0, (n-2)^{n-b-1}, (n-2b-2)^{b-1}, -n$.

Theorem 3.3. *Let $\mathcal{T}$ be a T_H- graph with the binary string $t = 0^{a_1}1^{b_1} \cdots 0^{a_k}1^{b_k}$. Then $\mathcal{T}$ has exactly two distinct Seidel Laplacian eigenvalues if and only if $\mathcal{T}$ is either complete graph K_n or complete split graph $\overline{K_n} + K_n$.*

Proof. Assume that $\mathcal{T}$ has exactly two distinct Seidel Laplacian eigenvalues. Then $k = 1$ and $t = 0^a1^b$ is the binary string of $\mathcal{T}$ by Theorem 2.2. 0 and $-(a+b)$ are the only Seidel Laplacian eigenvalues. We consider two cases.

Case I: If $a > 0$ and $a - b = 0$ or $-(a+b)$.
If $a-b = -(a+b) \implies 2a = 0$, which is not possible. Hence $a-b = 0 \implies a = b$. Therefore $t = 0^a1^a$ is the string of T_H- graph $\mathcal{T}$. If $t = 0^a1^a$, then $\mathcal{T}$ is a complete split graph $\overline{K_n} + K_n$ where $n = a$.

Case II: If $a = 1$ then $a - b \notin Spec_{\mathcal{SL}_{\mathcal{T}}}$. So $t = 01^b$ is the string of T_H- graph $\mathcal{T}$. If $t = 01^b$ is the complete graph K_n where $n = 1 + b$.
Conversely if $t = 0^a1^a$ or 01^b, then by Theorem 2.2 $\mathcal{SL}_{\mathcal{T}}$ has exactly two distinct eigenvalues namely 0 and $-(a+b)$.

□

Theorem 3.4. *Let $\mathcal{T}$ be the T_H- graph with binary string $t = 0^{a_1}1^{b_1} \cdots 0^{a_k}1^{b_k}$. Then $\mathcal{T}$ has exactly three distinct Seidel Laplacian eigenvalues if and only if one of the following conditions hold*

(1) $t = 0^{a_1}1^{b_1}$; $a_1 > 1$ *and* $a_1 \neq b_1$.
(2) $t = 01^{b_1}0^{b+1}1^{b-b_1}$.
(3) $t = 01^{b_1}0^{a_2}1^{a_2+b_1+1}$

Proof. If $\mathcal{T}$ has exactly three distinct Seidel Laplacian eigenvalues, then $k \leq 2$. For a T_H-graph, the Seidel Laplacian matrix has atleast two Seidel Laplacian eigenvalues, they are 0 and $-(a+b)$. Then the third Seidel Laplacian eigenvalue of $\mathcal{T}$ will be one among $a-b, a-b+2b_1$ and $a - b - 2a_1$. We consider three cases.

Case 1: $a - b$ is the third Seidel Laplacian eigenvalue of $\mathcal{T}$.

Then $a_1 > 1$ and $a \neq b$.
But $a - b + 2b_1 \neq a - b - 2a_1$ and its value must be 0 or $-(a+b)$. If $a - b + 2b_1 = -(a+b)$ then $a = -b_1$, which is not possible. If $a - b + 2a_1 = -(a+b)$, then $a = a_1 \implies a_2 = 0$.

Hence the binary string of the T_H- graph is of the form $t = 0^{a_1}1^{b_1}$ with $a_1 > 1$ and $a \neq b$.

Case 2: $a - b + 2b_1$ is the third Seidel Laplacian eigenvalue.

Then $a_2 \neq 0$, $a - b + 2b_1 \neq 0$, $a - b + 2b_1 \neq -(a + b)$ and $a - b = 0$ or $a_1 = 1$. Now if $a - b = 0$ then

- $a - b - 2a_1 = 0 \implies -2a_1 = 0$, which is not possible.
- $a - b - 2a_1 = -(a + b) \implies a = a_1$,in this case $\mathcal{T}$ has exactly two distinct Seidel Laplacian eigenvalues, a contradiction.

Therefore $a \neq b$. If $a_1 > 1$ and $a \neq b$, then this case reduces to **Case 1**. Hence $a \neq b$ and $a_1 = 1$.
If $a - b - 2a_1 = -(a + b)$, then $a = a_1$. So $\mathcal{T}$ has exactly two Seidel Laplacian eigenvalues by Theorem 3.3. If $a - b - 2a_1 = 0 \implies a - b = 2$, then the binary string is of the form $t = 01^{b_1}0^{b+1}1^{b-b_1}$.

Case 3: $a - b - 2a_1$ is the third Seidel Laplacian eigenvalue.

Then $a - b - 2a_1 \neq 0$, $a - b - 2a_1 \neq -(a + b)$ and $a - b = 0$ or $a_1 = 1$. Now if $a - b = 0$ then

- $a - b + 2b_1 = 0 \implies 2b_1 = 0$, which is not possible.
- $a - b + 2b_1 = -(a + b) \implies b_1 = -a$,which is not possible.

So $a \neq b$ and $a_1 = 1$. If $a - b + 2b_1 = -(a + b)$ then $a + b_1 = 0$, which is impossible. If $a - b + 2b_1 = 0$, then the binary string of $\mathcal{T}$ is of the form $t = 01^{b_1}0^{a_2}1^{a_2+b_1+1}$

Conversely if $t = 0^{a_1}1^{b_1}$ with $a_1 > 1$ and $a_1 \neq b_1$, $t = 01^{b_1}0^{b+1}1^{b-b_1}$ or $t = 01^{b_1}0^{a_2}1^{a_2+b_1+1}$, then $\mathcal{T}$ has exactly three distinct Seidel Laplacian eigenvalues. □

Theorem 3.5. *Let $\mathcal{T}$ be the T_H- graph with binary string $t = 0^{a_1}1^{b_1} \cdots 0^{a_k}1^{b_k}$. Then $\mathcal{T}$ has exactly four distinct Seidel Laplacian eigenvalues if and only if one of the following conditions hold*

(1) $t = 0^{a_1}1^{b_1}0^{a_2}1^{b_2}$; $a_1 > 1$ *and* $a = b$.
(2) $t = 0^{a_1}1^{b_1}0^{a_2}1^{b_2}$; $a_1 > 1$ *and* $a = b - 2b_1$.
(3) $t = 0^{a_1}1^{b_1}0^{a_2}1^{b_2}$; $a_1 > 1$ *and* $a = b + 2a_1$
(4) $t = 0^{a_1}1^{b_1}0^{a_2}1^{b_2}$; $a_1 = 1$, $a \neq b - 2b_1$ *and* $a \neq b + 2a_1$

Proof. If $\mathcal{T}$ has exactly four distinct Seidel Laplacian eigenvalues, then $K = 2$. So the binary string of the T_H-graph $\mathcal{T}$ is of the form $t = 0^{a_1}1^{b_1}0^{a_2}1^{b_2}$. We have $0, -(a + b) \in Spec_{\mathcal{SL}_{\mathcal{T}}}$. Consider the following cases:

Case I: If $a_1 > 1$ and $a \neq b$.

Then $a - b \in Spec_{\mathcal{SL}_{\mathcal{T}}}$. The fourth eigenvalue is either $a - b + 2b_1$ or $a - b - 2a_1$. We consider the following subcases;

Subcase I: $a - b - 2a_1$ is the fourth Seidel Laplacian eigenvalue.

Then $a-b-2a_1 \neq 0$, $a-b-2a_1 \neq -(a+b)$ and $a-b-2a_1 \neq (a-b)$. If $a-b+2b_1 = (a-b)$ or $-(a+b) \implies b_1 = 0$ or $a+b_1 = 0$, which is not possible. Hence $a-b+2b_1 = 0$. Then the binary string is of the form $t = 0^{a_1}1^{b_1}0^{a_2}1^{b_2}$ with $a = b - 2b_1$.

Subcase II: $a-b+2b_1$ is the fourth Seidel Laplacian eigenvalue.

Then $a-b+2b_1 \neq 0$, $a-b+2b_1 \neq -(a+b)$ and $a-b+2b_1 \neq (a-b)$. Now, $a-b-2a_1 = (a-b) \implies -2a_1 = 0$, which is impossible. But $a-b-2a_1 = -(a+b) \implies a = a_1$, then by Theorem 3.4, $\mathcal{T}$ has exactly three distinct Seidel Laplacian eigenvalues. Therefore $a-b-2a_1 = 0$. Then the binary string is $t = 0^{a_1}1^{b_1}0^{a_2}1^{b_2}$ with $a = b - 2a_1$.

Case II: If $a_1 > 1$ and $a = b$.

Then $0 \in Spec_{\mathcal{SL}_{\mathcal{T}}}$ with multiplicity a_1. So the values of $a-b-2a_1$ and $a-b+2b_1$ are not equal to 0 and $-(a+b)$. We have $a-b-2a_1$, $a-b+2b_1 \neq -(a+b)$. Hence the binary string is of the form $t = 0^{a_1}1^{b_1}0^{a_2}1^{b_2}$ with $a = b, a \neq b - 2b_1$ and $a \neq b + 2a_1$

Case III: If $a_1 = 1$ and $a \neq b$
Then $a-b \notin Spec_{\mathcal{SL}_{\mathcal{T}}}$. Hence the binary string of the T_H- graph is of the form $t = 0^{a_1}1^{b_1}0^{a_2}1^{b_2}$ with $a_1 = 1, a \neq b - 2b_1$ and $a \neq b + 2$.

Case IV: If $a_1 = 1$ and $a = b$

Then the values of $a-b-2a_1$, $a-b+2b_1$ are not equal to 0 and $-(a+b)$. Thus the binary string of the T_H- graph is of the form $t = 0^{a_1}1^{b_1}0^{a_2}1^{b_2}$ with $a_1 = 1, a \neq b - 2b_1$ and $a \neq b + 2$.
Conversely if the binary string t satisfies these four cases, then Theorem 2.2 $\mathcal{T}$ has exactly four distinct Seidel Laplacian eigenvalues. □

4. Conclusion

In this study, we have studied the Seidel Laplacian spectrum of T_H-graphs. Additionally, we have characterized T_H-graphs with the binary string $t = 0^{a_1}1^{b_1} \cdots 0^{a_k}1^{b_k}$ for small values of k. Further scope of study, we can find their applications in chemical graph theory.

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