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Research Paper

FINDING A GENERALIZED POSITIVE SOLUTION EQUATION FOR A TRAPEZOIDAL FULLY FUZZY SYLVESTER MATRIX

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ABSTRACT

The determination of the solvability of Sylvester matrix equations is crucial in various domains of control theory and systems theory. In numerous applications, employing fuzzy numbers is more suitable for representing certain system parameters compared to using exact values. Previous studies predominantly focus on solutions that incorporate triangular fuzzy numbers for the fuzzy Sylvester matrix problem. This work presents two analytical methods aimed at addressing the Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation. By utilizing contemporary fuzzy arithmetic multiplication operations, we reformulate this equation into a corresponding system of crisp Sylvester Matrix Equations. The investigation covers the necessary and sufficient conditions for the existence of positive fuzzy solutions and their uniqueness for the Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation. Moreover, the relationship between the Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation and the system of Sylvester Matrix Equations is explored. An illustrative example is provided to demonstrate the effectiveness of the proposed approaches.

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1. Introduction

The Sylvester matrix equation of the form PY + YQ = R plays a vital role in various disciplines, such as control systems [1, 2, 3], orbital theory [4], nonlinear control system model reduction [5], system design [6], and medical imaging acquisition systems [7, 8, 9]. Traditional linear systems often struggle with imprecision and uncertainty, which can arise from incomplete information [10]. This uncertainty is frequently represented using fuzzy numbers instead of precise values [11]. When both the constant matrix R and the solution matrix Y are fuzzy, the Sylvester Matrix Equation can be generalized to a Fuzzy Sylvester Matrix Equation expressed as $\tilde{P}\tilde{Y} - \tilde{Y}\tilde{Q} = \widetilde{R}$.

The study of the Fuzzy Sylvester Matrix Equation was explored in works [12, 13], where it was reformulated as a fuzzy linear system through the application of the Kronecker product. However, this approach is limited to small-sized Fuzzy Sylvester Matrix Equations. To overcome this limitation, the authors in [14] proposed a conversion of the Fuzzy Sylvester Matrix Equation into a standard Sylvester Matrix Equation by utilizing Dubois and Prade's arithmetic operations [15]. They then applied Bartle's Stewart method to derive the fuzzy solution.

The term Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation is used when all its parameters are represented as positive trapezoidal fuzzy matrices. Several studies have focused on solving the Triangular Fully Fuzzy Sylvester Matrix Equation, given by $\tilde{P}\tilde{Y} - \tilde{Y}\tilde{Q} = \widetilde{R}$.

2. Basic Operations with Fuzzy Numbers

In this section, we will discuss some fundamental arithmetic operations associated with fuzzy numbers.

Definition 2.1: A fuzzy number $\tilde{P} = (n, a, \beta, \gamma)$ is classified as an LR-fuzzy trapezoidal number if its membership function is expressed as:

$$\mu_{\tilde{P}}(y) = \begin{cases} 1 - \frac{n-y}{\beta} & \text{for } n - \beta \leq y < n, \\ 1 & \text{for } n < y < a, \\ 1 - \frac{y-m}{\gamma} & \text{for } m \leq y < m + \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2: The classification of the sign of $\tilde{P} = (n, a, \beta, \gamma)$ is as follows:

- $\tilde{P}$ is termed positive (negative) if $n - \beta \geq 0$ (or $\gamma + a \leq 0$).
- $\tilde{P}$ is classified as zero if $n = 0$, $a = 0$, $\beta = 0$, and $\gamma = 0$.
- $\tilde{P}$ is considered near zero if $n - \beta \leq 0$ and $0 \leq \gamma + m$.

Definition 2.3: A fuzzy number $\tilde{P} = (n, a, \beta, \gamma)$ is termed a zero fuzzy trapezoidal number if $n = 0$, $a = 0$, $\gamma = 0$, and $\beta = 0$.

Definition 2.4: Two fuzzy numbers $\tilde{P} = (n, a, \beta, \gamma)$ and $\tilde{B} = (u, v, \delta, \alpha)$ are equal if:

$$n = u, \quad a = v, \quad \beta = \delta, \quad \text{and} \quad \gamma = \alpha.$$

Definition 2.5: The arithmetic operations for fuzzy numbers are defined as follows. For two trapezoidal fuzzy numbers $\tilde{P} = (n, a, \beta, \gamma)$ and $\tilde{Q} = (u, v, \delta, \alpha)$:

Addition:

(i) $$(n, a, \beta, \gamma) + (u, v, \delta, \alpha) = (n + u, a + v, \beta + \delta, \gamma + \alpha)$$

Subtraction:

(ii) $$(n, a, \beta, \gamma) - (u, v, \delta, \alpha) = (n - v, a - u, \beta + \delta, \gamma + \alpha)$$

Multiplication:

- **Case I:** If $\tilde{P} > 0$ and $\tilde{Q} > 0$, then:

(iii) $$\tilde{P} \cdot \tilde{Q} = (n, a, \beta, \gamma) \cdot (u, v, \delta, \alpha) = (nu, nv, n\delta + a\alpha, n\alpha + a\gamma)$$

Definition 2.10: The trapezoidal fully fuzzy matrix equation can be expressed as:

(x) $$\tilde{P}\tilde{Y} - \tilde{Y}\tilde{Q} = \tilde{C}$$

where $\tilde{P} = (a_{ij})_{n\times n}$, $\tilde{Q} = (b_{ij})_{m\times m}$, $\tilde{C} = (c_{ij})_{n\times m}$, and $\tilde{Y} = (y_{ij})_{n\times m}$ represents a trapezoidal fully fuzzy Sylvester matrix equation. This can also be reformulated as:

(xi) $$\sum_{i,j=1}^{n} a_{ij}y_{ij} - \sum_{i,j=1}^{m} y_{ij}b_{ij} = c_{ij}$$

Definition 2.11: The Schur factorization of a matrix P is represented as:

$$P = QRQ^T$$

where:

- R is an upper triangular matrix referred to as the Schur form of P.
- Q is a unitary matrix satisfying $QQ^T = I$.

3. MAIN RESULTS

The Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation solution is covered in this section. By using arithmetic fuzzy multiplication procedures, the Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation in Equation (x) is expanded to a system of four Sylvester Matrix Equations. Afterwards, the positive fuzzy solution is produced by creating two distinct techniques. The Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation is transformed into an equivalent set of matrix equations in the ensuing Theorem 3.1.

Theorem 3.1: If $\tilde{\mathrm{P}} = (\tilde{u}_{ij})_{n\times n} = (n_{ij}, m_{ij}, \beta_{ij}, \gamma_{ij}) > 0, \tilde{\mathrm{Q}} = (\tilde{v}_{ij})_{m\times m} = (u_{ij}, v_{ij}, \delta_{ij}, \alpha_{ij}) > 0, \tilde{Y} = (\tilde{x}_{ij})_{n\times m} = (y_{ij}, z_{ij}, t_{ij}, b_{ij}) > 0$ and $\tilde{C} = (\tilde{w}_{ij})_{n\times m} = (w_{ij}, g_{ij}, h_{ij}, f_{ij})$, then the Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation in $\tilde{P}\tilde{Y} - \tilde{Y}\tilde{D} = \tilde{E}$ is equivalent to the x following SME:

(3.1) $$\begin{cases} n_{ij}y - z_{ij}b_{ij} = c_{ij} \\ m_{ij}t_{ij} - y_{ij}u_{ij} = g_{ij} \\ n_{ij}t_{ij} + \beta_{ij}y_{ij} + z_{ij}\alpha_{ij} + q_{ij}v_{ij} = h_{ij} \\ m_{ij}q_{ij} + \gamma_{ij}z_{ij} + y_{ij}\delta_{ij} + t_{ij}a_{ij} = f_{ij} \end{cases}$$

Proof: Let $\tilde{\mathrm{P}} = (\tilde{u}_{ij})_{m\times m} = (n_{ij}, m_{ij}, \beta_{ij}, \gamma_{ij}) > 0, \tilde{\mathrm{Q}} = (\tilde{v}_{ij})_{n\times n} = (u_{ij}, v_{ij}, \delta_{ij}, \alpha_{ij}) > 0, \tilde{\mathrm{Y}} = (\tilde{x}_{ij})_{n\times m} = (y_{ij}, z_{ij}, t_{ij}, b_{ij}) > 0$ and $\tilde{C} = (\tilde{c}_{ij})_{n\times m} = (w_{ij}, g_{ij}, h_{ij}, f_{ij})$.

We have from Definition 2.5 and by Equation (2.3),

$$\begin{aligned}\tilde{\mathrm{P}}\tilde{Y} &= (\tilde{u}_{ij})(\tilde{y}_{ij}) = (n_{ij}, m_{ij}, \beta_{ij}, \gamma_{ij})(y_{ij}, z_{ij}, t_{ij}, b_{ij}).\\ &= (n_{ij}y_{ij}, m_{ij}z_{ij}, n_{ij}t_{ij} + \beta_{ij}y_{ij}, m_{ij}b_{ij} + \gamma_{ij}z_{ij}).\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathrm{Y}}\tilde{\mathrm{Q}} &= (\tilde{x}_{ij})(\tilde{v}_{ij}) = (y_{ij}, z_{ij}, t_{ij}, b_{ij})(u_{ij}, v_{ij}, \delta_{ij}, \alpha_{ij}).\\ &= (y_{ij}u_{ij}, z_{ij}v_{ij}, y_{ij}\delta_{ij} + t_{ij}u_{ij}, z_{ij}\alpha_{ij} + b_{ij}v_{ij})\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{\mathrm{P}}\tilde{Y} - \tilde{Y}\tilde{Q} &= (n_{ij}y_{ij}, m_{ij}z_{ij}, n_{ij}t_{ij} + \beta_{ij}y_{ij}, m_{ij}b_{ij} + \gamma_{ij}z_{ij})\\ &- (y_{ij}u_{ij}, z_{ij}v_{ij}, y_{ij}\delta_{ij} + t_{ij}u_{ij}, z_{ij}\alpha_{ij} + b_{ij}v_{ij}).\end{aligned}$$

Which can be written as

$$\begin{aligned}&\tilde{\mathrm{P}}\tilde{Y} - \tilde{Y}\tilde{Q} = (m_{ij}x_{ij} - y_{ij}b_{ij}, n_{ij}y_{ij} - x_{ij}a_{ij}, m_{ij}z_{ij} + \alpha_{ij}x_{ij} + y_{ij}\delta_{ij} + q_{ij}b_{ij}, n_{ij}q_{ij}\\ &+\beta_{ij}y_{ij} + x_{ij}\gamma_{ij} + z_{ij}a_{ij})\\ &= (n_{ij}y_{ij} - y_{ij}u_{ij}, m_{ij}z_{ij} - z_{ij}v_{ij}, n_{ij}t_{ij} - y_{ij}\delta_{ij}, \beta_{ij}y_{ij} - t_{ij}u_{ij}, m_{ij}b_{ij} - z_{ij}\alpha_{ij}, \gamma_{ij}z_{ij}\\ &-b_{ij}v_{ij})\end{aligned}$$

Since $\tilde{P}\tilde{Y} - \tilde{Y}\tilde{Q} = \tilde{C}$ can be written as

$$\sum_{i,j=1}^{n} \tilde{u}_{ij}\tilde{y}_{ij} - \sum_{i,j=1}^{m} \tilde{y}_{ij}\tilde{v}_{ij} = \tilde{c}_{ij}$$

Therefore, the Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation P P $\tilde{Y} - \tilde{Y}\tilde{D} = \tilde{E}$ is equivalent to the following system of Sylvester Matrix Equation:

$$\begin{cases} n_{ij}y - z_{ij}b_{ij} = c_{ij}\\ m_{ij}t_{ij} - y_{ij}u_{ij} = g_{ij}\\ n_{ij}t_{ij} + \beta_{ij}y_{ij} + z_{ij}\alpha_{ij} + q_{ij}v_{ij} = h_{ij}\\ m_{ij}q_{ij} + \gamma_{ij}z_{ij} + y_{ij}\delta_{ij} + t_{ij}a_{ij} = f_{ij}\end{cases}$$

We take into consideration the appropriate Sylvester Matrix Equation in Equation (3.1) in order to solve the Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation in Equation (x). There are numerous traditional techniques that can be used to derive the analytical solution of the SME system in equation (3.1). The Bartels Stewart Method [36] is extended in the fuzzy environment that follows.

real and have real Schur decompositions $n_{ij} = U_1R_1U_1^T, u_{ij} = V_1S_1V_1^T, m_{ij} = U_2R_2U_2^T, v_{ij} = V_2S_2V_2^T$, where U and V are orthogonal and R and S are upper quasi-triangular. Then the first two equations in Equation (3.1) can be transformed to:

$$\begin{aligned}&U_1^Tn_{ij}U_1 \cdot U_2^Ty_{ij}V_1 - U_1^Tz_{ij}V_2 \cdot V_2^Tv_{ij}V_2 = U_1^Tw_{ij}V_2\\ &U_2^Tm_{ij}U_2 \cdot U_1^Tz_{ij}V_2 - U_2^Ty_{ij}V_1 \cdot V_1^Tu_{ij}V_1 = U_2^Tg_{ij}V_1.\end{aligned}$$

Consequently, they can be written as

$$\begin{cases} R_1W_1 - W_2S_2 = D_1 \\ R_2W_2 - W_1S_1 = D_2 \end{cases}$$

where $R_1 = U_1^T n_{ij} U_1, R_2 = U_2^T m_{ij} U_2, W_1 = U_2^T y_{ij} V_1, W_2 = U_1^T \mathrm{z} V_2, S_1 = V_1^T u_{ij} V_1, S_2 = V_2^T v_{ij} V_2, D_1 = U_1^T w_{ij} V_2$ and $D_2 = U_2^T g_{ij} V_1$.

Then, this system can be written as

$$P_1 w_1 = d_1$$

where $P_1 = \begin{pmatrix} I_n \otimes R_1 & -S_2^T \otimes I_m \\ -S_1^T \otimes I_m & I_n \otimes R_2 \end{pmatrix}, w_1 = \begin{pmatrix} \mathrm{vec}\,(W_1) \\ \mathrm{vec}\,(W_2) \end{pmatrix}$ and $d_1 = \begin{pmatrix} \mathrm{vec}\,(D_1) \\ \mathrm{vec}\,(D_2) \end{pmatrix}$.

Gaussian elimination and back substitution are applied to obtain w_1.

Case 2: The values of y_{ij} and z_{ij} can be computed as follows:

$$y_{ij} = U_2 W_1 V_1^T$$
$$z_{ij} = U_1 W_2 V_2^T$$

Case 3: The third and fourth equations in Equation (3.1) can be written as follows:

$$\begin{cases} n_{ij}t_{ij} + b_{ij}bv_{ij} = h_{ij} - \beta_{ij}y_{ij} - z_{ij}\alpha_{ij} \\ m_{ij}b_{ij} + t_{ij}u_{ij} = f_{ij} - \gamma_{ij}z_{ij} - y_{ij}\delta_{ij} \end{cases} \tag{3.2}$$

If we let,

$$h_1^\beta = h_{ij} - \beta_{ij}y_{ij} - z_{ij}\alpha_{ij}$$

and

$$f_1^\beta = f_{ij} - \gamma_{ij}z_{ij} - y_{ij}\delta_{ij}$$

Then Equation (3.2) can be written as

$$\begin{cases} n_{ij}t_{ij} + b_{ij}v_{ij} = h_1^\beta \\ m_{ij}u_{ij} + t_{ij}u_{ij} = f_1^\beta \end{cases} \tag{3.3}$$

Since Equation (3.3) has the same structure as the first two equations in Equation (3.1), it can be transformed to:

$$U_1^T n_{ij} U_1 \cdot U_2^T t_{ij} V_1 + U_1^T b_{ij} V_2 \cdot V_2^T v_{ij} V_2 = U_1^T h_{1}{}_{ij}^{\beta} V_2.$$
$$U_2^T m_{ij} U_2 \cdot U_1^T b_{ij} V_2 + U_2^T t_{ij} V_1 \cdot V_1^T u_{ij} V_1 = U_2^T f_1^\beta V_1.$$

that is,

$$\begin{cases} R_1W_3 + W_4S_2 = D_3 \\ R_2W_4 + W_3S_1 = D_4 \end{cases}$$

or equivalently

$$P_2 w_2 = d_2$$

where $P_2 = \begin{pmatrix} I_n \otimes R_1 & S_2^T \otimes I_m \\ S_1^T \otimes I_m & I_n \otimes R_2 \end{pmatrix}$, $w_2 = \begin{pmatrix} \operatorname{vec}(W_3) \\ \operatorname{vec}(W_4) \end{pmatrix}$ and $d_2 = \begin{pmatrix} \operatorname{vec}(D_3) \\ \operatorname{vec}(D_4) \end{pmatrix}$.
Gaussian elimination and back substitution are applied to obtain w_2.

Case 4: The values of t_{ij} and b_{ij} can be computed as follows:

$$t_{ij} = U_2 W_3 V_1^T$$
$$b_{ij} = U_1 W_4 V_2^T$$

Case 5: Combining the values of y_{ij}, z_{ij}, t_{ij} and b_{ij} which obtained in step 2 and step 4 . The solution of TrFFSME is represented by:

$$\tilde{Y} = (\tilde{y}_{ij})_{m\times m} = (y_{ij}, z_{ij}, t_{ij}, b_{ij}), \forall\{1 \leq i, j \leq m, n\}$$

The following Matrix Vectorization Method is based on the concept of Kronecker product, Kronecker difference and Vec-operator. It is an extension of the method proposed for solving Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation in the form $\tilde{P}\tilde{Y} + \tilde{Y}\tilde{Q} = \tilde{C}$ by Elsayed et al. [24].

Now: MVM for solving Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation.

In this method we apply the concept of Kronecker product and Vec-operator to Equation (3.1) as follows:

Case 1: Applying subtraction property of equality on the third and fourth equations in Equation (3.1) we get:

$$(3.4)\qquad \begin{cases} n_{ij}y_{ij} - z_{ij}v_{ij} = w_{ij} \\ m_{ij}z_{ij} - y_{ij}u_{ij} = g_{ij} \\ n_{ij}t_{ij} + b_{ij}v_{ij} = h_{ij} - \beta_{ij}y_{ij} - z_{ij}\alpha_{ij} \\ m_{ij}b_{ij} + t_{ij}u_{ij} = f_{ij} - \alpha_{ij}z_{ij} - y_{ij}\delta_{ij} \end{cases}$$

have

$$(3.5)\qquad \begin{cases} \operatorname{Vec}(n_{ij}y_{ij} - z_{ij}v_{ij}) = \operatorname{Vec}(w_{ij}), \\ \operatorname{Vec}(m_{ij}z_{ij} - y_{ij}u_{ij}) = \operatorname{Vec}(g_{ij}), \\ \operatorname{Vec}(n_{ij}t_{ij} + b_{ij}v_{ij}) = \operatorname{Vec}(h_{ij} - \beta_{ij}y_{ij} - z_{ij}\alpha_{ij}), \\ \operatorname{Vec}(m_{ij}b_{ij} + t_{ij}u_{ij}) = \operatorname{Vec}(f_{ij} - \alpha_{ij}z_{ij} - y_{ij}\delta_{ij}) \end{cases}$$

Using Definition 2.8 on Equation (3.5) we get

(3.6)
$$\begin{cases} \begin{pmatrix} I_m \otimes n_{ij} & -v_{ij}^T \otimes I_n \\ -u_{ij}^T \otimes I_n & I_m \otimes m_{ij} \end{pmatrix} \begin{pmatrix} \operatorname{Vec}(y_{ij}) \\ \operatorname{Vec}(z_{ij}) \end{pmatrix} = \begin{pmatrix} \operatorname{Vec}(w_{ij}) \\ \operatorname{Vec}(g_{ij}) \end{pmatrix} \\ \begin{pmatrix} I_m \otimes n_{ij} & v_{ij}^T \otimes I_n \\ u_{ij}^T \otimes I_n & I_m \otimes m_{ij} \end{pmatrix} \begin{pmatrix} \operatorname{Vec}(t_{ij}) \\ \operatorname{Vec}(u_{ij}) \end{pmatrix} = \begin{pmatrix} \operatorname{Vec}(h_{ij}) \\ \operatorname{Vec}(f_{ij}) \end{pmatrix} - \begin{pmatrix} I_m \otimes \beta_{ij} & \alpha_{ij}^T \otimes I_n \\ \delta_{ij}^T \otimes I_n & I_m \otimes \gamma_{ij} \end{pmatrix} \begin{pmatrix} \operatorname{Vec}(y_{ij}) \\ \operatorname{Vec}(z_{ij}) \end{pmatrix} \end{cases}$$

Case 2: Let,

$$R_1 = \begin{pmatrix} I_m \otimes n_{ij} - v_{ij}^T \otimes I_n \\ -u_{ij}^T \otimes I_n I_m \otimes m_{ij} \end{pmatrix} = \begin{pmatrix} (n_{ij})_{11} & 0 & 0 & v_{11} I_m & \dots & v_{n1} I_m \\ 0 & (n_{ij})_{22} & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & (n_{ij})_{nn} & \mathrm{v}_{1m} I_n & \dots & \mathrm{v}_{nn} I_m \\ -\mathrm{u}_{11} I_m & \dots & -\mathrm{u}_{m1} I_n & (m_{ij})_{11} & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & (m_{ij})_{22} & -\mathrm{v}_{21} \\ -\mathrm{u}_{1n} I_m & \dots & -\mathrm{u}_{nn} I_m & 0 & -\mathrm{v}_{12} & (m_{ij})_{nn} \end{pmatrix},$$

$$R_2 = \begin{pmatrix} I_n \otimes n_{ij} & \mathrm{v}_{ij}^T \otimes I_n \\ \mathrm{u}_{ij}^T \otimes I_n & I_m \otimes m_{ij} \end{pmatrix} = \begin{pmatrix} (n_{ij})_{11} & 0 & 0 & \mathrm{v}_{11} I_m & \dots & \mathrm{v}_{n1} I_m \\ 0 & (n_{ij})_{22} & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & (n_{ij})_{nn} & \mathrm{v}_{1m} I_n & \dots & \mathrm{v}_{nn} I_n \\ \mathrm{u}_{11} I_m & \dots & \mathrm{u}_{m1} I_n & (m_{ij})_{11} & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & (m_{ij})_{22} & -\mathrm{v}_{21} \\ \mathrm{u}_{1n} I_m & \dots & \mathrm{u}_{nn} I_n & 0 & -\mathrm{v}_{12} & (m_{ij})_{nn} \end{pmatrix},$$

$$S_1 = \begin{pmatrix} \mathrm{Vec}\,(y_{ij}) \\ \mathrm{Vec}\,(z_{ij}) \end{pmatrix} = \begin{pmatrix} y_{11} \\ \vdots \\ y_{mn} \\ z_{11} \\ \vdots \\ z_{mn} \end{pmatrix}, T_1 = \begin{pmatrix} \mathrm{Vec}\,(w_{ij}) \\ \mathrm{Vec}\,(g_{ij}) \end{pmatrix} = \begin{pmatrix} w_{11} \\ \vdots \\ w_{mn} \\ g_{11} \\ \vdots \\ g_{mn} \end{pmatrix}, S_2 = \begin{pmatrix} \mathrm{Vec}\,(t_{ij}) \\ \mathrm{Vec}\,(b_{ij}) \end{pmatrix} = \begin{pmatrix} t_{11} \\ \vdots \\ t_{mn} \\ b_{11} \\ \vdots \\ b_{mn} \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} \mathrm{Vec}\,(h_{ij}) \\ \mathrm{Vec}\,(f_{ij}) \end{pmatrix} - \begin{pmatrix} I_m \otimes \beta_{ij} \alpha_{ij}^T \otimes I_n \\ \delta_{ij}^T \otimes I_n I_m \otimes \gamma_{ij} \end{pmatrix} \begin{pmatrix} \mathrm{Vec}\,(y_{ij}) \\ \mathrm{Vec}\,(z_{ij}) \end{pmatrix} = \begin{pmatrix} T_{111}^{\beta} \\ \vdots \\ T_{1mn}^{\beta} \\ T_{211}^{\beta} \\ \vdots \\ T_{2mn}^{\beta} \end{pmatrix}$$

Case 3: With the assumption that R_1 and R_2 are non- singular, the system of equations in Equation (3.6) can be written as follows:

$$\left\{ \begin{array}{l} R_1 S_1 = T_1 \\ R_2 S_2 = T_2 \end{array} \right. \tag{3.7}$$

By the multiplicative inverse of R_1 and R_2 we obtain the following:

$$\left\{ \begin{array}{l} S_1 = R_1^{-1} \cdot T_1 \\ S_2 = R_2^{-1} \cdot T_2 \end{array} \right. \tag{3.8}$$

Case 4: The solution of the TrFFSME is represented by:

$$\tilde{Y} = (\tilde{y}_{ij})_{n \times m} = (y_{ij}, z_{ij}, t_{ij}, b_{ij})\,, \forall \{1 \leq i, j \leq m, n$$

4. CONCLUDING REMARKS

The Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation can be solved using methods such as the Generalized Bartels-Stewart Method and the Matrix Vectorization Method, among others, ensuring a comprehensive approach to deriving analytical solutions. This research introduces an analytical solution to the Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation given by

$$\tilde{P}\tilde{Y} - \tilde{Y}\tilde{Q} = \widetilde{R}.$$

To derive a positive fuzzy solution, we employ two analytical techniques: the Matrix Vectorization Method and the Generalized Bartels-Stewart Method. We also explore the conditions for the uniqueness and existence of the positive fuzzy solutions.

Both methodologies provide positive fuzzy solutions to the Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation with a focus on accuracy and reliability. Additionally, these approaches are applicable to solve similar Trapezoidal Fully Fuzzy Sylvester Matrix Equations. For large systems, the implementation of either method can be facilitated using computational software such as Matlab or Mathematica.

One notable limitation of the Matrix Vectorization Method is its intensive computational resource requirement, which can lead to extended processing times and substantial memory usage. Future studies should aim to enhance the existing arithmetic fuzzy techniques to streamline the system of Sylvester Matrix Equations that arises. Moreover, optimization strategies will need to be developed to overcome the challenges posed by the current methodologies.

Looking ahead, subsequent research will focus on adapting the findings of this study to address the Trapezoidal Fully Fuzzy Sylvester Matrix Equation involving nearly zero trapezoidal fuzzy numbers, broadening the scope and applicability of the established solutions

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