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Research Paper

GEODESIC VECTORS OF (α, β)-METRICS ON HYPERCOMPLEX 4-DIMENSIONAL LIE GROUPS

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ABSTRACT

In this paper, we consider invariant (α, β)-metrics and describe all geodesic vectors and investigate the set of all homogeneous geodesics on left invariant hypercomplex four dimensional simply connected Lie groups. Also, we study the conditions for the Douglas and Berwald type of (α, β)-metrics on the left invariant hypercomplex four dimensional simply connected Lie groups.

1. Introduction

The geometry based on the element of arc

$$ds = F(x^1, \ldots, x^n; dx^1, \ldots, dx^n), \tag{1.1}$$

where F is positively homogeneous of degree 1 in dx^i is called Riemannian-Finsler geometry or Finsler geometry for short. Roughly speaking, F is a collection of Minkowskian norms F_x in the tangent space at x such that F_x varies smoothly in x. In fact, metric (1.1) was introduced by Riemann in his famous 1854 Habilitationsvortag " Uber die Hypotheser

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welche der geometric zugrund liegen". He paid particular attention to a metric defined by the positive square root of a positive definite quadratic differential form, i.e.

$$F^2(x, dx) = g_{ij}(x)dx^i dx^j.$$

In the famous Paris address of 1900, Hillbert devoted the last problem to variational calculus of $\int ds$ and its geometrical overtone. A few years later, the general development took a curious turn away from the basic aspect and methods of the theory as developed by P.Finsler. Finsler's thesis, which treats curves and surfaces of (1.1), must be regarded as the first step in this direction. The name "Finsler geometry" comes from his thesis in 1918 [12].

Now we consider the one of important classes of Finsler metrics known as (α, β)-metrics. For a first time, the notion of (α, β)-metrics are introduced by Matsumoto [11]. If we set $F = \alpha + \beta$, then we get the Randers metric such that it is one of the most famous (α, β)-metric. It is worth noting that an (α, β)-metric is a Finsler metric of the form

$$F = \alpha\varphi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth n-dimensional manifold M and $\beta = b_i(x)y^i$ is a 1-form on M [16]. Among other important and famous (α, β)-metrics, we can mention the Kropina metric $F = \alpha^2/\beta$, infinite series metric $F = \frac{\beta^2}{\beta - \alpha}$, square metric $F = (\alpha + \beta)^2/\alpha$, exponential metric $F = \alpha \exp(\beta/\alpha)$ and Matsumoto metric $F = \alpha^2/(\alpha - \beta)$.

A connected Finsler manifold (M, F) is said to be homogeneous if its full isometry group $I(M)$ acts transitively on M. Just as the concept of a line is very important in Euclidean geometry and has many applications, this concept will naturally be considered important in other geometries. Geodesics are actually considered to be generalizations of lines in Finsler geometry. Geodesics are very important concepts of Finsler geometry and there have been many studies in this field. A geodesic in a homogeneous Finsler space $(G/H, F)$ is called homogeneous geodesic if it is an orbit of a one-parameter subgroup of G. In the Riemannian setting, homogeneous geodesics have been studied by many authors and many results have also been obtained. In [8], the second author for the first time has extended the concept of homogeneous geodesics in homogeneous Finsler spaces.

Suppose (M, F) be a connected homogeneous Finsler space, G is a connected transitive group of isometries of M and H is the isotropy subgroup at a point $o \in M$. Therefore, M is naturally identified with the coset space G/H with G-invariant Finsler metric F. Also, in this case the Lie algebra $\mathfrak{g}$ of G has a reductive decomposition

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{h},$$

where $\mathfrak{m} \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ isomorphic to the T_oM and $\mathfrak{h}$ is the Lie algebra of H.

For the first time, hypercomplex manifolds were introduced by Boyer [3]. He was able to provide a classification of compact hypercomplex manifolds for $dim_{\mathbb{H}}M = 1$ where $\mathbb{H}$ is the quaternion Lie algebra. Let (M, I, J, K) be a manifold equipped with an action of the quaternion algebra $\mathbb{H}$ on TM. The manifold M is called hypercomplex if the operators

$I,J,K\in\mathbb{H}$ define integrable complex structures on M. In [13], Obata showed that this condition is hold if and only if M admits a torsion-free connection ∇ such that

$$\nabla I=\nabla J=\nabla K=0.$$

Obata's connection on (M,I,J,K) is necessarily unique ([13]).

In [16], we describe all geodesic vectors of invariant infinite series metric on the left invariant hypercomplex four dimensional simply connected Lie groups. In this paper, we extend these results to any (α,β)-metrics. Indeed, we study homogeneous geodesics of left invariant (α,β)-metrics on left invariant hypercomplex 4-dimensional simply connected Lie groups and also we study the conditions for the Douglas and Berwald type of (α,β)-metrics on this Lie groups. For more details on geodesic vectors see [5, 9, 14, 15].

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x\in M$ and by $TM:=\cup_{x\in M}T_xM$ the tangent bundle of M. The dual space of T_xM is T_x^*M, called the cotangent space at x. The union $T^*M:=\cup_{x\in M}T_x^*M$ is the cotangent bundle of M.

Definition 2.1. A function $F:TM\to[0,\infty)$ is called a Finsler structure, if, in a local coordinate system (x^i,y^i),

$$F(x,y)=F(y^i\frac{\partial}{\partial x^i}|_x),$$

satisfies

(1) $F(x,y)$ is C^∞ for $y\neq 0$.
(2) $F(x,\lambda y)=\lambda F(x,y)$; $\lambda>0$.
(3) $\frac{1}{2}(F^2)_{y^iy^j}\quad (y\neq 0)$ is positive definite.

A C^∞ manifold M with its Finsler structure F is said a Finsler manifold or Finsler space [12].

Let $\alpha=\sqrt{\widetilde{a}_{ij}\,(x)\,y^iy^j}$ be a norm induced by a Riemannian metric $\widetilde{a}$ and $\beta\,(x,y)=b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Suppose

$$b:=\|\beta(x)\|_\alpha:=\sqrt{\widetilde{a}^{ij}(x)b_i(x)b_j(x)},$$

and let the function F is defined as follows

$$F:=\alpha\varphi(s),\quad s=\frac{\beta}{\alpha},\tag{2.1}$$

where $\varphi=\varphi(s)$ is a positive C^∞ function on $(-b_0,b_0)$ satisfying

$$\varphi\,(s)-s\varphi'\,(s)+\left(b^2-s^2\right)\varphi''\,(s)>0,\qquad |s|\leq\ b<b_0.$$

Then F is a Finsler metric if $\|\beta(x)\|_\alpha<b_0$ for any $x\in M$. A Finsler metric in the form (2.1) is called an (α,β)-metric [16].

Proposition 2.2. *Suppose that (M,α) be a Riemannian space. Then the (α,β)-metric $F=\alpha\phi(\beta/\alpha)$ where $\beta=b_iy^i$, is a 1-form with $\|\beta(x)\|=\sqrt{b_ib^i}<b_0$, consists of a Riemannian metric α along with a smooth vector field X on M with $\alpha(X|_x)<b_0$, $\forall x\in M$, i.e.,*

$$F(x,y) = \alpha(x,y)\phi\left(\frac{\langle X|_x, y\rangle}{\alpha(x,y)}\right), \quad x \in M, \quad y \in T_xM,$$

where $\langle , \rangle$ *is the inner product induced by the Riemannian metric* α.

Proof. We know that, from the definition of Riemannian metric, the form

$$\langle m, n\rangle = a_{ij}m^i n^j, \quad m, n \in T_xM,$$

is an inner product on T_xM. On the other hand, this inner product induces an inner product on the cotangent space T_x^*M such that

$$\langle dx^i(x), dx^j(x)\rangle = a^{ij}(x).$$

Now with this inner product we can define a linear isomorphism between T_x^*M and T_xM. Therefore, the 1-form β corresponds to a vector field X on M such that

$$X|_x = b^i\frac{\partial}{\partial x_i}, \quad b^i = a^{ij}b_j,$$

and

$$\langle X|_x, y\rangle = \langle b^i\frac{\partial}{\partial x_i}, y^j\frac{\partial}{\partial x_j}\rangle = b^i y^j a_{ij} = b_j y^j = \beta(y).$$

Also, we have

$$\alpha(X|_x) = \|\beta\| < b_0.$$

□

Suppose that (M, F) be a Finsler space. The pulled-back bundle π^*TM admits a unique linear connection, called the Chern connection. It's connection forms are characterized by the torsion freeness and almost g-compatibility structural equations. Let the coefficients of Chern connection are denoted by Γ^i_{jk}. Now suppose that $\sigma(t)$ be a smooth regular curve in M, with velocity field T and $W(t) := W^i(t)\frac{\partial}{\partial x^i}$ be a vector field along σ. Then the expression

$$\left[\frac{dW^i}{dt} + W^jT^k\Gamma^i_{jk}\right]\frac{\partial}{\partial x^i}|_{\sigma(t)},$$

would have defined the covariant derivative D_TW, had Γ not had a directional y-dependence. If T is plugged into the direction slot y, we get

$$\left[\frac{dW^i}{dt} + W^jT^k(\Gamma^i_{jk})_{(\sigma,T)}\right]\frac{\partial}{\partial x^i}|_{\sigma(t)},$$

such that we call it D_TW with reference vector T.
A curve $\sigma(t)$ with the velocity T, is a Finslerian geodesic if

$$D_T\left[\frac{T}{F(T)}\right] = 0, \quad \text{With reference vector } T.$$

We note that, the constant speed geodesics are precisely the solutions of

$$D_TT = 0, \quad \text{With reference vector } T.$$

A Finsler structure F is said to be of Berwald type if the Chern connection coefficients Γ^i_{jk} in natural coordinates have no y dependence. We note that, Berwald spaces are just a bit more general than Riemannian and locally Minkowskian spaces. They provide examples that are more properly Finslerian, but only slightly so.

Now let,

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right).$$

We call $D := D^i_{jkl} dx^j \otimes dx^k \otimes dx^l$ the Douglas tensor. Douglas metric is characterized by the curvature equation $D = 0$. We note that all Berwald spaces are Douglas spaces, but there are many non-Berwald Douglas metrics. The following Theorem give a necessary and sufficient condition for an invariant Randers metric to be a Douglas metric on a homogeneous manifold.

Theorem 2.3. [1] *Let α be an invariant Riemannian metric on G/H, $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to inner product $\langle , \rangle$ on $\mathfrak{g}$ induced by α. Then there exists a bijection between the set of all invariant Randers metrics on G/H with the underlying Riemannian metric α and the set*

$$V_1 = \{X \in \mathfrak{m} : Ad(h)X = X,\ \langle X, X\rangle < 1,\ \forall h \in H\}.$$

Moreover, the Randers metric is of Douglas type if and only if X satisfies

$$\langle [Y, Z]_{\mathfrak{m}}, X\rangle = 0, \quad \forall Y, Z \in \mathfrak{m}.$$

Here, $[Y, Z]_{\mathfrak{m}}$ denotes the projection of $[Y, Z]$ to $\mathfrak{m}$.

For left invariant (α, β)-metrics on a Lie group G, we have the following Theorem:

Theorem 2.4. [4] *Let F be a left invariant (α, β)-metric on a Lie group G, arising from a left invariant Riemannian metric $\langle , \rangle$ and a left invariant vector filed X. Then F is of Berwald type if and only if the following two conditions are valid:*

$$\big\langle [y, X], z\big\rangle + \big\langle [z, X], y\big\rangle = 0, \qquad \big\langle [y, z], X\big\rangle = 0, \qquad \forall y, z \in \mathfrak{g}. \tag{2.2}$$

Definition 2.5. Suppose $(G/H, F)$ be a homogeneous Finsler manifold with a fixed origin o. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition. Therefore, a homogeneous geodesic through the $o \in G/H$ is a geodesic $\gamma(t)$ of the form

$$\gamma(t) = \exp(tZ)(o), \quad t \in \mathbb{R}, \tag{2.3}$$

where Z is a nonzero vector of $\mathfrak{g}$.

In [7] and in the Riemannian setting, we have:

Lemma 2.6. [7] *A nonzero vector $X \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\langle [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle = 0, \quad \forall Y \in \mathfrak{m}. \tag{2.4}$$

Here $\mathfrak{m}$ written as subscript denotes the m-component of a vector from $\mathfrak{g}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$.

After this, the second author in Finsler setting shown that:

Lemma 2.7. [8] *Suppose $(G/H, F)$ be a homogeneous Finsler space with a reductive decomposition*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}.$$

Therefore, $Y \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if

$$g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}}) = 0, \quad \forall Z \in \mathfrak{m}, \tag{2.5}$$

where the subscript $\mathfrak{m}$ indicates the projection of a vector from $\mathfrak{g}$ to $\mathfrak{m}$.

Now we give the definition of a complex manifold.

Definition 2.8. Complex manifolds are differentiable manifolds with a holomorphic atlas. They are necessarily of even dimension, say $2n$, and allow for a collection of charts (U_i, z_i) that are one to one maps of the corresponding U_i to $\mathbb{C}^n$ such that for every non-empty intersection $U_i \cap U_j$ the maps are $z_i z_j^{-1}$ are holomorphic.

For example, the (unit) two-sphere S^2, which is the subset of $\mathbb{R}^3$, defined by

$$x^2 + y^2 + z^2 = 1,$$

is a complex manifold.

As the word already indicates, almost complex means it is not quite complex. Indeed, we have:

Definition 2.9. If a real manifold M admits a globally defined tensor J of rank $(1, 1)$ with the property

$$J^2 = -\mathbf{1}, \tag{2.6}$$

then M is called an almost complex manifold. Here, $\mathbf{1}$ is the identity operator and J is a tensor field of type $(1, 1)$; both operators are maps from the tangent bundle TM into itself. Also, a globally defined $(1, 1)$ tensor satisfying (2.6) is called an almost complex structure.

We note that, Locally, this implies that at each given point $p \in M$, there is an endomorphism $J_p : T_pM \to T_pM$ which satisfies $(J_p)^2 = \mathbf{1}_p$ and which depends smoothly on $p \in M$.

3. Hypercomplex manifolds and geodesic vectors of (α, β)-metrics on four dimensional Lie groups

In this section we will study the hypercomplex manifolds. Then we describe all geodesics vectors of (α, β)-metrics on the left invariant hypercomplex four dimensional simply connected Lie groups.

3.1. **Hypercomplex manifolds.** As previously stated, an almost complex structure on a real differentiable manifold M is a tensor field J which is, at every point x of M, an endomorphism of the tangent space T_xM such that $J^2 = -\mathbf{1}$, where $\mathbf{1}$ denotes the identity transformation of T_xM.
We remind that, the Lie bracket is defined on the space of vector fields and acts on functions according to:

$$[X, Y]f = X(Y(f)) - Y(X(f)).$$

Definition 3.1. Let (M, J) be an almost complex manifold. Then for any two vector fields X and Y, we define the Nijenhuis tensor N as

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]. \tag{3.1}$$

Definition 3.2. Let (M, J) be an almost complex manifold. If the Lie bracket of any two holomorphic vector field is again a holomorphic vector filed, then the almost complex structure is said to be integrable.

Definition 3.3. A manifold M with three globally-defined, integrable complex structures I, J, K satisfying the quaternion identities

$$I^2 = J^2 = K^2 = -1, \quad \text{and} \quad IJ = K = -JI, \tag{3.2}$$

is called a hypercomplex manifold.

We note that for a four dimensional manifold M, a **hypercomplex structure** on M is a family $\mathbb{H} = \{J_\alpha\}_{\alpha=1,2,3}$ of fiber-wise endomorphism of TM such that

$$-J_2 J_1 = J_1 J_2 = J_3, \quad J_\alpha^2 = -Id_{TM}, \quad \alpha = 1, 2, 3, \tag{3.3}$$

$$N_\alpha = 0, \quad \alpha = 1, 2, 3, \tag{3.4}$$

where N_α is the Nijenhuis tensor (torsion) corresponding to J_α.

Also, an almost complex structure is a complex structure if and only if it has no torsion [6]. Then the complex structures $J_\alpha,\ \alpha = 1, 2, 3$, on a four dimensional manifold M form a hypercomplex if they satisfy in the relation (3.3).

Definition 3.4. A Riemannian metric $\langle , \rangle$ on a hypercomplex manifold $(M, \mathbb{H})$ is called hyper-Hermitian if for all vector fields X and Y on M and for all $\alpha = 1, 2, 3$ satisfy in the following relation:

$$\langle J_\alpha X, J_\alpha Y \rangle = \langle X, Y \rangle.$$

Definition 3.5. A hypercomplex structure $\mathbb{H} = \{J_\alpha\}_{\alpha=1,2,3}$ on a Lie group G is said to be left invariant if for any $t \in G$ we have

$$J_\alpha = Tl_t \circ J_\alpha \circ Tl_{t^{-1}},$$

where Tl_t is the differential function of the left translation l_t.

3.2. **Geodesics vectors of (α, β)-metrics.** In this section, we consider left invariant hyper-Hermitian Riemannian metrics on left invariant hypercomplex four dimensional simply connected Lie groups. In [2], Barberis shown that in this spaces, $\mathfrak{g}$ is either Abelian or isomorphic to one of the following Lie algebras:

$$\text{Case (1): } [e_2, e_3] = e_4, \quad [e_3, e_4] = e_2, \quad [e_4, e_2] = e_3, \quad e_1 : \text{central}. \tag{3.5}$$

$$\text{Case (2): } [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_4] = e_2, \quad [e_2, e_4] = -e_1. \tag{3.6}$$

$$\text{Case (3): } [e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = e_4. \tag{3.7}$$

$$\text{Case (4): } [e_1, e_2] = e_2, \quad [e_1, e_3] = \frac{1}{2} e_3, \quad [e_1, e_4] = \frac{1}{2} e_4, \quad [e_3, e_4] = \frac{1}{2} e_2. \tag{3.8}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis.

As mentioned earlier in Proposition (2.2), we can also write the (α, β)-metrics as follows:

$$F(y) = \sqrt{\langle y, y\rangle}\phi\Big(\frac{\langle X, y\rangle}{\sqrt{\langle y, y\rangle}}\Big).$$

Now by using the formula

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}F^2(x, y + su + tv)|_{s=t=0},$$

and some computations we get:

$$\begin{aligned} g_y(u, v) &= \langle u, v\rangle\phi^2(r) + \langle y, u\rangle\phi(r)\phi'(r)\Big(\frac{\langle X, v\rangle}{\sqrt{\langle y, y\rangle}} - \frac{\langle X, y\rangle\langle y, v\rangle}{(\langle y, y\rangle)^{\frac{3}{2}}}\Big) \\ &+ \Big((\phi'(r))^2 + \phi(r)\phi''(r)\Big)\Big(\frac{\langle X, v\rangle}{\sqrt{\langle y, y\rangle}} - \frac{\langle X, y\rangle\langle y, v\rangle)}{(\langle y, y\rangle)^{\frac{3}{2}}}\Big) \\ &\times \Big(\langle X, u\rangle\sqrt{\langle y, y\rangle} - \frac{\langle y, u\rangle\langle X, y\rangle}{\sqrt{\langle y, y\rangle}}\Big) \\ &+ \frac{\phi(r)\phi'(r)}{\sqrt{\langle y, y\rangle)}}\Big(\langle X, u\rangle\langle y, v\rangle - \langle u, v\rangle\langle X, y\rangle\Big), \end{aligned} \tag{3.9}$$

where $r = \frac{\langle X,y\rangle}{\sqrt{\langle y,y\rangle}}$. Therefore for all $z \in \mathfrak{g}$ we have:

$$\begin{aligned} g_{y_{\mathfrak{m}}}(y_{\mathfrak{m}}, [y, z]_{\mathfrak{m}}) &= \langle X, [y, z]_{\mathfrak{m}}\rangle\Big(\phi'(r)F(y)\Big) \\ &+ \langle y_{\mathfrak{m}}, [y, z]_{\mathfrak{m}}\rangle\Big(\phi^2(r) - \phi(r)\phi'(r)r\Big) \\ &= \langle SX + Qy_{\mathfrak{m}}, [y, z]_{\mathfrak{m}}\rangle, \end{aligned} \tag{3.10}$$

where

$$S = \phi'(r)F(y), \quad Q = \phi^2(r) - \phi(r)\phi'(r)r.$$

In [10] and in Theorem (3.4), we show that a non-zero vector $y \in \mathfrak{g}$ is a geodesic vector if and only if

$$\langle SX + Qy_{\mathfrak{m}}, [y, z]_{\mathfrak{m}}\rangle = 0, \quad \forall z \in \mathfrak{g}. \tag{3.11}$$

Now, by using equation (3.11) a vector $y = \sum_{i=1}^{4} y_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if for each $j = 1, 2, 3, 4,$

$$\Big\langle S\sum_{i=1}^{4} x_i e_i + Q\sum_{i=1}^{4} y_i e_i, \Big[\sum_{i=1}^{4} y_i e_i, e_j\Big]\Big\rangle = 0. \tag{3.12}$$

Therefore, we get the following cases:

3.2.1. *Case (1).*

$$\begin{cases} j = 2 \;\rightarrow\; & S(x_3y_4 - x_4y_3) = 0, \\ j = 3 \;\rightarrow\; & S(x_4y_2 - x_2y_4) = 0, \\ j = 4 \;\rightarrow\; & S(x_2y_3 - x_3y_2) = 0. \end{cases}$$

As a special case, if $X = x_1e_1$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_1\}$.

Corollary 3.6. *Let (M, F) be a Finsler space with (α,β)-metric F defined by an invariant metric $\langle,\rangle$ and an invariant vector field $X = \sum_{i=1}^{4} x_i e_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.5) holds. Then geodesic vectors depending on x_2, x_3 and x_4.*

Theorem 3.7. *Let (M, F) be a Finsler space with (α,β)-metric F defined by an invariant metric $\langle,\rangle$ and an invariant vector field $X = x_1 e_1$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.5) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \langle,\rangle)$.*

Proof. Let $y \in \sum_{i=1}^{4} y_i e_i \in \mathfrak{g}$. Let y is a geodesic vector of $(M, \langle,\rangle)$. By using (2.4) we have $\langle y, [y, e_i]\rangle = 0$ for each $i = 1, 2, 3, 4$. Therefore by using (3.12), y is a geodesic of (M, F). Conversely, let $y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{g}$ is a geodesic vector of (M, F), because $\langle X, [y, e_i]\rangle = 0$ for each $i = 1, 2, 3, 4$, by using (3.12) we have $\langle y, [y, e_i]\rangle = 0$. □

In the following, we shall give a necessary and sufficient condition for an invariant (α,β)-metric to be Berwald metric on a left invariant hypercomplex 4-dimensional simply connected Lie groups with (3.5) holds.

Theorem 3.8. *Suppose (M, F) be the left invariant (α,β)-metric induced by the Riemannian metric $\langle,\rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.5) holds. If (M, F) is of Berwald type, then $\langle X, e_2\rangle = \langle X, e_3\rangle = \langle X, e_4\rangle = 0$.*

Proof. From Theorem (2.4), if the corresponding left invariant (α,β)-metric is of Berwald type, then X satisfy

$$\langle [y, z], X\rangle = 0, \qquad \forall y, z \in \mathfrak{m}. \tag{3.13}$$

Therefore, by using formula (3.5) the proof is complete. □

Corollary 3.9. *Suppose (M, F) be the left invariant (α,β)-metric induced by the Riemannian metric $\langle,\rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.5) holds. If $\langle X, e_2\rangle = \langle X, e_3\rangle = \langle X, e_4\rangle = 0$ and $\big\langle [y, X], z\big\rangle + \big\langle [z, X], y\big\rangle = 0$ holds, then (M, F) must be Berwaldian.*

Now, we shall give a necessary and sufficient condition for an invariant Randers metric to be Douglas metric on a left invariant hypercomplex 4-dimensional simply connected Lie groups with (3.5) holds.

Theorem 3.10. *Suppose (M, F) be the Randers metric induced by the Riemannian metric $\langle,\rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.5) holds. Then (M, F) is of Douglas type if and only if $\langle X, e_2\rangle = \langle X, e_3\rangle = \langle X, e_4\rangle = 0$.*

Proof. From Theorem (2.3), the corresponding Randers metric is of Douglas type if and only if X satisfy

$$\langle [y, z], X\rangle = 0, \qquad \forall y, z \in \mathfrak{m}. \tag{3.14}$$

Therefore, by using formula (3.5) the proof is complete. □

3.2.2. *Case (2).*

$$\begin{cases} j=1 \ \rightarrow \quad Sx_1y_3+Qy_1y_3+Sx_2y_4+Qy_2y_4=0, \\ j=2 \ \rightarrow \quad Sx_1y_4+Qy_1y_4-(Sx_2y_3+Qy_2y_3)=0, \\ j=3 \ \rightarrow \quad Sx_1y_1+Qy_1^2+Sx_2y_2+Qy_2^2=0, \\ j=4 \ \rightarrow \quad S(x_2y_1-x_1y_2)=0. \end{cases}$$

As a special case, if $X = x_3e_3 + x_4e_4$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_3, e_4\}$.

Corollary 3.11. *Let (M, F) be a Finsler space with (α, β)-metric F defined by an invariant metric $\langle , \rangle$ and an invariant vector field $X = \sum_{i=1}^{4} x_ie_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.6) holds. Then geodesic vectors depending on x_1 and x_2.*

Theorem 3.12. *Let (M, F) be a Finsler space with (α, β)-metric F defined by an invariant metric $\langle , \rangle$ and an invariant vector field $X = x_3e_3 + x_4e_4$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.6) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \langle , \rangle)$.*

Proof. The proof is the same as before. □

In the following, we shall give a necessary and sufficient condition for an invariant (α, β)-metric to be Berwald metric on a left invariant hypercomplex 4-dimensional simply connected Lie groups with (3.6) holds.

Theorem 3.13. *Suppose (M, F) be the left invariant (α, β)-metric induced by the Riemannian metric $\langle , \rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.6) holds. If (M, F) is of Berwald type, then $\langle X, e_1\rangle = \langle X, e_2\rangle = 0$.*

Proof. From Theorem (2.4), if the corresponding left invariant (α, β)-metric is of Berwald type, then X satisfy

$$\langle [y, z], X\rangle = 0, \qquad \forall y, z \in \mathfrak{m}. \tag{3.15}$$

Therefore, by using formula (3.6) the proof is complete. □

Corollary 3.14. *Suppose (M, F) be the left invariant (α, β)-metric induced by the Riemannian metric $\langle , \rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.6) holds. If $\langle X, e_1\rangle = \langle X, e_2\rangle = 0$ and $\big\langle [y, X], z\big\rangle + \big\langle [z, X], y\big\rangle = 0$ holds, then (M, F) must be Berwaldian.*

Now, we shall give a necessary and sufficient condition for an invariant Randers metric to be Douglas metric on a left invariant hypercomplex 4-dimensional simply connected Lie groups with (3.6) holds.

Theorem 3.15. *Suppose (M, F) be the Randers metric induced by the Riemannian metric $\langle , \rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.6) holds. Then (M, F) is of Douglas type if and only if $\langle X, e_1\rangle = \langle X, e_2\rangle = 0$.*

Proof. From Theorem (2.3), the corresponding Randers metric is of Douglas type if and only if X satisfy

$$\langle [y, z], X\rangle = 0, \qquad \forall y, z \in \mathfrak{m}. \tag{3.16}$$

Therefore, by using formula (3.6) the proof is complete. □

3.2.3. *Case (3).*

$$\begin{cases} j = 1 \ \rightarrow \quad S(x_2y_2 + x_3y_3 + x_4y_4) + Q(y_2^2 + y_3^2 + y_4^2) = 0, \\ j = 2 \ \rightarrow \quad Sx_2y_1 + Qy_2y_1 = 0, \\ j = 3 \ \rightarrow \quad Sx_3y_1 + Qy_3y_1 = 0, \\ j = 4 \ \rightarrow \quad Sx_4y_1 + Qy_4y_1 = 0. \end{cases}$$

As a special case, if $X = x_1e_1$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_1\}$.

Corollary 3.16. *Let (M, F) be a Finsler space with (α, β)-metric F defined by an invariant metric $\langle,\rangle$ and an invariant vector field $X = \sum_{i=1}^{4} x_ie_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.7) holds. Then geodesic vectors depending on x_2, x_3 and x_4.*

Theorem 3.17. *Let (M, F) be a Finsler space with (α, β)-metric F defined by an invariant metric $\langle,\rangle$ and an invariant vector field $X = x_1e_1$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.7) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \langle,\rangle)$.*

Proof. The proof is the same as before. □

Now, we shall give a necessary and sufficient condition for an invariant (α, β)-metric to be Berwald metric on a left invariant hypercomplex 4-dimensional simply connected Lie groups with (3.7) holds.

Theorem 3.18. *Suppose (M, F) be the left invariant (α, β)-metric induced by the Riemannian metric $\langle,\rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.7) holds. If (M, F) is of Berwald type, then $\langle X, e_2\rangle = \langle X, e_3\rangle = \langle X, e_4\rangle = 0$.*

Proof. From Theorem (2.4), if the corresponding left invariant (α, β)-metric is of Berwald type, then X satisfy

$$\langle [y, z], X\rangle = 0, \qquad \forall y, z \in \mathfrak{m}. \tag{3.17}$$

Therefore, by using formula (3.7) the proof is complete. □

Corollary 3.19. *Suppose (M, F) be the left invariant (α, β)-metric induced by the Riemannian metric $\langle,\rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.7) holds. If $\langle X, e_2\rangle = \langle X, e_3\rangle = \langle X, e_4\rangle = 0$ and $\big\langle [y, X], z\big\rangle + \big\langle [z, X], y\big\rangle = 0$ holds, then (M, F) must be Berwaldian.*

Now, we shall give a necessary and sufficient condition for an invariant Randers metric to be Douglas metric on a left invariant hypercomplex 4-dimensional simply connected Lie groups with (3.7) holds.

Theorem 3.20. *Suppose (M, F) be the Randers metric induced by the Riemannian metric $\langle,\rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.7) holds. Then (M, F) is of Douglas type if and only if $\langle X, e_2\rangle = \langle X, e_3\rangle = \langle X, e_4\rangle = 0$.*

Proof. From Theorem (2.3), the corresponding Randers metric is of Douglas type if and only if X satisfy

$$\langle [y, z], X\rangle = 0, \qquad \forall y, z \in \mathfrak{m}. \tag{3.18}$$

Therefore, by using formula (3.7) the proof is complete. □

3.2.4. *Case (4).*

$$\begin{cases} j = 1 \;\rightarrow\; & S(2x_2y_2 + x_3y_3 + x_4y_4) + Q(2y_2^2 + y_3^2 + y_4^2) = 0, \\ j = 2 \;\rightarrow\; & Sx_2y_1 + Qy_2y_1 = 0, \\ j = 3 \;\rightarrow\; & S(x_3y_1 - x_2y_4) + Q(y_3y_1 - y_2y_4) = 0, \\ j = 4 \;\rightarrow\; & S(x_2y_3 + x_4y_1) + Q(y_4y_1 + y_2y_3) = 0. \end{cases}$$

As a special case, if $X = x_1e_1$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_1\}$.

Corollary 3.21. *Let (M, F) be a Finsler space with infinite series metric F defined by an invariant metric $\langle,\rangle$ and an invariant vector field $X = \sum_{i=1}^{4} x_ie_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.8) holds. Then geodesic vectors depending on x_2, x_3 and x_4.*

Theorem 3.22. *Let (M, F) be a Finsler space with (α, β)-metric F defined by an invariant metric $\langle,\rangle$ and an invariant vector field $X = x_1e_1$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.8) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \langle,\rangle)$.*

Proof. The proof is the same as before. □

Now, we shall give a necessary and sufficient condition for an invariant (α, β)-metric to be Berwald metric on a left invariant hypercomplex 4-dimensional simply connected Lie groups with (3.8) holds.

Theorem 3.23. *Suppose (M, F) be the left invariant (α, β)-metric induced by the Riemannian metric $\langle,\rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.8) holds. If (M, F) is of Berwald type, then $\langle X, e_2\rangle = \langle X, e_3\rangle = \langle X, e_4\rangle = 0$.*

Proof. From Theorem (2.4), if the corresponding left invariant (α, β)-metric is of Berwald type, then X satisfy

$$\langle [y, z], X\rangle = 0, \qquad \forall y, z \in \mathfrak{m}. \tag{3.19}$$

Therefore, by using formula (3.8) the proof is complete. □

Corollary 3.24. *Suppose (M, F) be the left invariant (α, β)-metric induced by the Riemannian metric $\langle , \rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.8) holds. If $\langle X, e_2\rangle = \langle X, e_3\rangle = \langle X, e_4\rangle = 0$ and $\big\langle [y, X], z\big\rangle + \big\langle [z, X], y\big\rangle = 0$ holds, then (M, F) must be Berwaldian.*

Now, we shall give a necessary and sufficient condition for an invariant Randers metric to be Douglas metric on a left invariant hypercomplex 4-dimensional simply connected Lie groups with (3.8) holds.

Theorem 3.25. *Suppose (M, F) be the Randers metric induced by the Riemannian metric $\langle , \rangle$ and the left invariant vector field X on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.8) holds. Then (M, F) is of Douglas type if and only if $\langle X, e_2\rangle = \langle X, e_3\rangle = \langle X, e_4\rangle = 0$.*

Proof. From Theorem (2.3), the corresponding Randers metric is of Douglas type if and only if X satisfy

$$\langle [y, z], X\rangle = 0, \qquad \forall y, z \in \mathfrak{m}. \tag{3.20}$$

Therefore, by using formula (3.8) the proof is complete. □

References

[1] H. An and S. Deng, *Invariant (α, β)-metric on homogeneous manifolds*, Monatsh. Math., **154**, (2008), 89-102.

[2] M. L. Barberis, *Hypercomplex structures on four-dimensional Lie groups*, Proc. Am. Math. Soc., **125**, (1997), 1043-1054.

[3] Boyer and P. Charles, *A note on hyper-Hermitian four-manifolds*, Proc. Amer. Math. Soc., **102** (1), (1988), 157164.

[4] S. Deng, M. Hosseini, H. Liu and H. R. Salimi Moghaddam, *On the left invariant (α, β)-metrics on some Lie groups*, Houston J. Math., **45** (4), (2019), 1071-1088.

[5] M. Ebrahimi and D. Latifi, *On flag curvature and homogeneous geodesics of left invariant Randers metrics on the semi-direct product $a \oplus_p r$*, Journal of Lie Theory, **29** (3), (2019), 619-627.

[6] S. Kobayashi and K. Nomizu, *Foundations of differential Geometry*, Interscience Publishers, 1969.

[7] O. Kowalski and J. Szenthe, *On the Existence of Homogeneous Geodesics in Homogeneous Riemannian manifolds*, Geom. Dedicata, **81**, (2000), 209-214.

[8] D. Latifi, *Homogeneous geodesics in homogeneous Finsler spaces*, J. Geom. Phys., **57**, (2007), 1421-1433.

[9] D. Latifi and A. Razavi, *Homogeneous geodesics of left invariant Randers metrics on a three-dimensional Lie group*, Int. J. Contemp. Math. Sciences, **4** (18), (2009), 873-881.

[10] D. Latifi and M. L. Zeinali, *Geodesic vectors of invariant (α, β)-metrics on nilpotent Lie groups of five dimensional*, Caspian Journal of Mathematical Sciences, **12** (2) (2023), 211-223.

[11] M. Matsumoto, *Theory of Finsler spaces with (α, β)-metric*, Rep. Math. Phys., **31**, (1992), 33-65.

[12] X. H. MO, *An Introduction to Finsler Geometry*, Vol.1, World Scientific Publishing Co. Pte. Ltd., 2006.

[13] M. Obata, *Affine connections on manifolds with almost complex quaternion or Hermitian structure*, Jap. J. Math, **26**, (1956), 43-79.

[14] M. Parhizkar and D. Latifi, *Geodesic vectors of Randers metrics on nilpotent Lie groups of dimension five*, Global. J. Adv. Res. Class. Moder. Geom, **7**, (2018), 92-101.

[15] M. Parhizkar and D. Latifi, *On invariant Matsumoto metrics*, Vietnam Journal of Mathematics, **47**, (2019), 355-365.

[16] M. L. Zeinali, *Geodesic vectors of infinite series (α, β)-metric on hypercomplex four dimensional Lie groups*, Journal of Finsler Geometry and its Applications, **4** (2), (2023), 103-112.