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Research Paper

ON THE GENERALIZATION OF PSEUDO P-CLOSURE IN PSEUDO BCI-ALGEBRAS

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ABSTRACT

In this paper, the notion of generalization of pseudo p-closure, denoted by gcl, is introduced and its related properties are investigated. The gcl of subalgebras and pseudo-ideals is discussed. Also, a necessary and sufficient condition for an element to be minimal; and for pseudo BCI-algebra to be nilpotent are given. It is proved that the set of all nilpotent elements of a pseudo BCI-algebra A, denoted by $\mathcal{N}_A$, is the least closed pseudo-ideal with the property $\text{gcl}(\mathcal{N}_A) = \mathcal{N}_A$. Finally, it is shown that the mentioned notion, as a function, defines a closure operation on pseudo-ideals.

1. Introduction

The notion of BCK/BCI-algebras was introduced by Y. Imai and K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi [7]. In 2008, W. A. Dudek and Y. B. Jun extended the idea of BCI-algebras to introduce pseudo BCI-algebras [5]. Y. B. Jun et al. introduced the idea of pseudo-ideal and pseudo-homomorphism in a pseudo BCI-algebra, and then they investigated several related properties. G. Dymek introduced the idea of p-semisimple pseudo BCI-algebras and established its characterizations [3]. For example, it is shown the p-semisimple pseudo BCI-algebras and the groups are categorically equivalent. The idea of minimal elements in a pseudo BCI-algebras was

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defined by Y. H. Kim and K. S. So [10]. Furthermore, it was shown that the collection of all minimal elements of a pseudo BCI-algebra form a subalgebras of A. In [12], H. Moussei et al. introduced the idea of p-closure in BCI-algebras and investigated some related properties. In the sequel, H. Harizavi introduced the notion of p-closure in pseudo BCI-algebras and discussed several related properties [6]. In this paper, following [6], we defined the generalization of p-closure, denoted by gcl(C) for a non-empty subset of a pseudo BCI-algebra, and discuss several related properties. We determined the gcl of subalgebras and pseudo-ideals. We discus the relationship between the gcl and the minimal elements. Also, using the notion of gcl, we give a necessary and sufficient condition for a pseudo BCI-algebra to be nilpotent. Moreover, we prove that the set of all nilpotent elements of a pseudo BCI-algebra A, denoted by $\mathcal{N}_A$, is the least closed pseudo-ideal with the property $\mathrm{gcl}(\mathcal{N}_A) = \mathcal{N}_A$. Finally, we showed that the gcl, as a function, acts on closed pseudo-ideals as the same as a closure operation.

2. Preliminaries

In this section, we present some fundamental results which are useful in this paper, and for additional details, the reader is referred to [5, 11].

By a BCI-algebra, we mean an algebra $(A, *, 0)$ of type $(2, 0)$ satisfying the following axioms for all $r, s, t \in A$:
BCI-1: $((r * s) * (r * t)) * (t * s) = 0$,
BCI-2: $r * r = 0$,
BCI-3: $r * s = 0$ and $s * r = 0$ imply $r = s$.

A BCI-algebra $(A, *, \diamond)$ that satisfies the property $0 * r = 0$ for all $r \in A$ is known as a BCK-algebra [13].

Definition 2.1. [5] A pseudo BCI-algebra is the structure $A = (A, \preceq, *, \diamond, 0)$ consists of $\preceq$ as a binary relation on set A, $*$ and $\diamond$ as binary operations on A, and 0 as an element of A satisfying the following axioms: for all $r, s, t \in A$,
(a_1) $(r * s) \diamond (r * t) \preceq t * s$, $(r \diamond s) * (r \diamond t) \preceq t \diamond s$,
(a_2) $r * (r \diamond s) \preceq s$, $r \diamond (r * s) \preceq s$,
(a_3) $r \preceq r$,
(a_4) $r \preceq s$, $s \preceq r \Longrightarrow r = s$,
(a_5) $r \preceq s \Longleftrightarrow r * s = 0 \Longleftrightarrow r \diamond s = 0$.

A pseudo BCI-algebra $A = (A, \preceq, *, \diamond, 0)$ satisfying the property $0 * a = 0 = 0 \diamond a$ for all $a \in A$ is known as a pseudo BCK-algebra. It is clear that every pseudo BCI-algebra (respectively, pseudo BCK-algebra) satisfying the property $r * s = r \diamond s$ for any $r, s \in A$ is a BCI-algebra (respectively, BCK-algebra).

Example 2.2. [3] Consider $A = [0, \infty)$ with the usual order $\leqslant$. Define binary operations $*$ and $\circ$ on A as:

$$r * s = \begin{cases} 0 & \text{if } r \leqslant s \\ \frac{2r}{\pi} \arctan(\ln(\frac{r}{s})) & \text{if } s < r, \end{cases}$$

$$r \circ s = \begin{cases} 0 & \text{if } r \leqslant s \\ re^{-\tan(\frac{\pi s}{2r})} & \text{if } s < r, \end{cases}$$

for all $r, s \in A$. Then $(A, \leqslant, *, \circ, 0)$ is a pseudo BCK-algebra, and hence it is a pseudo BCI-algebra.

Proposition 2.3. [5] *Any pseudo BCI-algebra A satisfies the following conditions: for any $r, s, t \in A$,*
(p_1) $r \preceq 0 \Longrightarrow r = 0$,
(p_2) $r \preceq s \Longrightarrow r * t \preceq s * t, r \diamond t \preceq s \diamond t$,
(p_3) $r \preceq s \Longrightarrow t * s \preceq t * r, t \diamond s \preceq t \diamond r$,
(p_4) $r \preceq s, s \preceq t \Longrightarrow r \preceq t$,
(p_5) $(r * s) \diamond t = (r \diamond t) * s$,
(p_6) $r * s \preceq t \Leftrightarrow r * t \preceq s$,
(p_7) $(r * s) * (t * s) \preceq r * t, (r \diamond s) \diamond (t \diamond s) \preceq r \diamond t$,
(p_8) $r * (r \diamond (r * s)) = r * s$ *and* $r \diamond (r * (r \diamond s)) = r \diamond s$,
(p_9) $r * 0 = r = r \diamond 0$,
(p_{10}) $r * r = 0 = r \diamond r$,
(p_{11}) $0 * (r \diamond s) \preceq s \diamond r$ *and* $0 \diamond (r * s) \preceq s * r$,
(p_{12}) $0 * r = 0 \diamond r$,
(p_{13}) $0 * (r * s) = (0 * r) \diamond (0 * s)$ *and* $0 \diamond (r \diamond s) = (0 \diamond r) * (0 \diamond s)$.

Notation 2.4. For any elements r, s of a pseudo BCI-algebra A and a natural number p, we denote

$$r * s^{(\diamond,p)} = ((\ldots(r * \underbrace{s) \diamond s) \diamond s)\ldots) \diamond s}_{p-times}.$$

$$r * s^{(\diamond,*,p)} = ((\ldots(r * \underbrace{s) \diamond s) * s) \diamond \ldots)}_{p-times}.$$

Let $(A, \preceq, *, \circ, 0)$ be a pseudo BCI-algebra and S a non-empty subset of A. Then S is called a subalgebra of A if $r * s \in S$ and $r \diamond s \in S$ for any $r, s \in S$. It can be checked that $K(A) := \{r \in A \mid 0 * r = 0 = 0 \circ r\}$ is a subalgebra of A, which implies that $(K(A), \preceq, *, \diamond, 0)$ forms a pseudo BCK-algebra.

In a pseudo BCI-algebra A, an element m is called minimal if the following condition holds:

$$(\forall r \in A)\ r \preceq m \Longrightarrow r = m.$$

The set of all minimal elements of A will be denoted by $M(A)$. Clearly, $0 \in M(A)$. In [8], it has showed that $m \in M$ if and only if $m = 0 * (0 \diamond m)$.
Hence $M(A) = \{m \in A \mid m = 0 * (0 \diamond m)\}$. It can be shown that $K(A) \bigcap M(A) = \{0\}$.

A pseudo BCI-algebra A is called *p-semisimple* if every element in A is minimal, that is $M(A) = A$.

Proposition 2.5. [2] *Considering a pseudo BCI-algebra A and elements $r, s \in A$, the following conditions are equivalent:*
(i) *A is p-semisimple,*
(ii) $r * (r \diamond s) = s = r \diamond (r * s)$,
(iii) $0 * (0 \diamond r) = r = 0 \diamond (0 * r)$.

An element a of a pseudo BCI-algebra A is said to be *nilpotent* if $0 * a^{(*,p)} = 0$ (or equivalently $0 \diamond a^{(\diamond,p)} = 0$) for some $p \in \mathbb{N}$. The set of all nilpotent elements of A is denoted by $\mathcal{N}_A$. A pseudo BCI-algebra A is called *nilpotent* if all its elements are nilpotent, that is $\mathcal{N}_A = A$.

If m is a minimal element of A, then the set $V(m) := \{a \in A \mid m \preceq a\}$ is called the branch of m. It has proved that for any pseudo BCI-algebra A, $A = \cup_{m \in M(A)} V(m)$ [4].

Definition 2.6. [9] In a pseudo BCI-algebra A, a subset I of A is called a pseudo-ideal if it satisfies the following conditions:
$(I1)$ $0 \in I$,
$(I2)$ $(\forall s \in I)(*(s, I) \subseteq I$ and $\diamond(s, I) \subseteq I)$,
where $*(s, I) := \{r \in A \mid r * s \in I\}$ and $\diamond(s, I) := \{r \in A \mid r \diamond s \in I\}$.

Theorem 2.7. [9] *Let A be a pseudo BCI-algebra and I a pseudo-ideal of A. Then, the following statements hold:*
(i) $(\forall r, s \in A)$ $s * r \in I$ *(or* $s \diamond r \in I$*) and* $r \in I \Longrightarrow s \in I$,
(ii) $(\forall r, s \in A)$ $s \preceq r$ *and* $r \in I \Longrightarrow s \in I$,
(ii) I is closed if and only if $0 * r = 0 \diamond r \in I$ *for any* $r \in I$.

A mapping $p : E \to E$ is said to be a closure operation on an ordered set $(E, \leq)$ if it satisfies the following properties: for all $x, y \in E$,
(i) $x \leq p(x)$,
(ii) $x \leq y \Rightarrow p(x) \leq p(y)$,
(iii) $p(p(x)) = p(x)$.

3. On the generalization of pseudo p-closure

We will start by defining the concept of gcl(C) for a non-empty subset C of a pseudo BCI-algebra A, and then investigate some related properties. Throughout, let A be a pseudo BCI-algebra unless specified otherwise.

Definition 3.1. Assuming C is a non-empty subset of A, the generalization of pseudo p-closure of C, represented by gcl(C), is defined as follows:

$$\text{gcl(C)}:=\{a\in A \mid c * a^{(*,p)},\ c \diamond a^{(\diamond,p)} \in C \text{ for some } c \in C \text{ and } p \in \mathbb{N}\}.$$

Example 3.2. [1] Let $A = \{0, a, b, x, y, g\}$ be a pseudo BCI-algebra in which $*$ and $\diamond$ are binary operations that defined by the following Cayley's tables:

$*$	0	a	b	x	y	g
0	0	0	0	g	g	g
a	a	0	a	g	y	y
b	b	b	0	x	g	x
x	x	g	x	0	b	b
y	y	y	g	a	0	a
g	g	g	g	0	0	0

$\diamond$	0	a	b	x	y	g
0	0	0	0	g	g	g
a	a	0	a	x	g	x
b	b	b	0	g	y	y
x	x	x	g	0	a	a
y	y	g	y	b	0	b
g	g	g	g	0	0	0

By taking $C := \{a\}$, it is straightforward to verify that gcl(C)$=\{0, b, g\}$.

From Definition 3.1 and Proposition 2.3(p_9), (p_{10}), the following lemma is clear.

Lemma 3.3. *If C, D are non-empty subsets of A, then we have:*
(i) $0 \in gcl(C)$,
(ii) if $C \subseteq D$, then $gcl(C) \subseteq gcl(D)$,
(iii) if $0 \in C$, then $C \subseteq gcl(C)$.

The next theorem characterizes the minimal elements within A.

Theorem 3.4. *$m \in A$ is minimal if and only if $gcl(\{m\}) = \mathcal{N}_A$.*

Proof. Assume that $a \in \mathrm{gcl}(\{m\})$ for an element minimal m of A. Then there exists $p \in \mathbb{N}$ such that $m * a^{(*,p)} = m = m \diamond a^{(\diamond,p)}$. Consequently, by Proposition 2.3(p_5), we get $0 = (m * a^{(*,p)}) \diamond m = (m \diamond m) * a^{(*,p)} = 0 * a^{(*,p)}$, that is $0 * a^{(*,p)} = 0$ which yield $a \in \mathcal{N}_A$. Hence $\mathrm{gcl}(\{m\}) \subseteq \mathcal{N}_A$. To prove the reverse inclusion, let $a \in \mathcal{N}_A$. Thus $0 * a^{(*,q)} = 0$ for some $q \in \mathbb{N}$, and so, we have

$$
\begin{aligned}
&\text{by the minimality of } m && m * a^{(*,q)} = (0 \diamond (0 * m)) * a^{(*,q)} \\
&\text{by Proposition 2.3}(p_5) && \qquad\qquad = (0 * a^{(*,q)}) \diamond (0 * m) \\
& && \qquad\qquad = 0 \diamond (0 * m) \\
&\text{by the minimality of } m && \qquad\qquad = m,
\end{aligned}
$$

that is $m * a^{(*,q)} = m$. Similarly, we have $m \diamond a^{(\diamond,q)} = m$ which yield $a \in \mathrm{gcl}(\{m\})$ and so $\mathcal{N}_A \subseteq \mathrm{gcl}(\{m\})$. Therefore $\mathrm{gcl}(\{m\}) = \mathcal{N}_A$.

Conversely, suppose that $\mathrm{gcl}(\{m\}) = \mathcal{N}_A$ for $m \in A$. Let $d \in A$ with $d \preceq m$. Hence, by Proposition 2.3(p_2) we get $0 \preceq m * d$, therefore, this implies that $m * d \in K(A)$. It is not difficult to see that $K(A) \subseteq \mathcal{N}_A$. Hence $m * d \in \mathcal{N}_A$, and so, by hypothesis, $m * d \in \mathrm{gcl}(\{m\})$ which implies $m * (m * d)^{(*,p)} = m = m \diamond (m * d)^{(\diamond,p)}$ for some $p \in \mathbb{N}$. Thus, we have

$$
\begin{aligned}
&\text{by Proposition 2.3}(p_5) && m * d = (m \diamond (m * d)^{(\diamond,p)}) * d = (m * d) \diamond (m * d)^{(\diamond,p)} \\
& && \qquad = ((m * d) \diamond (m * d)) \diamond (m * d)^{(\diamond,p-1)} \\
& && \qquad = 0 \diamond (m * d)^{(\diamond,p-1)} \\
& && \qquad = (0 \diamond (m * d)) \diamond (m * d)^{(\diamond,p-2)} \\
&\text{by Proposition 2.3}(p_{11}) && \qquad \preceq (d * m) \diamond (m * d)^{(\diamond,p-2)} \\
& && \qquad = 0 \diamond (m * d)^{(\diamond,p-2)} = \ldots = 0 \diamond (m * d) \preceq d * m = 0.
\end{aligned}
$$

Consequently, we have $m * d = 0$, which implies $m \preceq d$. Therefore $m = d$, and so m is a minimal element of A. $\square$

Proposition 3.5. *If C is a subset of A containing a minimal element of A, then $\mathcal{N}_A \subseteq gcl(C)$.*

Proof. Let m be an arbitrary minimal element of C. By Lemma 3.3, $\mathrm{gcl}(\{m\}) \subseteq \mathrm{gcl(C)}$. On other hand, by Theorem 3.4, we have $\mathcal{N}_A = \mathrm{gcl}(\{m\})$. Therefore $\mathcal{N}_A \subseteq \mathrm{gcl(C)}$. $\square$

Theorem 3.6. *For any pseudo BCI-algebra A, $gcl\ (M(A)) = A$.*

Proof. Suppose initially that $a \in A$. It follows from $A = \cup_{m \in M(A)} V(m)$ that $a \in V(m)$ for some $m \in M(A)$, and therefore $m \preceq a$. Consequently, $m * a = 0 \in M(A)$, which implies $a \in \mathrm{gcl}(M(A))$. Hence, $A = \mathrm{gcl}(M(A))$.

$\square$

Corollary 3.7. *For any subset C of A, if $M(A) \subseteq C$, then $gcl(C) = A$.*

Proof. This is a direct consequence of Lemma 3.3(ii) and Theorem 3.6. □

The converse of Corollary 3.7 is not universally true as shown in the following example.

Example 3.8. Consider $A = \{0, a, b, x, y, g\}$ as a pseudo BCI-algebra in Example 3.2. By taking $C = \{a, b, x, y, g\}$, it can be checked that $M(A) = \{0, g\}$ and $\mathrm{gcl}(C) = A$, but $M(A) \nsubseteq C$.

Theorem 3.9. *If A is a pseudo BCK-algebra, then $gcl(\{0\}) = A$.*

Proof. The proof is straightforward by using Proposition 2.3(p_{12}). □

The following example show that the converse of Theorem 3.9 is not true in general.

Example 3.10. Consider $A = \{0, a, b, x, y, g\}$ as a pseudo BCI-algebra in Example 3.2. With a simple calculation, we get $\mathrm{gcl}(\{0\}) = A$, but A is not a pseudo BCK-algebra.

Theorem 3.11. *A pseudo BCI-algebra A is nilpotent if and only if $gcl(\{0\}) = A$.*

Proof. The proof is straightforward by using Definition 3.1. □

The following lemma is useful for the proof of the next theorems.

Lemma 3.12. *For any $a, d \in A$ and $p, q \in \mathbb{N}$, the following conditions hold:*
(i) $0 * a^{(*,p)} = 0 \diamond a^{(\diamond,p)}$,
(ii) $0 * (0 * a^{(*,p)}) = 0 * (0 * a)^{(*,p)}$,
(iii) *If* $0 \diamond (0 * a^{(\diamond,p)}) = 0$, *then* $0 * a^{(*,p)} = 0$,
(iv) $0 * (d \diamond a^{(\diamond,p)}) = (0 * d) * (0 * a^{(*,p)})$,
(v) $0 * (d * a^{(*,p)}) = (0 * d) \diamond (0 * a)^{(\diamond,p)}$,
(vi) $0 * (0 * a^{(*,p)})^{(*,q)} = 0 \diamond (0 \diamond a)^{(\diamond,pq)} = 0 * (0 * a^{(*,pq)})$.

Proof. (i)-(iii) The proofs are straightforward by using the induction method and proposition 2.3(p_5).

(iv) We proceed by induction on $p \geq 1$. For $p = 1$, the result holds by Proposition 2.3(p_{13}). Suppose that the statement is true for p, that is

$$0 * (d \diamond a^{(\diamond,p)}) = (0 * d) * (0 * a^{(*,p)}), \tag{3.1}$$

and prove it for $p + 1$. For this purpose, we have

$$\begin{aligned}
& 0 * (d \diamond a^{(\diamond,p+1)}) = 0 * ((d \diamond a^{(\diamond,p)}) \diamond a) \\
\text{by Proposition 2.3}(p_{13}) \quad & = (0 * (d \diamond a^{(\diamond,p)})) * (0 * a) \\
\text{by (1)} \quad & = ((0 * d) * (0 * a^{(*,p)})) * (0 * a) \\
\text{by Proposition 2.3}(p_{13}) \text{ and (ii)} \quad & = ((0 * (0 * a)^{(*,p)}) * (0 * a)) \diamond d \\
& = (0 * (0 * a)^{(*,p+1)}) \diamond d \\
\text{by (ii)} \quad & = (0 * (0 * a^{(*,p+1)})) \diamond d \\
\text{by Proposition 2.3}(p_5) \quad & = (0 * d) * (0 * a^{(*,p+1)})
\end{aligned}$$

This completes the proof.

(v) The proof is similar to the proof of (iv).

(vi) We proceed by induction on $p \geq 1$. For $p = 1$, the result clear for any $q \in \mathbb{N}$. Suppose that the statement is true for p and any $q \in \mathbb{N}$ and prove it for $p + 1$. We have

$$\begin{aligned} 0 * (0 * a^{(*,q)})^{(*,p+1)} &= (0 * (0 * a^{(*,q)})^{(*,p)}) * (0 * a^{(*,q)}) \\ \text{by the induction hypothesis} \quad &= (0 \diamond (0 \diamond a)^{(\diamond,pq)}) * (0 * a^{(*,q)}) \\ \text{by Proposition 2.3}(p_5) \quad &= (0 * (0 * a^{(*,q)})) \diamond (0 \diamond a)^{(\diamond,(pq)} \\ \text{by (ii)} \quad &= (0 \diamond (0 \diamond a)^{(\diamond,q)}) \diamond (0 \diamond a)^{(\diamond,(pq)} \\ &= 0 \diamond (0 \diamond a)^{(\diamond,(p+1)q)}. \end{aligned}$$

Therefore the statement holds for every $p, q \in \mathbb{N}$. □

Lemma 3.13. *For any subalgebra C of A, we have*

$$a \in gcl(C) \iff 0 * a^{(*,q)} \in C \text{ for some } q \in \mathbb{N}$$

Proof. ($\Rightarrow$) Assume that $a \in \mathrm{gcl}(C)$. Then, there exist $c \in C$ and $q \in \mathbb{N}$ such that $c * a^{(*,q)} \in C$. By closedness of C, we get $(c * a^{(*,q)}) \diamond c \in A$. Then it follows from Proposition 2.3(p_5) that $(c \diamond c) * a^{(*,q)} \in A$, and consequently $0 * a^{(*,q)} \in C$.

($\Leftarrow$) It is clear by Definition 3.1 and Lemma 3.12(i). □

In the sequel, we introduce a condition on pseudo BCI-algebras and obtain several results about gcl.

Definition 3.14. A pseudo BCI-algebra A is called a pseudo BCI-algebra with condition (Z) if it satisfies the following equation:

$$(0 * a) * d = (0 * a) \diamond d \text{ for all } a, d \in A.$$

Example 3.15. Consider $(A = \{0, a, b, x, y, g\}, *, \diamond, 0)$ as a pseudo BCI-algebra in Example 3.2. Some routine calculations show that $0 * (u * t) = 0 * (u \diamond t)$ for any $u, t \in A$. Therefore A satisfies condition (Z).

Some pseudo BCI-algebras does not satisfy the condition (Z) as shown in the following example.

Example 3.16. Consider $(A = \{0, a, b, c, d, e\}, *, \diamond, 0)$ as a pseudo BCI-algebra [1] in which the operations $*, \diamond$ are given by the following Cayley's tables:

$*$	0	a	b	c	d	e
0	0	a	b	d	c	e
a	a	0	c	e	b	d
b	b	d	0	a	e	c
c	c	e	a	0	d	b
d	d	b	e	c	0	a
e	e	c	d	b	a	0

$\diamond$	0	a	b	c	d	e
0	0	a	b	d	c	e
a	a	0	d	b	e	c
b	b	c	0	e	a	d
c	c	b	e	0	d	a
d	d	e	a	c	0	b
e	e	d	c	a	b	0

The pseudo BCI-algebra A doesn't satisfy condition (Z), because $(0*a)*b = c$, but $(0*a)\diamond b = d$.

Lemma 3.17. *Let A be a pseudo BCI-algebra with condition (Z). Then, for any $a, d \in A$ and $p \in N$,*

$$0 * (a * d)^{(*,p)} = (0 * a^{(*,p)}) \diamond (0 * d^{(*,p)})$$

Proof. We proceed by induction on $p \geq 1$. For $p = 1$, the result holds by Proposition 2.3(p_{13}). Suppose that the statement is true for p and prove it for $p + 1$. For this purpose, we have

$$\begin{aligned}
0 * (a * d)^{(*,p+1)} &= (0 * (a * d)^{(*,p)}) * (a * d) \\
\text{by the induction hypothesis} \quad &= ((0 * a^{(*,p)}) \diamond (0 * d^{(*,p)})) * (a * d) \\
&= ((0 * (a * d)) \diamond a^{(\diamond,p)}) \diamond (0 * d^{(*,p)}) \\
\text{by Proposition 2.3}(p_{13}) \quad &= (((0 * a) \diamond (0 * d)) \diamond a^{(\diamond,p)} \diamond (0 * d^{(*,p)}) \\
\text{by condition (Z) and and Proposition 2.3}(p_{13}) \quad &= ((0 * a^{(\diamond,p+1)}) * (0 * d)) \diamond (0 * d^{(*,p)}) \\
\text{by Proposition 2.3}(p_{13}) \quad &= ((0 * (0 * d^{(*,p)})) * a^{(*,p+1)}) * (0 * d) \\
\text{by condition (Z)} \quad &= ((0 * (0 * d^{(*,p)})) \diamond a^{(\diamond,p+1)}) * (0 * d) \\
\text{by Lemma 3.10(ii)} \quad &= ((0 * (0 * d)^{(*,p)}) * (0 * d)) \diamond a^{(\diamond,p+1)}) \\
&= (0 * (0 * d)^{(*,p+1)}) \diamond a^{(\diamond,p+1)} \\
\text{by Lemma 3.10(ii)} \quad &= (0 * (0 * d^{(*,p+1)})) \diamond a^{(\diamond,p+1)} \\
\text{by Proposition 2.3}(p_{13}) \quad &= (0 * a^{(*,p+1)}) \diamond (0 * d^{(*,p+1)}).
\end{aligned}$$

This completes the proof. □

Theorem 3.18. *Let A be a pseudo BCI-algebra with condition (Z). If C is a subalgebra of A, then so is $gcl(C)$.*

Proof. Let $c, d \in \mathrm{gcl}(C)$. Then, there exist $s, t, r \in C$ and $q \in \mathbb{N}$ such that $s * c^{(*,q)} = t$, $s \diamond c^{(\diamond,q)} = r$. So $(s \diamond t) * c^{(*,q)} = 0 = (s * r) \diamond c^{(\diamond,q)}$. By taking, $s \diamond t = a$, $s * r = b$, we obtain

$$a * c^{(*,q)} = 0 \textit{ and } 0 * c^{(*,q)} = 0 * a, \tag{3.2}$$

$$b \diamond c^{(\diamond,q)} = 0 \textit{ and } 0 \diamond c^{(\diamond,q)} = 0 \diamond b. \tag{3.3}$$

Thus, we have

$$\begin{aligned}
0 * c^{(*,kq)} &= ((a \diamond a) * c^{(*,q)}) * c^{(*,(k-1)q)} \\
\text{by Proposition 2.3}(p_5) \quad &= ((a * c^{(*,q)}) * c^{(*,(k-1)q)}) \diamond a \\
\text{by (3.2)} \quad &= (0 * c^{(*,(k-1)q)}) \diamond a = ((0 * c^{(*,q)}) * c^{(*,(k-2)q)}) \diamond a \\
\text{by (3.2)} \quad &= ((0 \diamond a) * c^{(*,(k-2)q)}) \diamond a = (0 * c^{(*,(k-2)q)}) \diamond a^{(\diamond,2)} \\
&\cdots \\
\text{by (3.2)} \quad &= (0 * c^{(*,q)}) \diamond a^{(\diamond,k-1)} = 0 \diamond a^{(\diamond,k)} \in C,
\end{aligned}$$

for any $k \in \mathbb{N}$. Thus

$$0 * c^{(*,kq)},\ 0 \diamond c^{(\diamond,kq)} \in C \textit{ for any } k \in \mathbb{N}. \tag{3.4}$$

Similarly, from $d \in \mathrm{gcl}(C)$, we get

$$0 * d^{(*,kp)}, 0 \diamond d^{(\diamond,kp)} \in C \textit{ for any } k \in \mathbb{N}. \tag{3.5}$$

Combining, (3.4), (3.5) and Lemma 3.17, by closedness of C, we get $0 * (c * d)^{(*,pq)} = (0 * c^{(*,pq)}) \diamond (0 * d^{(*,pq)}) \in C$, which implies $c * d \in \mathrm{gcl}(C)$. Therefore $\mathrm{gcl}(C)$ is a subalgebra of A. □

The converse of Theorem 3.18 may not hold as seen in the following example.

Example 3.19. Consider $A = \{0, a, b, x, y, g\}$ as a pseudo BCI-algebra with property (Z) in Example 3.2. By taking $C = \{0, a, x\}$, it can be checked that $\mathrm{gcl}(C) = A$. But C is not a subalgebra of A because $a * x = g \notin C$.

Proposition 3.20. *The following hold:*
(i) if $0 \in C \subseteq K(A)$, then $gcl(C) = \mathcal{N}_{\mathcal{A}}$,
(ii) for all $d \in A$, $gcl(\{A(d)\}) = \mathcal{N}_{\mathcal{A}}$, where $A(d) = \{a \in A | a \preceq d\}$.

Proof. (i) By Theorem 3.5, $\mathcal{N}_A \subseteq \mathrm{gcl}(C)$. To prove the reverse inclusion, assume that $d \in \mathrm{gcl}(C)$. Then there exist $a, b, c \in C$ and $q \in \mathbb{N}$ such that $a * d^{(*,q)} = b$ and $a \diamond d^{(\diamond,q)} = c$. Then we have

$$
\begin{aligned}
&\text{by } b \in K(A) && 0 = 0 * b = 0 * (a * d^{(*,q)})\\
&\text{by Lemma 3.12}(iv) && \quad = (0 * a) \diamond (0 * d^{(\diamond,q)})\\
&\text{by } a \in K(A) && \quad = 0 \diamond (0 * d^{(\diamond,q)}).
\end{aligned}
$$

Then from Lemma 3.12(iii), we obtain $0 * d^{(\diamond,q)} = 0$, and so by Lemma 3.12(i), we have $0 * d^{(*,q)} = 0$. Thus, $d \in \mathcal{N}_A$. Therefore $\mathrm{gcl}(C) = \mathcal{N}_A$.

(ii) Let $a \in \mathcal{N}_A$. Then $0 * a^{(*,q)} = 0 = 0 \diamond a^{(\diamond,q)}$ for some $q \in \mathbb{N}$ and so $(d * a^{(*,q)}) \diamond d = (d \diamond d) * a^{(*,q)} = 0 * a^{(*,q)} = 0$. It follows that $d * a^{(*,q)} \preceq d$, and consequently $d * a^{(*,q)} \in A(d)$. Similarly, we have $d \diamond a^{(\diamond,q)} \in A(d)$. But, $d \in A(d)$. Hence, $a \in \mathrm{gcl}(A(d))$ and so $\mathcal{N}_A \subseteq \mathrm{gcl}(A(d))$. To prove the reverse inclusion, suppose that $a \in \mathrm{gcl}(A(d))$. Then there exists $t \in A(d)$ such that $t * a^{(*,q)} \preceq d$, that is $(t * a^{(*,q)}) \diamond d = 0$ and so $(t \diamond d) * a^{(*,q)} = 0$. On other hand, from $t \in A(d)$ we have $t \diamond d = 0$. Hence, $0 * a^{(*,q)} = 0$, and so, by Lemma 3.12(i) we get $0 \diamond a^{(\diamond,q)} = 0$. Consequently, $a \in \mathcal{N}_A$. Therefore $\mathrm{gcl}(A(d)) \subseteq \mathcal{N}_A$. This completes the proof of (ii). □

In the following theorem, we introduce a sufficient condition for pseudo BCI-algebra to be p-semisimple.

Theorem 3.21. *For any pseudo BCI-algebra A, if $gcl(\{0\}) = \{0\}$, then A is p-semisimple.*

Proof. Assume that $\mathrm{gcl}(\{0\}) = \{0\}$. Then by Theorem 3.20, $\mathcal{N}_A = \{0\}$. Since $K(A) \subseteq \mathcal{N}_A$, we get $K(A) = \{0\}$. Using Proposition 2.3(p_{13}),(p_8), we obtain, for any $a \in A$

$$0 * (a \diamond (0 * (0 * a))) = (0 * a) * (0 * (0 * (0 * a))) = (0 * a) * (0 * a) = 0$$

This implies that $a \diamond (0 * (0 * a)) \in K(A)$ and so $a \diamond (0 * (0 * a)) = 0$. On other hand, $(0 * (0 * a)) \diamond a = 0$ for any $a \in A$. Therefore, $0 * (0 * a) = a$, and so, by Proposition 2.5, A is p-semisimple. □

Lemma 3.22. *Let A be a pseudo BCI-algebra and $a, c \in A$. If $0 * a = 0 * c$, then $0 * a^{(*,q)} = 0 * c^{(*,q)} = 0 * c^{(\diamond,q)}$, for all $q \in \mathbb{N}$.*

Proof. The proof is straightforward. □

Theorem 3.23. *Let A be a pseudo BCI-algebra with condition (Z). If C is a closed pseudo-ideal of A, then so is $gcl(C)$. Moreover, $\mathcal{N}_A \subseteq gcl(C)$.*

Proof. Clearly, $0 \in \mathrm{gcl}(C)$. Let $a, c * a \in \mathrm{gcl}(C)$. Then there exist $b, d \in C$ and $q \in \mathbb{N}$ such that $b * a^{(*,q)}$, $b \diamond a^{(\diamond,q)} \in C$ and $d * (c * a)^{(*,p)}$, $d \diamond (c * a)^{(\diamond,p)} \in C$. Thus, similar to the argument in Theorem 3.18, we get $0 * a^{(*,pq)}$, $0 * (c * a)^{(*,pq)} \in C$. It follows from Definition 2.1(a_2) that $c \diamond (c * a) \preceq a$ and so, by Proposition 2.3(p_2), we get $0 * a \preceq 0 * (c \diamond (c * a))$. Then, by the minimality of $0 * (c \diamond (c * a))$, we have $0 * (c \diamond (c * a)) = 0 * a$. From this, by Lemma 3.22, we obtain $0 * (c \diamond (c * a))^{(*,pq)} = 0 * a^{(*,pq)}$. Now, by Lemma 3.17, we have

$$(0 * c^{(*,pq)}) \diamond (0 * (c * a)^{(*,pq)}) = 0 * (c * (c * a))^{(*,pq)} = 0 * a^{(*,pq)} \in C.$$

Thus, since C is a pseudo-ideal of A and $0 * (c * a)^{(*,pq)} \in C$, we get $0 * c^{(*,pq)} \in C$ and so $c \in \mathrm{gcl}(C)$. Therefore $\mathrm{gcl}(C)$ is a pseudo-ideal of A. Moreover, due to Theorem 3.18, $\mathrm{gcl}(C)$ is closed. Finally, using Theorem 3.4 and Lemma 3.3(ii), we get $\mathcal{N}_{\mathcal{A}} = gcl(\{0\}) \subseteq gcl(C)$, and so the proof is completed. □

The converse of Theorem 3.23 may not hold as seen in the following example.

Example 3.24. Let $A = (\mathbb{Q} - \{0\}, *, \diamond, 1)$ be the pseudo BCI-algebra and $\div$ be the usual division, which $a * c = a \diamond c = a \div c$, [12]. By taking $C = \{2^{-n} | n = 0, 1, 2, ...\}$, it can be easily seen that $\mathrm{gcl}(C) = \{2^n | n \in \mathbb{Z}\}$. Clearly, $\mathrm{gcl}(C)$ is closed but C is not closed.

Theorem 3.25. *Let A be a pseudo BCI-algebra with condition (Z). If C is a closed pseudo-ideal of A, then $gcl(C) = gcl(gcl(C))$.*

Proof. By Lemma 3.3(iii), gcl(C)$\subseteq$ gcl(gcl(C)). For the reverse inclusion, suppose $a \in$ gcl(gcl(C)). By Theorem 3.23, gcl(C) forms a subalgebra of A. Therefore, applying Lemma 3.13, we get $0 * a^{(*,q)} \in$ gcl(C) for some $q \in \mathbb{N}$. Thus $0 * (0 * a^{(*,q)})^{(*,p)} \in C$ for some $p \in \mathbb{N}$. Then it follows from Lemma 3.12(vi) that $0 * (0 * a)^{(*,pq)} = 0 * (0 * a^{(*,q)})^{(*,p)} \in C$. Thus $0 * a \in gcl(C)$ and so, by closedness of gcl(C), we get $0 * (0 * a) \in gcl(C)$. On other hand, by the similar argument in Theorem 3.21, we have $0 * (0 * a) = a$. Therefore $a \in gcl(C)$ and so the proof is completed. □

Corollary 3.26. *For any pseudo BCI-algebra A with condition (Z), we have*

$$gcl(\mathcal{N}_{\mathcal{A}}) = \mathcal{N}_{\mathcal{A}}.$$

Proof. Using Theorems 3.20(i) and 3.25, we have

$$\mathcal{N}_{\mathcal{A}} = gcl(\{0\}) = gcl(gcl(\{0\})) = gcl(\mathcal{N}_{\mathcal{A}}).$$

This completes the proof. □

Theorem 3.27. *Let A be a pseudo BCI-algebra with condition (Z). Then, $\mathcal{N}_{\mathcal{A}}$ is the least closed pseudo-ideal of A satisfying $gcl(\mathcal{N}_{\mathcal{A}}) = \mathcal{N}_{\mathcal{A}}$.*

Proof. By Theorem 3.4, $\mathcal{N}_{\mathcal{A}} = gcl(\{0\})$. Clearly, $\{0\}$ is a closed pseudo-ideal of A. Then, by Theorem 3.23, $\mathcal{N}_{\mathcal{A}}$ is a closed pseudo-ideal of A too. Also, by Corollary 3.26, $\mathrm{gcl}(\mathcal{N}_{\mathcal{A}}) = \mathcal{N}_{\mathcal{A}}$. To complete the proof, assume that C is another closed pseudo-ideal of A satisfying $\mathrm{gcl}(C) = C$. It follows from Theorem 3.23 that $\mathcal{N}_{\mathcal{A}} \subseteq$ gcl(C). Therefore $\mathcal{N}_{\mathcal{A}} \subseteq C$, which completes the proof. □

Using the notion of the "gcl" on the set of all closed pseudo-ideals, denoted by $\mathcal{J}(A)$, we provide a closure operation as seen in the following theorem.

Theorem 3.28. *For any pseudo BCI-algebra satisfying condition (Z), the mapping p : $\mathcal{J}(A) \to \mathcal{J}(A)$ defined by $p(C)$ =gcl(C) for any $C \in \mathcal{J}(A)$, is a closure operation.*

Proof. The proof is clear by Lemma 3.3 and Theorem 3.25. □

4. Conclusions

In this work, we introduced several identities which was useful to prove more results. In the sequel, we defined the notion of generalization of pseudo p-closure (denoted by gcl), and study related properties. Using this notion, we gave a necessary and sufficient condition for an element to be minimal. Also, by using the mentioned notion, we gave a necessary and sufficient condition for pseudo BCI-algebra to be nilpotent. Moreover, the gcl of subalgebras and pseudo-ideals was determined. Finally, we showed that the gcl, as a function, acts on the closed pseudo-ideals as the same as a closure operation.

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References

[1] L. C. Ciungu *Pseudo BCI-algebras With Derivations,* Department of Mathematics, University of Iowa, USA. **14** (2019), 5–17.

[2] G. Dymek, *On pseudo BCI-algebras,* Annales Universalistic Mariae Curie-Sklodowska Lublin Poloia., **1** (2015), 59–71.

[3] G. Dymek, *On two classes of pseudo BCI-algebras,* Discussions Math, General Algebra and Application, **31** (2011), 217–229.

[4] G. Dymek, *p-semisimple pseudo BCI-algebras*, J. Mult-Valued Logic Soft Comput., **19** (2012), 461–474.

[5] W. A. Dudek, Y. B. Jun, *Pseudo BCI-algebras,* East Asian Math. J., **24** (2008), 187–190.

[6] H. Hrizavi, *P-closure in pseudo BCI-algebras,* Journal of Algebraic Systems, **7** No.**2** (2020), 155–165.

[7] Y. Imai and K. Iséki, *On axiom system of propositional calculi,* Proc. Japan. Acad., **42** (1966), 19–22.

[8] K. Iséki, *An algebra related with a propositional calculus,* Japan. Acad., **42** (1966), 26–29.

[9] Y. B. Jun, H. S. Kim and J. Neggers, *On pseudo-ideals of pseudo BCI-algebras,* Matemat. Bech., **58** (2006), 39–46.

[10] Y. H. Kim and K. S. So, *On Minimality in pseudo BCI-algebras,* Commun. Korean. Math. Soc., **1** (2012), No. 1, 7–13.

[11] J. Meng and Y. B. Jun, *BCK-Algebras,* Kyung Moon Sa Co., Seoul, 1994.

[12] H. Moussei, H. Harizavi and R. A. Borzooei, *p-closure ideals in BCI-algebras*, Soft Computing, **22**, (2018), 7901–7908.

[13] H. Yisheng, *BCI-Algebra,* Published by Science Press, 2006.