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Research Paper

SOME INEQUALITIES OF REFORMULATED AND ENTIRE INF-SOMBOR INDEX

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ABSTRACT

The concept of the Sombor index was extended by Reti *et*. al. [2] by introducing a p-Sombor index which can be seen as the p norm of the vector $x = (d(u), d(v))$ as $p \to \infty$, $\|x\|_\infty = max\{d(u), d(v)\}$. Inspired by this, we defined indices namely, Reformulated Inf- Sombor index, Entire Inf-Sombor index, and KG Inf-Sombor index. Also, we present lower and upper bounds by using some graph parameters and obtain exact values of these new topological indices in some graph families. Further, we evaluated the statistical behavior of these indices after computing index values for different types of dendrimers for various growth values k.

1. Introduction

We only take into account undirected, simple, and finite graphs. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ be the minimum and maximum vertex degreeof the graph G respectively. $g \sim h$ means that the edges g and h are adjacent, i.e., they share a common end-vertex in G and let $u \in$

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Topological indices	Mathematical expressions
Forgotten index [8, 9]	$F(G) = \sum_{uv\in E(G)}[d(u)^2 + d(v)^2]$
Randic index [11, 10]	$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u).d(v)}}$
First Zagreb index [5]	$M_1(G) = \sum_{uv\in E(G)}[d(u) + d(v)]$
Second Zagreb index [6, 7]	$M_2(G) = \sum_{uv\in E(G)}[d(u).d(v)]$
Entire Randic index [17]	$ER(G) = \sum_{\{g,h\}\subseteq B(G)} \frac{1}{\sqrt{d(g)d(h)}}$
p-Sombor index [2]	$SO_p(G) = \sum_{uv\in E(G)}[d(u)^p + d(v)^p]^{1/p}$
Sombor index [1]	$SO(G) = \sum_{uv\in E(G)}[\sqrt{d(u)^2 + d(v)^2}]$
Inf-Sombor index [12]	$SO_\infty(G) = \sum_{uv\in E(G)} max\{d(u), d(v)\}.$
KG Sombor index [22]	$KG(G) = \sum_{ue}[\sqrt{d(u)^2 + d(e)^2}].$
Reformulated Sombor index [20]	$RS(G) = \sum_{g\sim h} \sqrt{[d(g)^2 + d(h)^2]}$
Reformulated first Zagreb index [21]	$REM_1(G) = \sum_{g\sim h}[d(g) + d(h)]$
Platt index [13]	$Pl(G) = \sum_{u\in V(G)} d(u)\,(d(u) - 1)$
Reformulated second Zagreb index [21]	$REM_2(G) = \sum_{g\sim h}[d(g)d(h)]$
First Entire Zagreb index [23]	$EM_1(G) = \sum_{\{g,h\}\subseteq B(G)}[d(g) + d(h)]$
Second Entire Zagreb index [23]	$EM_2(G) = \sum_{\{g,h\}\subseteq B(G)}[d(g)d(h)]$
Entire Sombor index [18, 19]	$ES(G) = \sum_{\{g,h\}\subseteq B(G)} \sqrt{[d(g)^2 + d(h)^2]}$
Reformulated Inverse sum Indeg index[29]	$ISI_e(G) = \sum_{g\sim h} \frac{d(g)d(h)}{d(g)+d(h)}$

TABLE 1. Topological indices and their mathematical expressions

$V(G)$, $e \in E(G)$ then ue represents u is incident to e. If there is a path connecting each pair of vertices in a graph G, then that graph is said to be connected. The degree of a vertex v, shown by $d(v)$, is the number of vertices that are adjacent to it. Furthermore, $d(e) = d(u)+d(v)-2$ defines the edge ($e = uv$) degree. For further terminology and notation pertaining to graph theory, see [3, 4]. The Sombor index, defined as $\sum_{uv\in E(G)}[\sqrt{d(u)^2 + d(v)^2}]$, was initially defined by Gutman [1] in 2021. It is based on the degree radius of an edge $e = uv \in E(G)$. The concept of the Sombor index was extended by Reti *et*. al. [2]. by introducing a p-Sombor index which can be seen as the p norm of the vector $x = (d(u), d(v))$ as $p \to \infty$, $\|x\|_\infty = max\{d(u), d(v)\}$. Inspired by this, we defined indices namely, Reformulated Inf- Sombor index, Entire Inf-Sombor index, and KG Inf-Sombor index. The reformulated inf-Sombor index is defined as $RSO_\infty(G) = \sum_{g\sim h} max\{d(g), d(h)\}$ and Entire Inf-Sombor index which is defined as the sum of the maximum of the terms g and h, which are the two members of the set $B(G)$, where $B(G)$ is the collection of all subsets of two members $\{g, h\} \subseteq V(G) \cup E(G)$ such that g and h are adjacent or incident to each other and is represented as $ESO_\infty(G) = \sum_{\{g,h\}\subseteq B(G)} max\{d(g), d(h)\}$. KG Sombor index is defined by Kulli *et*.al.[22] which is defined as $KG(G) = \sum_{ue} \sqrt{d(u)^2 + d(e)^2}$, motivated by this we defined an index named KG Inf-Sombor index is defined as $KG_\infty(G) = \sum_{ue} max\{d(u), d(e)\}$.

Some well-studied topological descriptors have many applications in the field of chemical graph theory. For more details refer [25, 24, 27, 28, 34, 36, 35, 26].

2. Reformulated Inf-Sombor index

This section contains the precise values of the reformulated Inf-Sombor index for certain graph families.

Observation: Any isolated vertices in a graph G have no impact on $L(G)$, since $L(G)$ is defined on its edge set. It is obvious that $RSO_\infty(G) = SO_\infty(L(G))$.

Theorem 2.1. *Let G be r-regular graph. Then,*

$$RSO_\infty(G) = (r-1)^2 nr.$$

Proof. Let G be an r-regular graph. Then,

$$\begin{aligned} RSO_\infty(G) &= \sum_{g\sim h} max\{d(g), d(h)\} \\ &= \frac{Pl(G)}{2} 2(r-1) \\ RSO_\infty(G) &= (r-1)^2 nr \end{aligned}$$

□

Theorem 2.2. *(1) For complete graph K_n with $n \geq 3$, $RSO_\infty(K_n) = n(n-1)(n-2)^2$.*

(2) For cycle C_n with $n \geq 3$, $RSO_\infty(C_n) = 2n$.

(3) For complete bipartite graph $K_{m,n}$ with $1 \leq m \leq n$, $RSO_\infty(K_{m,n}) = \frac{(m+n-2)^2 mn}{2}$.

(4) For path P_n with $n \geq 2$, $RSO_\infty(P_n) = 2(n-2)$.

(5) For star graph S_n, with $n \geq 2$, $RSO_\infty(S_n) = \frac{(n-2)^2(n-1)}{2}$.

2.1. Inequalities related to Reformulated Inf-Sombor index.

Theorem 2.3. *Let G be any graph. Then,*

$$Pl(G)(\delta - 1) \leq RSO_\infty(G) \leq Pl(G)(\Delta - 1).$$

Proof. Let G be a connected. Then,

$$\begin{aligned} RSO_\infty(G) &= \sum_{g\sim h} max\{d(g), d(h)\} \\ &\leq 2(\Delta - 1)\sum_{g\sim h} 1 \\ &\leq Pl(G)(\Delta - 1). \end{aligned} \tag{2.1}$$

$$\text{Similarly, } RSO_\infty(G) = \sum_{g\sim h} max \quad \{d(g), d(h)\}$$

$$\geq Pl(G)(\delta - 1). \tag{2.2}$$

From equations (2.1) and (2.2), we get

$$Pl(G)(\delta - 1) \leq RSO_\infty(G) \leq Pl(G)(\Delta - 1). \tag{2.3}$$

Equality holds when graph G is regular. □

Lemma 2.4. [14] *For any connected graph G with $n \geq 2$,*

(1) $2m(\delta - 1) \leq Pl(G) \leq 2m(\Delta - 1)$.

(2) $m \leq Pl(G) \leq 2m(n-2)$.

Lemma 2.5. [15, 16] *For any graph G with $n \geq 2$, $Pl(G) = M_1(G) - 2m$.*

Corollary 2.6. *For a graph G,*

(1) $2m(\delta - 1)^2 \leq Pl(G) \leq 2m(\Delta - 1)^2$.

(2) $m(\delta-1) \leq RSO_{\infty}(G) \leq 2m(n-2)(\Delta-1)$.

Proof. By using Lemma 2.4 in equation (2.3), we get the results. □

Theorem 2.7. *For any graph G,*

$$\frac{1}{\sqrt{2}}RSO(G) \leq RSO_{\infty}(G) \leq RSO(G).$$

Proof. Let G be a graph. Then,

$$\begin{aligned} RSO(G) &= \sum_{g\sim h}\sqrt{d(g)^2+d(h)^2} \\ &\leq \sum_{g\sim h}\sqrt{2max\{d(g)^2, d(h)^2\}} \\ &\leq \sqrt{2}RSO_{\infty}(G). \end{aligned} \tag{2.4}$$

$$\text{Consider,} \quad \sqrt{max\{d(g)^2, d(h)^2\}} \leq \sqrt{d(g)^2+d(h)^2}$$

$$RSO_{\infty}(G) \leq RSO(G). \tag{2.5}$$

From equations (2.4) and (2.5), we get $\frac{1}{\sqrt{2}}RSO(G) \leq RSO_{\infty}(G) \leq RSO(G)$. □

Corollary 2.8. *For any graph G,*

$$\frac{m}{2} \leq RSO_{\infty}(G) \leq 2m(n-2)^2.$$

Proof. From Lemma (2.4), we have

$$\frac{Pl(G)}{2} \leq max\{d(e), d(f)\} \leq 2(n-2)\cdot\frac{Pl(G)}{2} \tag{2.6}$$

$$\frac{m}{2} \leq max\{d(e), d(f)\} \leq 2m(n-2)^2.$$

□

Corollary 2.9. *For any graph G*

$$\frac{M_1(G)-2m}{2} \leq RSO_{\infty}(G) \leq (n-1)(M_1(G)-2m).$$

Proof. By using Lemma (2.5) in equation (2.6), we get the result. □

Theorem 2.10. *For any graph G,*

$$\frac{REM_1(G)}{2} \leq RSO_{\infty}(G) \leq REM_1(G).$$

Proof. For any integer $x, y \geq 0$

$$\frac{g+h}{2} \leq max\{g,h\} \leq g+h$$

$$\sum_{g\sim h}\frac{d(g)+d(h)}{2} \leq \sum_{g\sim h}max\{d(g), d(h)\} \leq \sum_{g\sim h}d(g)+d(h1)$$

$$\frac{REM_1(G)}{2} \leq RSO_{\infty}(G) \leq REM_1(G).$$

□

Theorem 2.11. *Let G be any graph. Then,*

$$RM_1(G)-(\Delta-1)Pl(G) \leq RSO_{\infty}(G) \leq RM_1(G)-(\delta-1)Pl(G).$$

Proof. Let G be any graph. Then,

$$RM_1(G) = \sum_{g\sim h}[d(g) + d(h)]$$
$$\leq \sum_{g\sim h} max\{d(g), d(h)\} + 2(\Delta - 1)\frac{Pl(G)}{2}$$
$$\leq RSO_\infty(G) + (\Delta - 1)Pl(G). \tag{2.7}$$

$$\text{Similarly, } RM_1(G) \geq RSO_\infty(G) + (\delta - 1)Pl(G). \tag{2.8}$$

From equations (2.7) and (2.8), we get

$$RM_1(G) - (\Delta - 1)Pl(G) \leq RSO_\infty(G) \leq RM_1(G) - (\delta - 1)Pl(G)$$

□

Theorem 2.12. *Let G be (n, m)-graph with$\Delta \geq 2$. Then*

$$\frac{4(\delta - 1)ISI_e(G)}{Pl(G)(\Delta - 1)} \leq RSO_\infty(G) \leq \frac{4(\Delta - 1)ISI_e(G)}{Pl(G)(\delta - 1)}.$$

Equality holds if and only if G is regular.

Proof. Let G be any graph. Then,

$$ISI_e(G) = \sum_{g\sim h}\frac{d(g)d(h)}{d(g) + d(h)}$$
$$\sum_{g\sim h}\frac{d(g)2(\delta - 1)}{2 \times 2(\Delta - 1)} \leq \sum_{g\sim h}\frac{d(g)d(h)}{d(g) + d(h)} \leq \sum_{g\sim h}\frac{d(g)2(\Delta - 1)}{2 \times 2(\delta - 1)}$$
$$\frac{4(\delta - 1)ISI_e(G)}{Pl(G)(\Delta - 1)} \leq RSO_\infty(G) \leq \frac{4(\Delta - 1)ISI_e(G)}{Pl(G)(\delta - 1)}.$$

Equality is attained if the graph is regular. □

3. Entire Inf-Sombor index

This section contains the precise values of the reformulated Inf-Sombor index for certain graph families.

Observation : Let G be connected graph with $n \geq 3$. Then,

$$ESO_{\infty(G)} = SO_\infty(G) + RSO_\infty(G) + KG_\infty(G). \tag{3.1}$$

Theorem 3.1. *Let G be an r-regular graph with $n \geq 3$ and $r \geq 1$. Then*

$$ESO_\infty(G) = \frac{nr^2}{2} + 2n(r - 1)^2 + 2nr(r - 1).$$

Proof. From Theorem 3.3, we have

$$ESO_\infty(G) = SO_\infty(G) + RSO_\infty(G) + 2Pl(G)$$
$$ESO_\infty(G) = \frac{nr^2}{2} + 2n(r - 1)^2 + 2nr(r - 1).$$

□

Theorem 3.2. *(1) For complete graph K_n with $n \geq 3$, $ESO_\infty(K_n) = \frac{(n-1)^2 n}{2} + n(n - 2)^2(n - 1) + 2n(n - 1)(n - 2)$.*

(2) For cycle C_n with $n \geq 3$, $ESO_\infty(C_n) = 8n$.

(3) For complete bipartite graph $K_{m,n}$ with $1 < m \leq n$, $ESO_\infty(K_{m,n}) = max\{m,n\} \cdot mn + \frac{(m+n-2)^2 mn}{2} + 2(m+n-2)mn$.

(4) For path P_n with $n \geq 3$, $ESO_\infty(P_n) = 4(2n-3)$.

(5) For star S_n with $n \geq 3$, $ESO_\infty(S_n) = 2(n-1)^2 + \frac{(n-2)^2(n-1)}{2} + (n-2)(n-1)$.

Theorem 3.3. *Let G be any connected graph without pendant vertices with $n \geq 2$. Then,*

$$ESO_\infty(G) = SO_\infty(G) + RSO_\infty(G) + 2Pl(G).$$

Proof. Let G be a graph without pendant vertices. Then $d(u) \geq 2$ for all $u \in V(G)$. If $e = uv$ is any edge then $d(e) \geq d(u)$ and $d(e) \geq d(v)$.

$$\text{Hence, } \sum_{ue} max\{d(u), d(e)\} = 2Pl(G). \tag{3.2}$$

From equation (3.1), we have $ESO_\infty(G) = SO_\infty(G) + RSO_\infty(G) + \sum_{ue} max\{d(u), d(e)\}$. Using equation (3.2) in equation (3.1) we get, $ESO_\infty(G) = SO_\infty(G) + RSO_\infty(G) + 2Pl(G)$.
□

Theorem 3.4. *Let G be any connected graph of order $n \geq 3$ with $\tilde{p}$-pendant vertices. Then $ESO_\infty(G) = SO_\infty(G) + RSO_\infty(G) + 2Pl(G) - \tilde{p}$.*

Proof. Let $v_1, v_2, \ldots v_{\tilde{p}}$ be pendant vertices of a graph G and $e_1, e_2, \ldots e_{\tilde{p}}$ be the edges incident to $v_1, v_2, \ldots v_{\tilde{p}}$, respectively. Then

$$\begin{aligned} ESO_\infty(G) &= SO_\infty(G) + RSO_\infty(G) + KG_\infty(G) \\ &= SO_\infty(G) + RSO_\infty(G) + \sum_{v \text{ is incident to } e} max\{d(v), d(e)\} \\ &= SO_\infty(G) + RSO_\infty(G) + \sum_{i=1}^{\tilde{p}} [max\{d(v_i), d(e_i)\} + max\{d(u_i), d(e_i)\}] + \\ &\sum_{e_i \in E(G)/\{e_1, e_2, \ldots e_{\tilde{p}}\}} [max\{d(v_i), d(e_i)\} + max\{d(u_i), d(e_i)\}] \\ &= SO_\infty(G) + RSO_\infty(G) + d(e_1) + d(e_1) - 1 + d(e_2) + d(e_2) - 1 + \ldots d(e_{\tilde{p}}) + d(e_{\tilde{p}}) - 1 + \\ &2d(e_{\tilde{p}+1}) + \ldots 2d(e_m) \end{aligned}$$

$$ESO_\infty(G) = SO_\infty(G) + RSO_\infty(G) + 2Pl(G) - \tilde{p}.$$

□

3.1. Inequalities related to Entire Inf-Sombor index.

Theorem 3.5. *For any graph G with $\Delta \geq 2$,*

$$\frac{EM_2(G)}{2(\Delta - 1)} \leq ESO_\infty(G) \leq \frac{EM_2(G)}{\delta}.$$

Proof. Let G be any graph. Then

$$ESO_{\infty}(G) = \sum_{\{gh\}\subseteq B(G)} max\{d(g), d(h)\}$$

$$= \sum_{\{gh\}\subseteq B(G)} \frac{max\{d(g), d(h)\}d(x)d(y)}{d(x)d(y)}$$

$$\leq \frac{EM_2(G)}{\delta}$$

$$\text{Similarly,} \quad ESO_{\infty}(G) = \sum_{\{gh\}\subseteq B(G)} max\{d(g), d(h)\}$$

$$= \sum_{\{gh\}\subseteq B(G)} \frac{max\{d(g), d(h)\}d(x)d(y)}{d(x)d(y)}$$

$$\geq \frac{EM_2(G)}{2(\Delta - 1)}$$

$$\frac{EM_2(G)}{2(\Delta - 1)} \leq ESO_{\infty}(G) \leq \frac{EM_2(G)}{\delta}.$$

□

Theorem 3.6. *For any graph G with $\Delta \geq 2$,*

$$\frac{\delta}{8(\Delta - 1)^2} EF(G) \leq ESO_{\infty}(G) \leq \frac{\Delta - 1}{\delta^2} EF(G).$$

Proof. Let G be any graph. Then

$$ESO_{\infty}(G) = \sum_{\{g,h\}\subseteq B(G)} max\{d(g), d(h)\}$$

$$= \sum_{\{g,h\}\subseteq B(G)} \frac{max\{d(g), d(h)\}[d(x)^2 + d(y)^2]}{[d(x)^2 + d(y)^2]}$$

$$\leq \frac{(\Delta - 1)}{\delta^2} EF(G) \tag{3.3}$$

$$\text{Similarly,} \quad ESO_{\infty}(G) = \sum_{\{g,h\}\subseteq B(G)} \frac{max\{d(g), d(h)\}[d(x)^2 + d(y)^2]}{[d(x)^2 + d(y)^2]}$$

$$\geq \sum_{\{g,h\}\subseteq B(G)} \frac{[d(x)^2 + d(y)^2]\delta}{2 \cdot 4(\Delta - 1)^2}$$

$$\geq \frac{\delta}{8(\Delta - 1)^2} EF(G) \tag{3.4}$$

From equations (3.3) and (3.4), we get

$$\frac{\delta}{8(\Delta - 1)^2} EF(G) \leq ESo_{\infty}(G) \leq \frac{\Delta - 1}{\delta^2} EF(G).$$

□

Theorem 3.7. *For any graph G, $\delta^2 ER(G) \leq ESO_{\infty}(G) \leq 4(\Delta(G) - 1)^2 ER(G)$.*

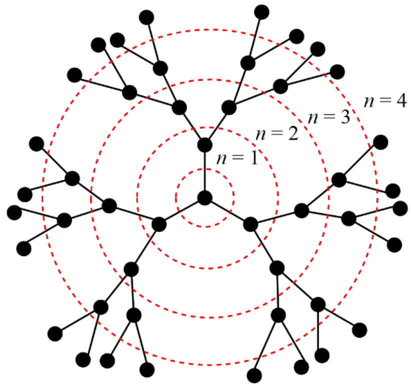


FIGURE 1. Regular Dendrimers

Proof. Let G be any graph. Then

$$\begin{aligned} ESO_\infty(G) &= \sum_{\{g,h\}\subseteq B(G)} max\{d(g), d(h)\} \\ &= \sum_{\{g,h\}\subseteq B(G)} \frac{max\{d(g), d(h)\}\sqrt{d(x)d(y)}}{\sqrt{d(x)d(y)}} \\ &\leq \sum_{\{g,h\}\subseteq B(G)} \frac{\sqrt{4(\Delta-1)^2}\cdot 2(\Delta-1)}{\sqrt{d(x)d(y)}} \\ &\leq 4(\Delta-1)^2 ER(G). \qquad (3.5) \end{aligned}$$

$$\text{Similarly, } ESO_\infty(G) = \sum_{\{g,h\}\subseteq B(G)} max\{d(g), d(h)\} \geq \delta^2 ER(G) \qquad (3.6)$$

(3.5) and (3.6), we get $\delta^2 ER(G) \leq ESO_\infty(G) \leq 4(\Delta-1)^2 ER(G)$. □

4. RESULTS ON DENDRIMERS

One of the most significant hyperbranched nanostructures is the dendrimer, which can be synthesized by convergent or divergent processes and assembled from monomers via a nanoscale production process. Nanotubes, nanolatex, chemical sensors, colored glass, micro and macro capsules, and photon funnels, such as artificial antennas are all made of dendrimers. [30, 31, 33, 37, 38, 39, 40, 41]

4.1. **Regular dendrimer.** Regular dendrimers, which are highly branched and scattered macro molecules, are a unique type of polymeric material found in chemical trees because of the significant influence they have on the physical and chemical properties of the molecules. There is a central vertex (v) in a regular dendrimer tree ($T_{k,d}$). Let the growth of the regular dendrimer be k $i.e$ the distance between the central vertex to the pendant vertex, and each nonpendant vertex of $T_{k,d}$ has a degree of $d \geq 2$. Figure 1 shows examples of dendrimers $T_{3,4}$.

Lemma 4.1. [32] *If $T_{k,d}$ is a tree with central vertex v, then*

(1) $T_{k,d}$ has d branches, $\frac{(d-1)^k-1}{d-2}$ vertices, $\frac{(d-1)^{k-1}-1}{d-2}$ nonpendant vertices, $(d-1)^{k-1}$ pendant vertices.
(2) The order of $T_{k,d}$ is $1+\frac{((d-1)^k-1)d}{d-2}$.

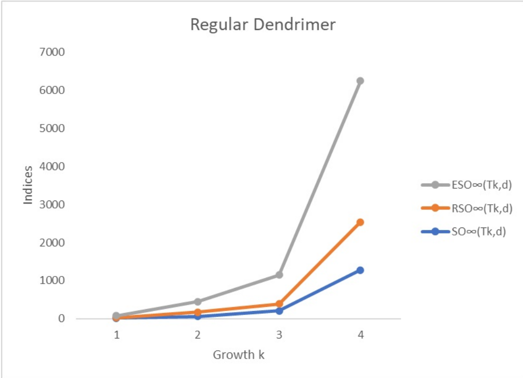


FIGURE 2. The particular values of the indices associated with the Inf-Sombor index

TABLE 2. The calculated values of the indices associated with the Inf-Sombor index

Indices	Calculated values
$Pl(T_{k,d})$	$\left[1+\frac{d[(d-1)^k-1]}{d-2}-d(d-1)^{k-1}\right]d(d-1)$
$SO_\infty(T_{k,d})$	$\frac{(d-1)^k d^2-d^2}{d-2}$
$RSO_\infty(T_{k,d})$	$2(d-1)\left\{\frac{d(d-1)}{2}\left[\frac{d[(d-1)^k-1]}{d-2}-d(d-1)^{k-1}+1\right]-\frac{d(d-1)^{k-1}(d-2)}{2}\right\}$
$ESO_\infty(T_{k,d})$	$\frac{d^2(d-1)^k-d^2}{d-2}+2(d-1)\left\{\frac{d(d-1)}{2}\left[1+\frac{d[(d-1)^k-1]}{d-2}-d(d-1)^{k-1}\right]-\frac{d(d-1)^{k-1}(d-2)}{2}\right\}$ $+2d(d-1)\left[1+\frac{d[(d-1)^k-1]}{d-2}-d(d-1)^{k-1}\right]-d(d-1)^{k-1}$

TABLE 3. The particular values of the indices associated with Inf-Sombor index

Growth k	Indices associated with Inf-Sombor related indices		
	$SO_\infty(T_{k,d})$	$RSO_\infty(T_{k,d})$	$ESO_\infty(T_{k,d})$
k=1	16	12	52
k=2	64	108	292
k=3	208	180	796
k=4	1280	1260	3812

Mathematically, the calculated values of the indices associated with the Inf-Sombor index are given in Table 2. We calculated particular values of the indices associated with the Inf-Sombor index of different growth k for $1 \leq k \leq 4$ as shown in Table 3 and its graphical comparison is shown in Figure 2.

4.2. **Polyamidoamine (PAMAM), Polypropylenimine (PPI) and Polypropyleneamine (POPAM) Dendrimers.** PAMAM dendrimers are characterized by their size, shape, and multifunctional terminal surface. They are hyperbranched dendrimers with a restricted molecular weight distribution and unmatched molecular homogeneity. It seems that the polypropylenimine dendrimers are appealing nonviral vectors for antisense oligonucleotides, small interfering RNA, and gene delivery. On the other hand, little is known about how PPI synthetic gene delivery vectors affect global gene expression. The Polypropyleneamine dendrimers closely resemble PPI dendrimers.

We have calculated the indices related to inf-Sombor index in table 4 and compared them graphically for growth values $k = 1, 2, 3, and\ 4$, which is shown in Figure 4a, 4b, 4c and 4d

TABLE 4. The calculated values of the indices associated with the different dendrimers

Dendrimers	Indices	Calculated values
D_1	$Pl(D_1)$	$114 \times 2^k - 60$
	$SO_\infty(D_1)$	$69 \times 2^k - 33$
	$RSO_\infty(D_1)$	$162 \times 2^k - 87$
	$ESO_\infty(D_1)$	$450 \times 2^k - 237$
D_2	$Pl(D_2)$	$152 \times 2^k - 72$
	$SO_\infty(D_2)$	$164 \times 2^k - 76$
	$RSO_\infty(D_2)$	$216 \times 2^k - 104$
	$ESO_\infty(D_2)$	$672 \times 2^k - 320$
D_3	$Pl(D_3)$	$76 \times 2^k - 72$
	$SO_\infty(D_3)$	$82 \times 2^k - 76$
	$RSO_\infty(D_3)$	$108 \times 2^k - 104$
	$ESO_\infty(D_3)$	$336 \times 2^k - 320$
D_4	$Pl(D_4)$	$36 \times 2^k - 28$
	$SO_\infty(D_4)$	$38 \times 2^k - 28$
	$RSO_\infty(D_4)$	$48 \times 2^k - 40$
	$ESO_\infty(D_4)$	$156 \times 2^k - 124$
D_5	$Pl(D_5)$	$72 \times 2^k - 28$
	$SO_\infty(D_5)$	$52 \times 2^k - 28$
	$RSO_\infty(D_5)$	$168 \times 2^k - 40$
	$ESO_\infty(D_5)$	$360 \times 2^k - 204$

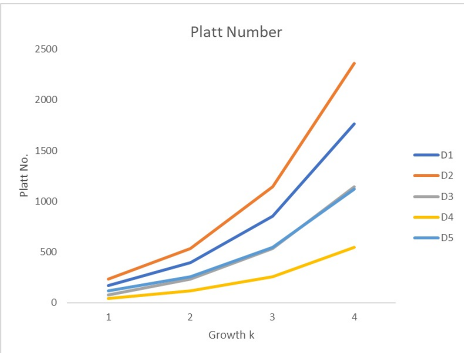
Based on the vertices degrees, we have three partitions of the vertex set: $V_3(D_j) = \{u : d(u) = 3\}$, $V_2(D_j) = \{u : d(u) = 2\}$ and $V_1(D_j) = \{u : d(u) = 1\}$.

Also, the four partitions of the edge set of D_j based on the degrees of end vertices are as follows:

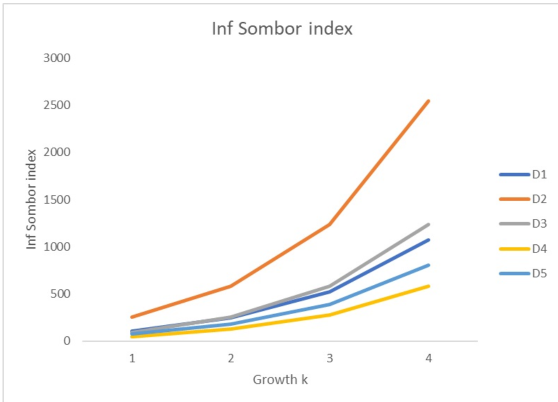
$$\begin{aligned}
E_1(D_j) &= \{e = uv \in E(D_j) : d(v) = 2 \ and \ d(u) = 1\} \\
E_2(D_j) &= \{e = uv \in E(D_j) : d(v) = 3 \ and \ duv) = 1\} \\
E_3(D_j) &= \{e = uv \in E(D_j) : d(v) = 2 \ and \ d(u) = 2\} \\
E_4(D_j) &= \{e = uv \in E(D_j) : d(v) = 3 \ and \ d(u) = 2\}.
\end{aligned}$$

Dendrimers	Class Number	Order of vertex and edge class	
	$1 \leq i \leq t$	$\lvert V_i(D_j)\rvert$	$\lvert E_i(D_j)\rvert$
D_1	$i = 1$	$9 \times 2^k - 3$	3×2^k
	$i = 2$	$30 \times 2^k - 15$	$6 \times 2^k - 3$
	$i = 3$	$9 \times 2^k - 5$	$18 \times 2^k - 9$
	$i = 4$	-	$21 \times 2^k - 12$
D_2	$i = 1$	$12 \times 2^k - 4$	4×2^k
	$i = 2$	$40 \times 2^k - 18$	$8 \times 2^k - 1$
	$i = 3$	$12 \times 2^k - 6$	$24 \times 2^k - 11$
	$i = 4$	-	$28 \times 2^k - 14$
D_3	$i = 1$	$6 \times 2^k - 4$	2^{k+1}
	$i = 2$	$20 \times 2^k - 18$	$4 \times 2^k - 4$
	$i = 3$	$6 \times 2^k - 6$	$12 \times 2^k - 11$
	$i = 4$	-	$14 \times 2^k - 14$
D_4	$i = 1$	2^{k+1}	2^{k+1}
	$i = 2$	$12 \times 2^k - 8$	-
	$i = 3$	$2 \times 2^k - 2$	$8 \times 2^k - 5$
	$i = 4$	-	$6 \times 2^k - 6$
D_5	$i = 1$	4×2^k	4×2^k
	$i = 2$	$24 \times 2^k - 8$	-
	$i = 3$	$4 \times 2^k - 2$	$16 \times 2^k - 5$
	$i = 4$	-	$12 \times 2^k - 6$

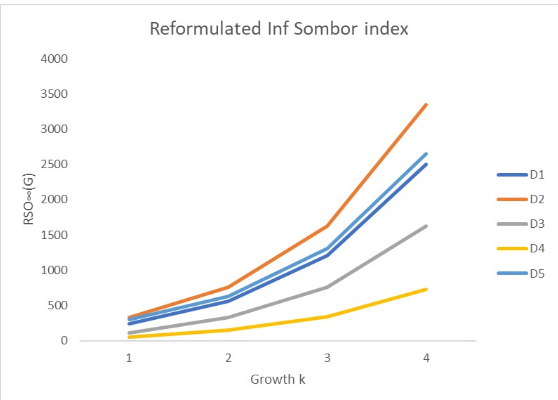
TABLE 5. Order of vertex and edge class of D_j



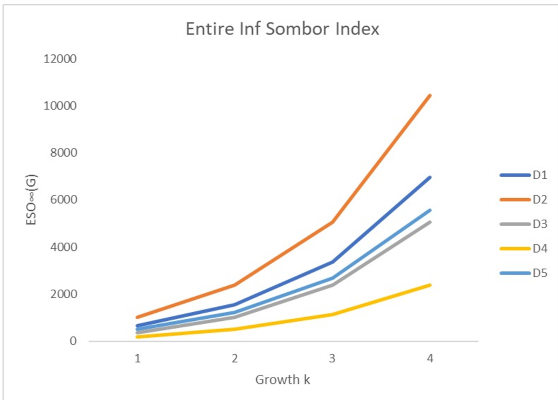
(A) Comparision of Platt Number of Dendrimers



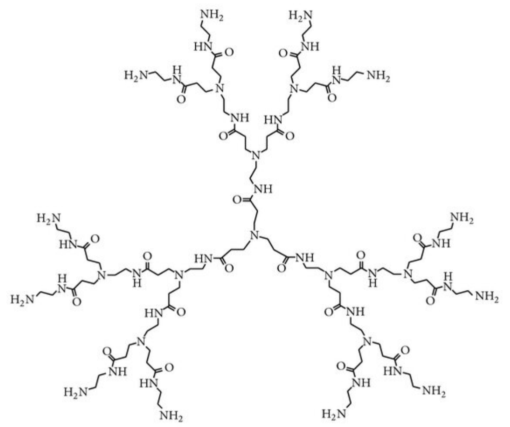
(B) Comparision of Infinite Sombor index of Dendrimers



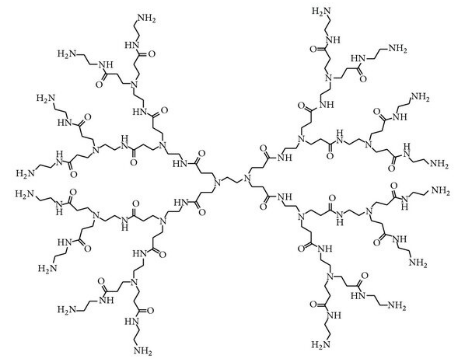
(C) Comparision of Reformulated Inf Sombor index of Dendrimers



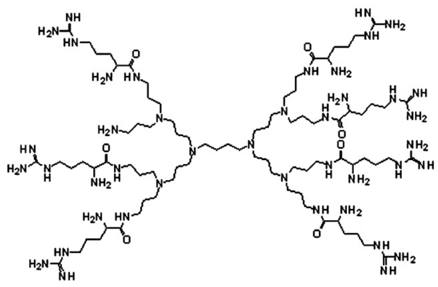
(D) Comparision of Entire Inf Sombor index of Dendrimers



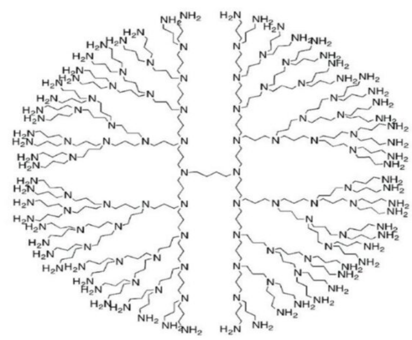
(A) First type of polyamidoamine dendrimer D_1



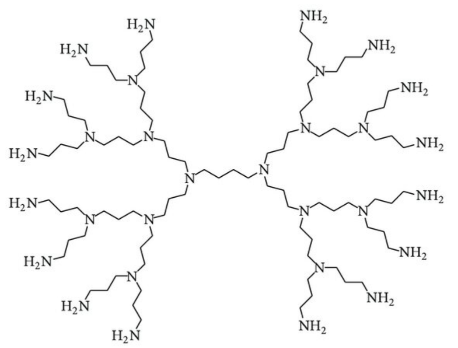
(B) Second type of polyamidoamine dendrimer D_2



(C) First type of polypropylenimine D_3



(D) Polypropylenimine octaamine dendrimer D_4



(E) Polypropyleneamine dendrimer (POPAM) D_5

5. Conclusions

To achieve more accurate estimates of intermolecular forces, one needs to consider the relations between edges and vertices in addition to the relation between vertices. This is because intermolecular forces occur not just between atoms but also between atoms and bonds. Hence, we studied the edge version of the Inf-Sombor and Entire Sombor indexes and gave some bounds. In the end, we calculated these indices for the different dendrimers and made a graphical comparison. The following can be taken into consideration for further study.

(1) Characterize these indices with respect to other degree-based topological indices.

(2) Finding extremal graphs for these indices will be an interesting task.

6. Declaration

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Conflicts of Interest: The authors declares that there are no conflicts of interest.

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