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Research Paper

ON THE DISTANCE-BASED INDICES OF MOBIUS FUNCTION GRAPH OF FINITE GROUPS

RANI JOSE[1,*] AND SUSHA D[2]

[1]Department of Mathematics, St. Dominic's College, Kanjirapally-686512, Kerala, India, ranijose@sdck.in

[2]Department of Mathematics, Catholicate College, Pathanamthitta - 689645, Kerala, India, sushad70@gmail.com

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ABSTRACT

In the domain of mathematical chemistry and graph theory, topological indices have emerged as vital tools for quantifying the structural properties of molecular graphs. Recently, the Möbius function graph of a finite group has earned significant attention due to its connections with algebraic and topological structures. However, determination of the topological indices of these graphs remain largely unexplored. In this paper we compute and investigate the relationships between several distance- based topological indices, including the Mostar index, weighted Mostar index, Szeged index, weighted Szeged index, PI index and weighted PI index, for the Möbius function graphs of finite groups.

1. Introduction

Algebraic graph theory deals with connections between algebraic structures and graphs [11, 12, 25, 27]. Several authors determined various graphs, including zero divisor graphs, coprime graphs, comaximal graph, etc., derived from algebraic structures such as groups, rings, and semigroups [1, 10, 32]. Based on this, we introduced the Möbius function graph of finite groups [20]. The Möbius function graph $M(G)$ of a finite group G is a simple graph whose vertex set is same as the elements

∗Address correspondence to Rani Jose; Department of Mathematics, St. Dominic's College, Kanjirapally-686512, Kerala, India, E-mail: ranijose@sdck.in.

of G and any two distinct vertices u, v are adjacent in $M(G)$ if and only if $\mu(|u||v|) = \mu(|u|)\mu(|v|)$, where μ is the Möbius function. In $M(G)$, the vertex associated to identity element of G is always adjacent to all other vertices. Hence $M(G)$ is always connected. Subsequently, its various properties such as spectral, coloring, Wiener index, and more [18–20] were also studied.

Topological indices play a crucial role in theoretical chemistry in analyzing the unique properties of molecules. Recently, a class of indices, namely distance-based indices, has gained significant attention in chemical graph theory [2, 6–9, 13, 23, 28, 31]. One such example is the Szeged index, introduced by Gutman [17] in 1994, which has been further explored in subsequent studies [16, 24]. Another notable index is the Padmakar-Ivan (PI) index, proposed by Khadikar [22] in 2000. More recently, Došlić *et al.* [14] introduced the Mostar index in 2018, a measure of bond peripherality that provides a global indicator of graph peripherality. A lot of works are being conducted in this area [3–5]

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For any two vertices $u, v \in \Gamma$, $d(u.v)$ denote the distance between u and v. In Γ, if the vertices u and v are adjacent (that is, uv is an edge in Γ), we first state the following definitions. Let

$$n_u(uv|\Gamma) = card\{x \in V(\Gamma) \mid d(u,x) < d(v,x)\} \tag{1.1}$$

$$n_v(uv|\Gamma) = card\{x \in V(\Gamma) \mid d(v,x) < d(u,x)\} \tag{1.2}$$

Where *card* refers to cardinality of the set. The two quantities described above appeared for the first time in [30].

$$M_O(\Gamma) = \sum_{uv \in E(\Gamma)} |n_u(uv|\Gamma) - n_v(uv|\Gamma)|$$

$$S_Z(\Gamma) = \sum_{uv \in E(\Gamma)} n_u(uv|\Gamma) n_v(uv|\Gamma)$$

$$PI(\Gamma) = \sum_{uv \in E(\Gamma)} (n_u(uv|\Gamma) + n_v(uv|\Gamma))$$

Giving importance to contributions of edges in Mostar index gives rise to weighted Mostar index [21]. Following the introduction of the Szeged and PI indices, which acquired considerable attention in the mathematical chemistry community, Ilic and Milosavljević proposed enhanced versions of these indices [26]. Drawing inspiration from the extension of the Wiener index, they introduced the weighted Szeged index and weighted PI index [15, 29], which are defined as

$$M_{O_w}(\Gamma) = \sum_{uv \in E(\Gamma)} (d(u) + d(v))|n_u(uv|\Gamma) - n_v(uv|\Gamma)|$$

$$S_{Z_w}(\Gamma) = \sum_{uv \in E(\Gamma)} (d(u) + d(v))(n_u(uv|\Gamma) n_v(uv|\Gamma))$$

$$PI_w(\Gamma) = \sum_{uv \in E(\Gamma)} (d(u) + d(v))(n_u(uv|\Gamma) + n_v(uv|\Gamma))$$

Where $d(u)$ represents the degree of vertex u in Γ. Also, let $\Gamma_1 = (V_1(\Gamma_1), E_1(\Gamma_1))$ and $\Gamma_2 = (V_2(\Gamma_2), E_2(\Gamma_2))$ be two simple graphs which are vertex- disjoint. The join $\Gamma_1 + \Gamma_2$ of Γ_1 and Γ_2 is defined as the graph where every vertex of Γ_1 is adjacent to every vertex of Γ_2.

This paper mainly focused on computing several distance - based topological indices – Mostar index, weighted Mostar index, Szeged index, weighted Szeged index, PI index and weighted PI index – for the Möbius function graphs of finite groups. By exploring these indices, we aim to find the relationships between the algebraic and topological aspects of these graphs. Also, we derive some

characterization theorems connecting these indices.
Note: Throughout this paper, we denote the identity element of a group by e.

2. Main Results

This section is devoted to investigating distance-based and weighted Mostar, Szeged and PI indices of Möbius function graphs associated with finite groups. Specifically, the next three theorems will provide an in-depth analysis of these indices for the special case of p-groups.

Theorem 2.1. *For any group of prime order p, the Möbius function graph have*

a) $M_O(M(G)) = p^2 - 3p + 2$
b) $M_{O_w}(M(G)) = p(p^2 - 3p + 2)$
c) $S_Z(M(G)) = (p-1)^2$
d) $S_{Z_w}(M(G)) = p(p-1)^2$
e) $PI_w(M(G)) = p^2(p-1)$

Proof. Let G be a group with $|G| = p$, where p is a prime number. In the Möbius function graph $M(G)$, the vertex corresponding to the identity element e is adjacent to every other vertex. Moreover, since the order of $|G|$ is prime, every non- identity element u in G has order p. For any two arbitrary elements $u, v \in G - \{e\}$, by using the definition of Möbius function graph $\mu(|u||v|) \neq \mu(|u|)\mu(|v|)$ and hence u is not adjacent to v. Therefore, $M(G)$ is isomorphic to the star graph $K_{1,p-1}$.
For any edge eu in $M(G)$ (from equation 1.1 and 1.2)

$$n_e(eu|M(G)) = card\{x \in V(M(G)) \mid d(e,x) < d(u,x)\} = (p-1) \tag{2.1}$$

$$n_u(eu|M(G)) = card\{x \in V(M(G)) \mid d(u,x) < d(e,x)\} = 1 \tag{2.2}$$

Here $d(e) = p - 1$ and $d(u) = 1$.
By using Equations 2.1 and 2.2 we get

a) Mostar index of $M(G)$ is

$$\begin{aligned} M_O(M(G)) &= \sum_{eu \in E(M(G))} |n_e(eu|M(G)) - n_u(eu|M(G))| \\ &= \sum_{eu \in E(M(G))} |(p-1) - 1| \\ &= p^2 - 3p + 2 \end{aligned}$$

b) Corresponding weighted Mostar index is

$$\begin{aligned} M_{O_w}(M(G)) &= \sum_{eu \in E(M(G))} (d(e) + d(u))|n_e(eu|M(G)) - n_u(eu|M(G))| \\ &= \sum_{eu \in E(M(G))} ((p-1)+1)|(p-1) - 1| \\ &= p(p^2 - 3p + 2) \end{aligned}$$

c) Szeger index of $M(G)$ is

$$\begin{aligned} S_Z(M(G)) &= \sum_{eu \in E(M(G))} (n_e(eu|M(G))n_u(eu|M(G))) \\ &= \sum_{eu \in E(M(G))} (p-1) \cdot 1 \end{aligned}$$

$$= (p-1)^2$$

d) Corresponding weighted Szeger index is

$$S_{Z_w}(M(G)) = \sum_{eu \in E(M(G))} (d(e)+d(u))(n_e(eu|M(G))n_u(eu|M(G)))$$
$$= p(p-1)^2$$

e) PI index of $M(G)$ is

$$PI(M(G)) = \sum_{eu \in E(M(G))} (n_e(eu|M(G)) + n_u(eu|M(G)))$$
$$= p(p-1)$$

f) Corresponding weighted PI index is

$$PI_w(M(G)) = \sum_{eu \in E(M(G))} (d(e)+d(u))(n_e(eu|M(G)) + n_u(eu|M(G)))$$
$$= p^2(p-1)$$

□

From the above Theorem 2.1 we can arrive to the following results

Corollary 2.2. *For a Möbius function graph of finite group of prime order* p*,*

a) $M_{O_w}(M(G)) = p(M_O(M(G)))$
b) $S_{Z_w}(M(G)) = p(S_z(M(G)))$
c) $PI_w(M(G)) = p(PI(M(G)))$

Theorem 2.3. *Let G be a finite abelian group of order* 2^k*, for some* $k \in \mathbb{Z}^+$*. Then*

a) $M_O(M(G)) = 0$
b) $M_{O_w}(M(G)) = 0$
c) $S_Z(M(G)) = 2^{k-1}(2^k-1)$
d) $S_{Z_w}(M(G)) = 2^k(2^k-1)^2$
e) $PI(M(G)) = 2^k(2^k-1)$
f) $PI_w(M(G)) = 2^{k+1}(2^k-1)^2$

Proof. Let G be a finite abelian group of order $n = 2^k$. We first establish that the graph $M(G)$ is complete. Since G is a 2-group, there exists an element u in G such that $|u| = 2$. Clearly, u is a self-inverse element in G. Moreover, every element in G, except for u and the identity e, has order a power of 2. It follows that every vertex in $M(G)$ satisfies the condition for a Möbius function graph. Consequently, $M(G)$ is a complete graph. For any edge uv in $M(G)$

$$n_u(uv|M(G)) = card\{x \in V(M(G)) \mid d(u,x) < d(v,x)\} = card\{u\} = 1$$
$$n_v(uv|M(G)) = card\{x \in V(M(G)) \mid d(v,x) < d(u,x)\} = card\{v\} = 1$$

Here in this case $d(u) = d(v) = n-1$

a) Mostar index of $M(G)$ is

$$M_0(M(G)) = \sum_{uv \in E(M(G))} |n_u(uv|M(G)) - n_v(uv|M(G))|$$

$$= \sum_{uv \in E(M(G))} |1-1| = 0$$

b) Corresponding weighted Mostar index of $M(G)$ is also zero.

c) Szeger index of $M(G)$ is

$$S_Z(M(G)) = \sum_{uv \in E(M(G))} (n_u(uv|M(G)) n_v(uv|M(G)))$$
$$= \binom{n}{2} = 2^{k-1}(2^k - 1)$$

d) Weighted Szeger index is calculated as

$$S_{Z_w}(M(G)) = \sum_{uv \in E(M(G))} (d(u)+d(v))(n_u(uv|M(G)) n_v(uv|M(G)))$$
$$= \binom{n}{2}(2n-2)(1) = 2^k(2^k-1)^2$$

e) PI index of $M(G)$ is

$$PI(M(G)) = \sum_{uv \in E(M(G))} (n_u(uv|M(G)) + n_v(uv|M(G)))$$
$$= \binom{n}{2} 2 = 2^k(2^k - 1)$$

f) Weighted PI index is calculated as

$$PI_w(M(G)) = \sum_{uv \in E(M(G))} (d(u)+d(v))(n_u(uv|M(G)) + n_v(uv|M(G)))$$
$$= \binom{n}{2}(2n-2)(2) = 2^{k+1}(2^k-1)^2$$

□

Using Theorem 2.3, we characterize the Möbius function graph of an abelian 2- group.

Corollary 2.4. *In Möbius function graph of an abelian group of order* 2^k

a) $PI(M(G)) = 2S_Z(M(G))$

b) $PI_w(M(G)) = 2S_{Z_w}(M(G))$

Theorem 2.5. *Let G be a finite group and $|G| = p^k$, where p is prime and $k > 1$. Then*

a) $M_O(M(G)) = mn(m-1)$

b) $M_{O_w}(M(G)) = m^3n + 2m^2n^2 - 2m^2n - 2mn^2 + mn$

c) $S_z(M(G)) = \binom{n}{2} + m^2n$

d) $S_{Z_w}(M(G)) = m^3n + 2m^2n^2 + n^3 + mn^2 - m^2n - 2n^2 - mn + n$

e) $PI(M(G)) = n(m^2 + (m+n) - 1)$

f) $PI_w(M(G)) = 2n^3 + 4mn^2 - 4n^2 - 3mn + 2n + m^3n + 2m^2n^2$

where n denote the number of elements of order p and $m = p^k - n$.

Proof. We can partition the vertex set of $M(G)$ into $\Psi_1 = \{x \in M(G) : |x| = p\}$ and $\Psi_2 = \{y \in M(G) : |y| \neq p\}$. Let $\Psi_1 = \{u_1, u_2, \ldots, u_m\}$ and $\Psi_2 = \{v_1, v_2, \ldots, v_n\}$. On applying the definition of Möbius function graph in Ψ_1, we get the subgraph induced by Ψ_1 in $M(G)$ is totally disconnected and is isomorphic to $\overline{K}_{|\Psi_1|}$. Similarly we get the subgraph induced by Ψ_2 in $M(G)$ is complete, which is isomorphic to $K_{|\Psi_2|}$. Next we take $x \in \Psi_1$ and $y \in \Psi_2$, $\mu(|x||y|) = \mu(|x|)\mu(|y|)$. Thus each element of Ψ_1 is adjacent to every element in Ψ_2. Hence $M(G)$ is a split graph with Ψ_1 as the independent set of cardinality m and Ψ_2 forms the clique with order n.

Case 1: For any $v_i, v_j \in \Psi_2$

$$\begin{aligned} n_{v_i}(v_iv_j|M(G)) &= card\{x \in V(M(G)) \mid d(v_i,x) < d(v_j,x)\} \\ &= card\{v_i\} = 1 \\ n_{v_j}(v_iv_j|M(G)) &= card\{x \in V(M(G)) \mid d(v_j,x) < d(v_i,x)\} \\ &= card\{v_j\} = 1 \end{aligned}$$

In this case $d(v_i) = d(v_j) = m + n - 1$

Case 2: For any $u_i \in \Psi_1$ and $v_j \in \Psi_2$

$$\begin{aligned} n_{u_i}(u_iv_j|M(G)) &= card\{x \in V(M(G)) \mid d(u_i,x) < d(v_j,x)\} \\ &= card\{u_i\} = 1 \\ n_{v_j}(u_iv_j|M(G)) &= card\{x \in V(M(G)) \mid d(v_j,x) < d(u_i,x)\} \\ &= card\{\{v_j\} \cup \{\Psi_1 - \{u_i\}\} = m \end{aligned}$$

Here $d(u_i) = n$ and $d(v_j) = m + n - 1$

By using the above two cases we get

a) Mostar index of $M(G)$ is

$$\begin{aligned} M_O(M(G)) &= \sum_{uv \in E(M(G))} |n_u(uv|M(G)) - n_v(uv|M(G))| \\ &= \sum_{v_i,v_j \in \Psi_2} |n_{v_i}(v_iv_j|M(G)) - n_{v_j}(v_iv_j|M(G))| + \\ &\quad \sum_{\substack{u_i \in \Psi_1 \\ v_j \in \Psi_2}} |n_{u_i}(u_iv_j|M(G)) - n_{v_j}(u_iv_j|M(G))| \\ &= \sum_{v_i,v_j \in \Psi_2} |1-1| + \sum_{\substack{u_i \in \Psi_1 \\ v_j \in \Psi_2}} |1-m| \\ &= 0 + mn|1-m| \\ &= mn(m-1) \end{aligned}$$

b) Weighted Mostar index is calculated as

$$\begin{aligned} M_{O_w}(M(G)) &= \sum_{uv \in E(M(G))} (d(u)+d(v))|n_u(uv|M(G)) - n_v(uv|M(G))| \\ &= \sum_{v_i,v_j \in \Psi_2} (d(v_i)+d(v_j))|n_{v_i}(v_iv_j|M(G)) - n_{v_j}(v_iv_j|M(G))| + \\ &\quad \sum_{\substack{u_i \in \Psi_1 \\ v_j \in \Psi_2}} (d(u_i)+d(v_j))|n_{u_i}(u_iv_j|M(G)) - n_{v_j}(u_iv_j|M(G))| \\ &= mn(m+2n-1)(m-1) \\ &= m^3n + 2m^2n^2 - 2m^2n - 2mn^2 + mn \end{aligned}$$

c) Szeger index of $M(G)$ is

$$\begin{aligned}
S_z(M(G)) &= \sum_{uv\in E(M(G))} (n_u(uv|M(G))n_v(uv|M(G))) \\
&= \sum_{v_i,v_j\in\Psi_2} (n_{v_i}(v_iv_j|M(G))n_{v_j}(v_iv_j|M(G)))+ \\
&\quad \sum_{\substack{u_i\in\Psi_1\\ v_j\in\Psi_2}} (n_{u_i}(u_iv_j|M(G))n_{v_j}(u_iv_j|M(G))) \\
&= \sum_{v_i,v_j\in\Psi_2} (1) + \sum_{\substack{u_i\in\Psi_1\\ v_j\in\Psi_2}} (m) \\
&= \binom{n}{2} + m^2n
\end{aligned}$$

d) Weighted Szeger index is calculated as

$$\begin{aligned}
S_{Z_w}(M(G)) &= \sum_{uv\in E(M(G))} (d(u)+d(v))(n_u(uv|M(G))n_v(uv|M(G))) \\
&= \sum_{v_i,v_j\in\Psi_2} (d(v_i)+d(v_j))(n_{v_i}(v_iv_j|M(G))n_{v_j}(v_iv_j|M(G)))+ \\
&\quad \sum_{\substack{u_i\in\Psi_1\\ v_j\in\Psi_2}} (d(u_i)+d(v_j))(n_{u_i}(u_iv_j|M(G))n_{v_j}(u_iv_j|M(G))) \\
&= \binom{n}{2}(2n+2m-2) + m^2n(m+2n-1) \\
&= m^3n + 2m^2n^2 + n^3 + mn^2 - m^2n - 2n^2 - mn + n
\end{aligned}$$

e) PI index of $M(G)$ is

$$\begin{aligned}
PI(M(G)) &= \sum_{uv\in E(M(G))} (n_u(uv|M(G))+n_v(uv|M(G))) \\
&= \sum_{v_i,v_j\in\Psi_2} (n_{v_i}(v_iv_j|M(G))+n_{v_j}(v_iv_j|M(G)))+ \\
&\quad \sum_{\substack{u_i\in\Psi_1\\ v_j\in\Psi_2}} (n_{u_i}(u_iv_j|M(G))+n_{v_j}(u_iv_j|M(G))) \\
&= \sum_{v_i,v_j\in\Psi_2} (1+1) + \sum_{\substack{u_i\in\Psi_1\\ v_j\in\Psi_2}} (1+m) \\
&= \binom{n}{2}2 + mn(m+1) \\
&= n(m^2+(m+n)-1)
\end{aligned}$$

f) Weighted PI index is calculated as

$$\begin{aligned}
PI_w(M(G)) &= \sum_{uv\in E(M(G))} (d(u)+d(v))(n_u(uv|M(G))+n_v(uv|M(G))) \\
&= \sum_{v_i,v_j\in\Psi_2} (d(u)+d(v))(n_{v_i}(v_iv_j|M(G))+n_{v_j}(v_iv_j|M(G)))+
\end{aligned}$$

$$\sum_{\substack{u_i \in \Psi_1 \\ v_j \in \Psi_2}} (d(u)+d(v))(n_{u_i}(u_iv_j|M(G)) + n_{v_j}(u_iv_j|M(G)))$$
$$= \binom{n}{2}(2n+2m-2)2 + mn(m+2n-1)(m+1)$$
$$= 2n^3 + 4mn^2 - 4n^2 - 3mn + 2n + m^3n + 2m^2n^2$$

□

In the next theorem, we identify the distance - based and weighted indices of Möbius function graph of group of order pq

Theorem 2.6. *Let G be a group with $|G| = pq$, where p and q are distinct primes. Then*

a) $M_O(M(G)) = n^2 + m^2 + t^2 + 2nt + 2mt + nm|m-n| - (n+m+t)$

b) $M_{O_w}(M(G)) = m^3 + n^3 + t^3 + 2mn(m+n) + 3nt(n+t) + 3mt(m+t) + 6mnt - 4mn - mt - (m+n+t) + (m^2n + mn^2 + 2mn)|m-n|$

c) $S_Z(M(G)) = n^2 + m^2 + t^2 + 2(n+m)t + n^2m^2$

d) $S_{Z_w}(M(G)) = m^3 + n^3 + t^3 + 2mn(m+n) + 3nt(n+t) + 3mt(m+t) + m^2n^2(m+n) + 2m^2n^2 + n^2 + m^2 + t^2 + 2nt + 2mt + 6mnt$

e) $PI(M(G)) = m^2 + n^2 + t^2 + (mn+2t)(m+n) + m + n + t$

f) $PI_w(M(G)) = m^3 + n^3 + t^3 + 4mn^2 + 4m^2n + 3n^2t + 3nt^2 + 3m^2t + 3mt^2 + m^3n + mn^3 + 2m^2n^2 + 2m^2 + 2n^2 + 2t^2 + 4nt + 4mt + 4mn + 6mnt + m + n + t$

where n, m, t denote the number of elements of order p, q, pq.

Proof. Let us consider a group G with $|G| = pq$, then we can partition the vertex set of Möbius function graph as $\Psi_1 = \{e\}$, $\Psi_2 = \{u : |u| = p\}$, $\Psi_3 = \{v : |v| = q\}$, $\Psi_4 = \{w : |w| = pq\}$. By the definition of Möbius function graph of finite group Ψ_2, Ψ_3, Ψ_4 are independent sets which are represented by $\overline{K_n}, \overline{K_m}, \overline{K_t}$ respectively where n, m, t denote the number of elements of order p, q, pq.
For any $u \in \Psi_2$ and $v \in \Psi_3$, $\mu(|u||v|) = \mu(|u|)\mu(|v|)$. Hence sets Ψ_2 and Ψ_3 forms a complete bipartate graph in $M(G)$ and it is represented by $\overline{K_n} + \overline{K_m}$. Hence, the graph so obtained is isomorphic to $(\overline{K_n} + \overline{K_m}) + K_1 + \overline{K_t}$, where $+$ denote the join of these graphs. Now we can find all the indices of $M(G)$
Case 1: For $e \in \Psi_1, u \in \Psi_2$

$$n_e(eu|M(G)) = card\{x \in V(M(G)) \mid d(e,x) < d(u,x)\}$$
$$= card\{\Psi_1 \cup \Psi_2 - \{u\} \cup \Psi_4\} = n + t$$
$$n_u(eu|M(G)) = card\{x \in V(M(G)) \mid d(u,x) < d(e,x)\}$$
$$= card\{u\} = 1$$

In this case $d(e) = m + n + t$ and $d(u) = m + 1$.
Case 2: For $e \in \Psi_1, v \in \Psi_3$

$$n_e(ev|M(G)) = card\{x \in V(M(G)) \mid d(e,x) < d(v,x)\}$$
$$= card\{\Psi_1 \cup \Psi_3 - \{v\} \cup \Psi_4\} = m + t$$
$$n_v(ev|M(G)) = card\{x \in V(M(G)) \mid d(v,x) < d(e,x)\}$$

$$= card\{v\} = 1$$

Here $d(e) = m + n + t$ and $d(v) = n + 1$.

Case 3: For $e \in \Psi_1, w \in \Psi_4$

$$\begin{aligned} n_e(ew|M(G)) &= card\{x \in V(M(G)) \mid d(e,x) < d(w,x)\} \\ &= card\{\Psi_1 \cup \Psi_2 \cup \Psi_3 \cup \Psi_4 - \{w\}\} = n + m + t \\ n_w(ew|M(G)) &= card\{x \in V(M(G)) \mid d(w,x) < d(e,x)\} \\ &= card\{w\} = 1 \end{aligned}$$

In this case $d(e) = m + n + t$ and $d(w) = 1$.

Case 4: For $u \in \Psi_2, v \in \Psi_3$

$$\begin{aligned} n_u(uv|M(G)) &= card\{x \in V(M(G)) \mid d(u,x) < d(v,x)\} \\ &= card\{\{u\} \cup \Psi_3 - \{v\}\} = m \\ n_v(uv|M(G)) &= card\{x \in V(M(G)) \mid d(v,x) < d(u,x)\} \\ &= card\{\{v\} \cup \Psi_2 - \{u\}\} = n \end{aligned}$$

Here $d(u) = m + 1$ and $d(v) = n + 1$.

By using above four cases;

a) Mostar index of $M(G)$ is

$$\begin{aligned} M_O(M(G)) &= \sum_{uv \in E(M(G))} |n_u(uv|M(G)) - n_v(uv|M(G))| \\ &= \sum_{\substack{e \in \Psi_1 \\ u \in \Psi_2}} |n_e(eu|M(G)) - n_u(eu|M(G))| + \\ &\quad \sum_{\substack{e \in \Psi_1 \\ v \in \Psi_3}} |n_e(ev|M(G)) - n_v(ev|M(G))| + \\ &\quad \sum_{\substack{e \in \Psi_1 \\ w \in \Psi_4}} |n_e(ew|M(G)) - n_v(ew|M(G))| + \\ &\quad \sum_{\substack{u \in \Psi_2 \\ v \in \Psi_3}} |n_u(uv|M(G)) - n_v(uv|M(G))| \\ &= n \cdot |(n+t) - 1| + m \cdot |(m+t) - 1| + \\ &\quad t \cdot |(n+m+t) - 1| + mn \cdot |m - n| \\ &= n^2 + m^2 + t^2 + 2nt + 2mt + nm|m-n| - (n+m+t) \end{aligned}$$

b) Corresponding weighted Mostar index can be calculated as

$$\begin{aligned} M_{O_w}(M(G)) &= \sum_{uv \in E(M(G))} (d(u) + d(v))|n_u(uv|M(G)) - n_v(uv|M(G))| \\ &= \sum_{\substack{e \in \Psi_1 \\ u \in \Psi_2}} (d(u) + d(v))|n_e(eu|M(G)) - n_u(eu|M(G))| + \\ &\quad \sum_{\substack{e \in \Psi_1 \\ v \in \Psi_3}} (d(u) + d(v))|n_e(ev|M(G)) - n_v(ev|M(G))| + \end{aligned}$$

$$\sum_{\substack{e\in\Psi_1\\ w\in\Psi_4}}(d(u)+d(v))|n_e(ew|M(G))-n_v(ew|M(G))|+$$

$$\sum_{\substack{u\in\Psi_2\\ v\in\Psi_3}}(d(u)+d(v))|n_u(uv|M(G))-n_v(uv|M(G))|$$

$$=n[(2m+n+t+1)|(n+t)-1|]+m[(m+2n+t+1)|$$
$$(m+t)-1|]+t[(m+n+t+1)|(n+m+t)-1|]+$$
$$mn[(m+n+2)|m-n|]$$
$$=m^3+n^3+t^3+2mn(m+n)+3nt(n+t)+3mt(m+t)+$$
$$6mnt-4mn-mt-(m+n+t)+$$
$$(m^2n+mn^2+2mn)|m-n|$$

c) Szeger index of $M(G)$ is

$$S_Z(M(G))=\sum_{uv\in E(M(G))}(n_u(uv|M(G))n_v(uv|M(G)))$$
$$=\sum_{\substack{e\in\Psi_1\\ u\in\Psi_2}}(n_e(eu|M(G))n_u(eu|M(G)))+$$
$$\sum_{\substack{e\in\Psi_1\\ v\in\Psi_3}}(n_e(ev|M(G))n_v(ev|M(G)))+$$
$$\sum_{\substack{e\in\Psi_1\\ w\in\Psi_4}}(n_e(ew|M(G))n_v(ew|M(G)))+$$
$$\sum_{\substack{u\in\Psi_2\\ v\in\Psi_3}}(n_u(uv|M(G))n_v(uv|M(G)))$$
$$=n(n+t)+m(m+t)+t(n+m+t)+n^2m^2$$
$$=n^2+m^2+t^2+2(n+m)t+n^2m^2$$

d) Weighted Szeger index is calculated as

$$S_{Z_w}(M(G))=\sum_{uv\in E(M(G))}(d(u)+d(v))(n_u(uv|M(G))n_v(uv|M(G)))$$
$$=\sum_{\substack{e\in\Psi_1\\ u\in\Psi_2}}(d(u)+d(v))(n_e(eu|M(G))n_u(eu|M(G)))+$$
$$\sum_{\substack{e\in\Psi_1\\ v\in\Psi_3}}(d(u)+d(v))(n_e(ev|M(G))n_v(ev|M(G)))+$$
$$\sum_{\substack{e\in\Psi_1\\ w\in\Psi_4}}(d(u)+d(v))(n_e(ew|M(G))n_v(ew|M(G)))+$$
$$\sum_{\substack{u\in\Psi_2\\ v\in\Psi_3}}(d(u)+d(v))(n_u(uv|M(G))n_v(uv|M(G)))$$
$$=m^3+n^3+t^3+2mn(m+n)+3nt(n+t)+$$
$$3mt(m+t)+20ptm^2n^2(m+n)+2m^2n^2+n^2+m^2+t^2+$$

$$2nt + 2mt + 6mnt$$

e) PI index of $M(G)$ is

$$\begin{aligned}
PI(M(G)) = & \sum_{uv\in E(M(G))} (n_u(uv|M(G)) + n_v(uv|M(G))) \\
= & \sum_{\substack{e\in\Psi_1 \\ u\in\Psi_2}} (n_e(eu|M(G)) + n_u(eu|M(G)))+ \\
& \sum_{\substack{e\in\Psi_1 \\ v\in\Psi_3}} (n_e(ev|M(G)) + n_v(ev|M(G)))+ \\
& \sum_{\substack{e\in\Psi_1 \\ w\in\Psi_4}} (n_e(ew|M(G)) + n_v(ew|M(G)))+ \\
& \sum_{\substack{u\in\Psi_2 \\ v\in\Psi_3}} (n_u(uv|M(G)) + n_v(uv|M(G))) \\
= & m^2 + n^2 + t^2 + (mn + 2t)(m + n) + m + n + t
\end{aligned}$$

f) Weighted PI index of $M(G)$ is

$$\begin{aligned}
PI(M(G)) = & \sum_{uv\in E(M(G))} (d(u) + d(v))(n_u(uv|M(G)) + n_v(uv|M(G))) \\
= & m^3 + n^3 + t^3 + 4mn^2 + 4m^2n + 3n^2t + 3nt^2 + 3m^2t+ \\
& 3mt^2 + m^3n + mn^3 + 2m^2n^2 + 2m^2 + 2n^2+ \\
& 2t^2 + 4nt + 4mt + 4mn + 6mnt + m + n + t
\end{aligned}$$

□

Theorem 2.7. *Let D_{2p} be the dihedral group with $p > 2$ a prime number, then*

a) $M_O(M(D_{2p})) = 3p^2 - 5p + 2$
b) $M_{O_w}(M(D_{2p})) = 8p^3 - 14p^2 + 6p$
c) $S_Z(M(D_{2p})) = p^4 - 2p^3 + 3p^2 - 2p + 1$
d) $S_{Z_w}(M(D_{2p})) = 2p^5 - 4p^4 + 5p^3 - 2p^2 + 3p$
e) $PI(M(D_{2p})) = 2p^3 - p^2 + p$
f) $PI_w(M(D_{2p})) = 4p^4 + 6p^3 - 2p^2$

Proof. Consider the dihedral group $G = D_{2p} = \{e, a, a^2, \ldots, a^{p-1}, b, ab, a^2b, \ldots, a^{p-1}b\}$, for a prime $p > 2$. Then $|e| = 1$, $|a| = |a^2| = \ldots = |a^{p-1}| = p$, $|b| = |ab| = |a^2b| = \ldots = |a^{p-1}b| = 2$. We can partition the vertex set of $M(G)$ into three sets say

$$\begin{aligned}
\Psi_1 &= \{e\} \\
\Psi_2 &= \{a^i : 1 \le i \le p-1\} \\
\Psi_3 &= \{a^jb : 0 \le j \le p-1\}
\end{aligned}$$

with $|\Psi_1| = 1, |\Psi_2| = p-1, |\Psi_3| = p$. In case of $M(G)$, the vertex associated with identity element e is adjacent to all other $(2p-1)$ vertices. That is Ψ_1 is adjacent to every elements of Ψ_2 and Ψ_3 For any two vertices $u, v \in \Psi_2$, $\mu(|u||v|) \neq \mu(|u|)\mu(|v|)$. Hence no two elements of Ψ_2 are adjacent. Similarly we can prove in case of Ψ_3, no two elements are adjacent.

Next we consider the case when $u \in \Psi_2$ and $w \in \Psi_3$ then

$$\mu(|u|)\mu(|w|) = \mu(p)\mu(2) = (-1)(-1) = 1$$
$$\mu(|u||w|) = \mu(p.2) = (-1)^2 = 1$$

Therefore, $\mu(|u||w|) = \mu(|u|)\mu(|w|)$. Here every vertex of Ψ_2 is adjacent to every vertex of Ψ_3. Hence we can conclude that $M(D_{2p})$, $p > 2$ prime number, is a complete tripartite graph.

Case 1: For $e \in \Psi_1, u \in \Psi_2$

$$\begin{aligned} n_e(eu|M(G)) &= card\{x \in V(M(G)) \mid d(e,x) < d(u,x)\} \\ &= card\{\Psi_1 \cup \Psi_2 - \{u\}\} = p - 1 \\ n_u(eu|M(G)) &= card\{x \in V(M(G)) \mid d(u,x) < d(e,x)\} \\ &= card\{u\} = 1 \end{aligned}$$

In this case $d(e) = 2p - 1$ and $d(u) = p + 1$

Case 2: For $e \in \Psi_1, v \in \Psi_3$

$$\begin{aligned} n_e(ev|M(G)) &= card\{x \in V(M(G)) \mid d(e,x) < d(v,x)\} \\ &= card\{\Psi_1 \cup \Psi_3 - \{v\}\} = p \\ n_v(ev|M(G)) &= card\{x \in V(M(G)) \mid d(v,x) < d(e,x)\} \\ &= card\{v\} = 1 \end{aligned}$$

Here $d(e) = 2p - 1$ and $d(v) = p$

Case 3: For $u \in \Psi_2, v \in \Psi_3$

$$\begin{aligned} n_u(uv|M(G)) &= card\{x \in V(M(G)) \mid d(u,x) < d(v,x)\} \\ &= card\{\{u\} \cup \Psi_3 - \{v\}\} = p \\ n_v(uv|M(G)) &= card\{x \in V(M(G)) \mid d(v,x) < d(u,x)\} \\ &= card\{\{v\} \cup \Psi_2 - \{u\}\} = p - 1 \end{aligned}$$

In this case $d(u) = p + 1$ and $d(v) = p$

By using all the above cases we get

a) Mostar index of $M(G)$ is

$$\begin{aligned} M_O(M(G)) &= \sum_{uv \in E(M(G))} |n_u(uv|M(G)) - n_v(uv|M(G))| \\ &= \sum_{\substack{e \in \Psi_1 \\ u \in \Psi_2}} |n_e(eu|M(G)) - n_u(eu|M(G))| + \\ &\quad \sum_{\substack{e \in \Psi_1 \\ v \in \Psi_3}} |n_e(ev|M(G)) - n_v(ev|M(G))| + \\ &\quad \sum_{\substack{u \in \Psi_2 \\ v \in \Psi_3}} |n_u(uv|M(G)) - n_v(uv|M(G))| \\ &= \sum_{\substack{e \in \Psi_1 \\ u \in \Psi_2}} |(p-1) - 1| + \sum_{\substack{e \in \Psi_1 \\ v \in \Psi_3}} |p - 1| + \sum_{\substack{u \in \Psi_2 \\ v \in \Psi_3}} |p - (p-1)| \\ &= (p-1)(p-2) + p(p-1) + p(p-1) \end{aligned}$$

$$= 3p^2 - 5p + 2$$

b) Corresponding weighted Mostar index of $M(G)$ is

$$\begin{aligned}
M_{O_w}(M(G)) &= \sum_{uv\in E(M(G))} (d(u)+d(v))|n_u(uv|M(G)) - n_v(uv|M(G))| \\
&= \sum_{\substack{e\in\Psi_1\\ u\in\Psi_2}} (d(e)+d(u))|n_e(eu|M(G)) - n_u(eu|M(G))| + \\
&\quad \sum_{\substack{e\in\Psi_1\\ v\in\Psi_3}} (d(e)+d(v))|n_e(ev|M(G)) - n_v(ev|M(G))| + \\
&\quad \sum_{\substack{u\in\Psi_2\\ v\in\Psi_3}} (d(u)+d(v))|n_u(uv|M(G)) - n_v(uv|M(G))| \\
&= \sum_{\substack{e\in\Psi_1\\ u\in\Psi_2}} 3p|(p-1)-1| + \sum_{\substack{e\in\Psi_1\\ v\in\Psi_3}} (3p-1)|p-1| + \\
&\quad \sum_{\substack{u\in\Psi_2\\ v\in\Psi_3}} (2p+1)|p-(p-1)| \\
&= 3p(p-1)(p-2) + p(3p-1)(p-1) + p(p-1)(2p+1) \\
&= 8p^3 - 14p^2 + 6p
\end{aligned}$$

c) Szeger index of $M(G)$ is

$$\begin{aligned}
S_Z(M(G)) &= \sum_{uv\in E(M(G))} (n_u(uv|M(G))n_v(uv|M(G))) \\
&= \sum_{\substack{e\in\Psi_1\\ u\in\Psi_2}} (n_e(eu|M(G))n_u(eu|M(G))) + \\
&\quad \sum_{\substack{e\in\Psi_1\\ v\in\Psi_3}} (n_e(ev|M(G))n_v(ev|M(G))) + \\
&\quad \sum_{\substack{u\in\Psi_2\\ v\in\Psi_3}} (n_u(uv|M(G))n_v(uv|M(G))) \\
&= \sum_{\substack{e\in\Psi_1\\ u\in\Psi_2}} (p-1) + \sum_{\substack{e\in\Psi_1\\ v\in\Psi_3}} p + \sum_{\substack{u\in\Psi_2\\ v\in\Psi_3}} (p-1)p \\
&= (p-1)^2 + p^2 + p^2(p-1)^2 \\
&= p^4 - 2p^3 + 3p^2 - 2p + 1
\end{aligned}$$

d) Corresponding weighted Szeger index of $M(G)$ is

$$\begin{aligned}
S_{Z_w}(M(G)) &= \sum_{uv\in E(M(G))} (d(u)+d(v))(n_u(uv|M(G))n_v(uv|M(G))) \\
&= \sum_{\substack{e\in\Psi_1\\ u\in\Psi_2}} (d(e)+d(u))(n_e(eu|M(G))n_u(eu|M(G))) + \\
&\quad \sum_{\substack{e\in\Psi_1\\ v\in\Psi_3}} (d(e)+d(v))(n_e(ev|M(G))n_v(ev|M(G))) +
\end{aligned}$$

$$\sum_{\substack{u\in\Psi_2\\ v\in\Psi_3}} (d(u)+d(v))(n_u(uv|M(G))n_v(uv|M(G)))$$
$$= \sum_{\substack{e\in\Psi_1\\ u\in\Psi_2}} 3p(p-1) + \sum_{\substack{e\in\Psi_1\\ v\in\Psi_3}} (3p-1)p + \sum_{\substack{u\in\Psi_2\\ v\in\Psi_3}} (2p+1)(p-1)p$$
$$= 3p(p-1)^2 + (3p-1)p^2 + (2p+1)p^2(p-1)^2$$
$$= 2p^5 - 4p^4 + 5p^3 - 2p^2 + 3p$$

e) PI index of $M(G)$ is

$$PI(M(G)) = \sum_{uv\in E(M(G))} (n_u(uv|M(G)) + n_v(uv|M(G)))$$
$$= \sum_{\substack{e\in\Psi_1\\ u\in\Psi_2}} (n_e(eu|M(G)) + n_u(eu|M(G)))+$$
$$\sum_{\substack{e\in\Psi_1\\ v\in\Psi_3}} (n_e(ev|M(G)) + n_v(ev|M(G)))+$$
$$\sum_{\substack{u\in\Psi_2\\ v\in\Psi_3}} (n_u(uv|M(G)) + n_v(uv|M(G)))$$
$$= p(p-1) + p(p+1) + p(p-1)(2p-1)$$
$$= 2p^3 - p^2 + p$$

f) Corresponding weighted PI index of $M(G)$ is

$$PI_w(M(G)) = \sum_{uv\in E(M(G))} (d(u)+d(v))(n_u(uv|M(G)) + n_v(uv|M(G)))$$
$$= \sum_{\substack{e\in\Psi_1\\ u\in\Psi_2}} (d(e)+d(u))(n_e(eu|M(G)) + n_u(eu|M(G)))+$$
$$\sum_{\substack{e\in\Psi_1\\ v\in\Psi_3}} (d(e)+d(v))(n_e(ev|M(G)) + n_v(ev|M(G)))+$$
$$\sum_{\substack{u\in\Psi_2\\ v\in\Psi_3}} (d(u)+d(v))(n_u(uv|M(G)) + n_v(uv|M(G)))$$
$$= 3p^2(p-1) + (3p-1)p(p+1) + p(p-1)(4p^2-1)$$
$$= 4p^4 + 6p^3 - 2p^2$$

□

3. Conclusions

In this paper, we compute the Mostar index, weighted Mostar index, Szeged index, weighted Szeged index, PI index, and weighted PI index of the Möbius function graph of finite groups. These calculations facilitate the identification of similarities between the Möbius function graph and other graphs derived from algebraic structures. There is a scope to study the Möbius function graph with other graphs using their respective indices.

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