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Research Paper

ON GENERALIZED BERWALD R-QUADRATIC METRICS

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ABSTRACT

Every Riemannian metric is R-quadratic while many Finsler metrics have not this property. A Finsler metric is called R-quadratic if its Riemannian curvature is quadratic in all direction at any points of the underlying manifold. A Finsler metric on a manifold is called a generalized Berwald metric if there exists a covariant derivative such that the parallel translations induced by it preserve the Finsler function. In this paper, we study the class of generalized Berwald (α, β)-manifolds with R-quadratic properties and prove a rigidity result. We show that such manifolds satisfy $\mathbf{S} = 0$ if and only if $\mathbf{B} = 0$.

1. Introduction

For a Finsler metric F an a manifold M, the second variation of geodesics gives rise to a family of linear maps $\mathbf{R}_y : T_xM \to T_xM$, at any point $y \in T_xM$ which is called the Riemann curvature in the direction y. One can see that it is not only a function of position but also depends on direction, while in Riemann geometry it only depends on position. If F is Riemannian, i.e., $F(y) = \sqrt{\mathbf{g}(y, y)}$ for some Riemannian metric $\mathbf{g}$, then $\mathbf{R}_y := \mathrm{R}(\cdot, y)y$, where $\mathrm{R}(u, v)z$ denotes the Riemannian curvature tensor of $\mathbf{g}$. In this case, $\mathbf{R}_y$ is quadratic in $y \in T_xM$. There are many Finsler metrics whose Riemann curvature in every direction is quadratic. A Finsler space is said to be R-quadratic if its Riemann curvature $\mathbf{R}_y$ is

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quadratic in $y \in T_xM$. Indeed a Finsler metric is R-quadratic if and only if the h-curvature of Berwald connection depends on position only in the sense of Bácsó-Matsumoto [1]. The notion of R-quadratic Finsler metrics was introduced by Shen, which can be considered as a generalization of R-flat metrics.

Every Berwald metric is a trivially R-quadratic. A Finsler metric F is called a Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratic in $y \in T_xM$ for any $x \in M$. Also, Berwald metrics belongs to the class of generalized Berwald metrics. A Finsler metric F on a manifold M is called a generalized Berwald metric if there exists a covariant derivative ∇ on M such that the parallel translations induced by ∇ preserve the Finsler function F [11][16]. In this case, (M, F) is called a generalized Berwald manifold. If ∇ is also torsion-free, then F reduces to a Berwald metric. Thus, we get the following

$$\{\text{Berwald metrics}\} \subseteq \{\text{R-quadratic metrics}\} \cap \{\text{generalized Berwald metrics}\}.$$

There is another quantity that is close to the Berwald metrics, namely, S-curvature. The S-curvature is constructed by Shen for given comparison theorems on Finsler manifolds [9]. A natural problem is to study and characterize Finsler metrics of vanishing S-curvature. It is known that some of Randers metrics are of vanishing S-curvature [7][14]. This is one of our motivations to consider Finsler metrics with vanishing S-curvature. Shen proved the following:

Theorem A. ([9] Shen Theorem) Every Berwald metric satisfies $\mathbf{S} = 0$.

Very soon, Tayebi-Rafie Rad generalized Shen theorem and proved that every isotropic Berwald metric has isotropic S-curvature [14]. However, in [3], Bao-Shen found a class of non-Berwaldian Randers metrics with vanishing S-curvature. Thus the converse of Shen's theorem is not true, generally. A natural question arises:

Question. *Under which conditions the converse of Shen's Theorem holds?*

To find some solutions for the above question, one can consider the class of (α, β)-metrics. An (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a positive-definite Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. The simplest (α, β)-metrics are the Randers metrics $F = \alpha + \beta$ which were discovered by G. Randers when he studied 4-dimensional general relativity. In [12], Tayebi-Eslami characterized the class of two-dimensional generalized Berwald (α, β)-metrics with vanishing S-curvature and prove the following.

Theorem B. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a two-dimensional generalized Berwald (α, β)-metric on a connected and orientable manifold M. Suppose that F has vanishing S-curvature and $\phi'(0) \neq 0$. Then one of the following holds:

- **:** (i) If F is a regular metric, then it reduces to a locally Minkowskian metric;
- **:** (ii) If F is an almost regular metric that is not locally Minkowskian, then ϕ is given by

$$\phi = c\exp\left[\int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}}dt\right], \tag{1.1}$$

where $c > 0$, $q > 0$, and k are real constants, and β satisfies

$$r_{ij} = 0, \qquad s_i = 0. \tag{1.2}$$

In this case, F is neither a Berwald nor Landsberg nor a Douglas metric.

Here, we consider generalized Berwald (α, β)-metric which are R-quadratic, and prove the following.

Theorem 1.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a regular generalized Berwald (α, β)-metric on a manifold M such that $\phi'(0) \neq 0$. Suppose that F is a R-quadratic. Then, F has vanishing S-curvature $\mathbf{S} = 0$ if and only if it is a Berwald metric $\mathbf{B} = 0$.*

In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively [13].

2. Preliminary

A Finsler metric on a manifold M is a nonnegative function F on TM having the following properties

(a) F is C^∞ on $TM_0 := TM \setminus \{0\}$;
(b) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$;
(c) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\Big[F^2(y + su + tv)\Big]\Big|_{s,t=0}, \qquad u, v \in T_xM.$$

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i(x, y)$ are local functions on TM_0 satisfying

$$G^i(x, \lambda y) = \lambda^2 G^i(x, y) \quad \lambda > 0.$$

$\mathbf{G}$ is called the associated spray to (M, F). The projection of an integral curve of G is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$. A Finsler metric F is called a Berwald metric if G^i are quadratic in $y \in T_xM$ for any $x \in M$ or equivalently the following Berwald curvature is vanishing.

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

For a non-zero vector $y \in T_xM_0$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by $R_y(u) := R^i{}_k(y)u^k \frac{\partial}{\partial x^i}$, where

$$R^i{}_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $R := \{R_y\}_{y \in TM_0}$ is called the Riemann curvature.

There are many Finsler metrics whose Riemann curvature in every direction is quadratic. A Finsler metric F is said to be R-quadratic if R_y is quadratic in $y \in T_xM$ at each point $x \in M$. Put

$$R_j{}^i{}_{kl}(y) := \frac{1}{3}\frac{\partial}{\partial y^j}\Big[\frac{\partial R^i{}_k}{\partial y^l} - \frac{\partial R^i{}_l}{\partial y^k}\Big].$$

$R_j{}^i{}_{kl}$ are the coefficients of the h-curvature of the Berwald connection, which are also denoted by $H_j{}^i{}_{kl}$ in literatures. We have

$$R^i{}_k(y) = y^j R_j{}^i{}_{kl}(y) y^l.$$

Thus $R^i{}_k(y)$ is quadratic in $y \in T_xM$ if and only if $R_j{}^i{}_{kl}(y)$ are functions of x only.

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}\big(B^n(1)\big)}{\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \ \Big|\ F\Big(y^i\frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}}.$$

In general, the local scalar function $\sigma_F(x)$ can not be expressed in terms of elementary functions, even F is locally expressed by elementary functions.

Let $G^i(x,y)$ denote the geodesic coefficients of F in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i\frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big].$$

where $\mathbf{y} = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric [7]. There are many non-Berwald metrics satisfying $\mathbf{S} = 0$ [3].

Given a Riemannian metric α, a 1-form β on a manifold M, and a C^∞ function $\phi = \phi(s)$ on $[-b_o, b_o]$, where $b_o := \sup_{x\in M}\|\beta\|_x$, one can define a function on TM by

$$F := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}.$$

If ϕ and b_o satisfy (2.1) and (2.2) below, then F is a Finsler metric on M. Finsler metrics in this form are called (α,β)-metrics. Randers metrics are special (α,β)-metrics.

Now we consider (α,β)-metrics. Let $\alpha = \sqrt{a_{ij}y^iy^j}$ be a Riemannian metric and $\beta = b_iy^i$ a 1-form on a manifod M. Let

$$\|\beta\|_x := \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

For a C^∞ function $\phi = \phi(s)$ on $[-b_o, b_o]$, where $b_o = \sup_{x\in M}\|\beta\|_x$, define

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha}.$$

By a direct computation, we obtain

$$g_{ij} = \rho a_{ij} + \rho_0 b_ib_j - \rho_1(b_i\alpha_j + b_j\alpha_i) + s\rho_1\alpha_i\alpha_j,$$

where $\alpha_i := a_{ij}y^j/\alpha$, and

$$\rho := \phi(\phi - s\phi'),$$
$$\rho_0 := \phi\phi'' + \phi'\phi',$$
$$\rho_1 := s(\phi\phi'' + \phi'\phi') - \phi\phi'.$$

By further computation, one obtains

$$\det(g_{ij}) = \phi^{n+1}\left(\phi - s\phi'\right)^{n-2}\Big[(\phi - s\phi') + (\|\beta\|_x^2 - s^2)\phi''\Big]\det(a_{ij}).$$

Using the continuity, one can easily show that

Lemma 2.1. *Let $b_o > 0$. $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric on M for any pair $\{\alpha, \beta\}$ with $\sup_{x\in M}\|\beta\|_x \le b_o$ if and only if $\phi = \phi(s)$ satisfies the following conditions:*

$$\phi(s) > 0, \qquad (|s| \le b_o) \tag{2.1}$$

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad (|s| \le b \le b_o). \tag{2.2}$$

Let

$$r_{ij} := \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big), \qquad s_{ij} := \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big).$$
$$r_j := b^i r_{ij}, \qquad s_j := b^i s_{ij}.$$

Let $r_{i0} := r_{ij}y^j$, $s_{i0} := s_{ij}y^j$, $r_0 := r_jy^j$ and $s_0 := s_jy^j$. Suppose that $G^i = G^i(x,y)$ and $\bar{G}^i = \bar{G}^i(x,y)$ denote the coefficients of F and α respectively in the same coordinate system. By definition, we obtain the following identity

$$G^i = \bar{G}^i + Py^i + Q^i,$$

where

$$\begin{aligned}
P &= \alpha^{-1}\Theta\Big[r_{00} - 2Q\alpha s_0\Big],\\
Q^i &= \alpha Q s^i{}_0 + \Psi\Big[r_{00} - 2Q\alpha s_0\Big]b^i,\\
Q &= \frac{\phi'}{\phi - s\phi'},\\
\Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)},\\
\Psi &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.
\end{aligned}$$

Clearly, if β is parallel with respect to α ($r_{ij} = 0$ and $s_{ij} = 0$), then $P = 0$ and $Q^i = 0$. In this case, $G^i = \bar{G}^i$ are quadratic in y, and F is a Berwald metric.

3. Proof of Theorem 1.1

In this section, we will prove a generalized version of Theorem 1.1. Indeed, we study

Theorem 3.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a regular generalized Berwald (α,β)-metric on an n-dimensional manifold M such that $\phi'(0) \neq 0$. Then F is a R-quadratic metric with isotropic S-curvature $\mathbf{S} = (n+1)cF$ if and only if it is a Berwald metric, where $c = c(x)$ is a scalar function on M.*

To prove Theorem 3.1, we need the following key lemma.

Lemma 3.2. ([15]) An (α,β)-metric satisfying $\phi'(0)\neq 0$ is a generalized Berwald manifold if and only if β has constant length with respect to α.

A Finsler metric F on an n-dimensional manifold M is called of isotropic S-curvature, if $\mathbf{S}=(n+1)cF$, where $c=c(x)$ is a scalar function on M. In [5], Cheng-Shen characterized (α,β)-metrics with isotropic S-curvature on a manifold M of dimension $n\geq 3$. Soon, they found that their result holds for the class of (α,β)-metrics with constant length one-forms, only. In [12], we give a new characterization of the class of generalized Berwald metrics with vanishing S-curvature and prove the following.

Lemma 3.3. ([12]) Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be a generalized Berwald (α,β)-metric on an n-dimensional manifold M. Suppose that $\phi'(0)\neq 0$. Then $\mathbf{S}=0$ if and only if β is a Killing form with constant length, namely

$$r_{ij}=0,\qquad s_j=0. \tag{3.1}$$

First, we remark the following well-known Bianchi identities.

Lemma 3.4. ([8]) For the Berwald connection, the following Bianchi identities hold:

$$R^i{}_{jkl|m}+R^i{}_{jlm|k}+R^i{}_{jmk|l}=B^i{}_{jku}R^u{}_{lm}+B^i{}_{jlu}R^u{}_{km}+B^i{}_{klu}R^u{}_{jm} \tag{3.2}$$

$$B^i{}_{jml|k}-B^i{}_{jkm|l}=R^i{}_{jkl,m} \tag{3.3}$$

$$B^i{}_{jkl,m}=B^i{}_{jkm,l}. \tag{3.4}$$

Now, we study the Berwald curvature of generalized Berwald (α,β)-metrics and prove the following.

Lemma 3.5. *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be a generalized Berwald (α,β)-metric on manifold M such that $\phi'(0)\neq 0$. Suppose that F has vanishing S-curvature. Then, the following hold*

$$b_mB^m{}_{jkl}=0, \tag{3.5}$$

where $b_m:=b_m(x)$ are the components of 1-form $\beta=b_i(x)y^i$.

Proof. The spray coefficients of an (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, are given by

$$G^i=\bar{G}^i+\alpha Qs^i{}_0+\frac{1}{\alpha}\Big(r_{00}-2Q\alpha s_0\Big)\Big(\Theta y^i+\alpha\Psi b^i\Big), \tag{3.6}$$

where $s^i_j:=a^{ih}s_{hj}$, $s^i_0:=s_iy^i$, $r_{00}=r_{ij}y^iy^j$ and

$$\Theta=\frac{Q-sQ'}{2\Delta},\qquad \Psi=\frac{Q'}{2\Delta}.$$

According to the assumption, F has vanishing S curvature. Putting (3.1) in (3.6) gives us

$$G^i=\bar{G}^i+\alpha Qs^i{}_0. \tag{3.7}$$

Multiplying (3.7) with b_i yields

$$b_iG^i=b_i\bar{G}^i. \tag{3.8}$$

The following hold

$$\frac{\partial^3 \bar{G}^i}{\partial y^j \partial y^k \partial y^l} = 0, \quad \frac{\partial b_i}{\partial y^j} = 0. \tag{3.9}$$

Then, taking three vertical derivation of (3.8) and using (3.9) gives us (3.5). □

In [4], Cheng consider regular (α, β)-metrics with isotropic S-curvature and prove the following.

Theorem C. ([4]) *A regular (α, β)-metric $F := \alpha\phi(\beta/\alpha)$, of non-Randers type on an n-dimensional manifold M is of isotropic S-curvature, $\mathbf{S} = (n+1)\sigma F$, if and only if β satisfies $r_{ij} = 0$ and $s_j = 0$. In this case, $\mathbf{S} = 0$, regardless of the choice of a particular $\phi = \phi(s)$.*

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1: By assumption, F is a regular (α, β)-metric. Then, by Theorem 3, the relations (3.1) hold. Taking a horizontal derivation of (3.5) implies that

$$b_m B^m_{\ jkl|s} = -b_{m|s} B^m_{\ jkl}. \tag{3.10}$$

By assumption F is R-quadratic metric. Thus

$$R^i_{\ jkl,m} = 0. \tag{3.11}$$

Then, by (3.3) and (3.11) we get

$$B^i_{\ jkl|m} - B^i_{\ jkm|l} = 0. \tag{3.12}$$

Multiplying (3.12) with b_i yields

$$b_i B^i_{\ jkl|m} = b_i B^i_{\ jkm|l}. \tag{3.13}$$

Comparing (3.10) and (3.13) gives us

$$b_{i|m} B^i_{\ jkl} = b_{i|l} B^i_{\ jkm}. \tag{3.14}$$

The following holds

$$b_{i|m} = r_{im} + s_{im}, \tag{3.15}$$

which by considering $r_{ij} = 0$, it reduces to following

$$b_{i|m} = s_{im}. \tag{3.16}$$

Multiplying (3.14) with y^l and considering (3.16) we obtain

$$s_{i0} B^i_{\ jkm} = 0. \tag{3.17}$$

Taking three times vertical derivation of (3.7) gives us the following

$$B^i_{\ jkl} = \left[\alpha Q s^i_{\ 0}\right]_{y^j y^k y^l}. \tag{3.18}$$

By (3.17) and (3.18) we have

$$s_{i0} \left[\alpha Q s^i_{\ 0}\right]_{y^j y^k y^l} = 0. \tag{3.19}$$

(3.19) is equal to

$$\Big[\alpha\ Q\Big]_{y^jy^ky^l}s_{i0}s^i{}_0+\Big[\alpha\ Q\Big]_{y^jy^k}s_{i0}s^i{}_l+\Big[\alpha\ Q\Big]_{y^jy^l}s_{i0}s^i{}_k+\Big[\alpha\ Q\Big]_{y^ky^l}s_{i0}s^i{}_j=0. \tag{3.20}$$

According to (3.1), $s^i = s_i = 0$. Then, multiplying (3.20) with $b^jb^kb^l$ yields

$$\Big[b^jb^kb^l[\alpha Q]_{y^jy^ky^l}\Big]s_{i0}s^i{}_0=0. \tag{3.21}$$

By (3.21), we get

$$s_{i0}s^i{}_0=0. \tag{3.22}$$

Since α is a positive-definite Riemannian metric, then by (3.22) it follows that

$$s^i{}_j=0. \tag{3.23}$$

(3.23) means that β is a closed 1-form, and by considering (3.1), we conclude that β is a parallel 1-form. In this case, F reduces to a Berwald metric. $\square$

Finally, we conclude the following.

Corollary 3.6. *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be a regular generalized Berwald (α,β)-metric on a 2-dimensional manifold M such that $\phi'(0)\neq 0$. Suppose that F is a R-quadratic. Then, F has vanishing S-curvature $\mathbf{S}=0$ if and only if it is a locally Minkowskian metric.*

Proof. The well-known Szabó rigidity theorem says that every 2-dimensional Berwald surface is either locally Minkowskian or Riemannian. On the other hand, every Riemannian metric satisfies $\phi(s)=constant$, and then $\phi'(0)=0$. By the assumption and using Theorem 1.1, it follows that F must be a locally Minkowskian metric. $\square$

References

[1] S. Bácsó and M. Matsumoto, *Finsler spaces with h-curvature tensor H dependent on position alone*, Publ. Math. Debrecen, **55**(1999), 199-210.

[2] D. Bao, S.S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer, 2000.

[3] D. Bao and Z. Shen, *Finsler metrics of constant positive curvature on the Lie group S^3*, J. London. Math. Soc. **66**(2002), 453-467.

[4] X. Cheng, *The (α,β)-metrics of scalar flag curvature*, Differ. Geom. Appl, **35**(2014), 361-369.

[5] X. Cheng and Z. Shen, *A class of Finsler metrics with isotropic S-curvature*, Israel J. Math. **169**(2009), 317-340.

[6] X. Cheng, Z. Shen and Y. Zhou, *On a class of locally dually flat Finsler metrics*, Int. J. Math. **21**(11) (2010), 1-13.

[7] Z. Shen, *Riemann-Finsler geometry with applications to information geometry*, Chin. Ann. Math. **27**(2006), 73-94.

[8] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.

[9] Z. Shen, *Volume comparison and its applications in Riemann-Finsler geometry*, Adv. Math. **128**(1997), 306-328.

[10] Z. Shen, *On R-quadratic Finsler spaces*, Publ. Math. Debrecen. **58**(2001), 263-274.

[11] A. Tayebi and M. Barzegari, *Generalized Berwald spaces with (α,β)-metrics*, Indagationes. Math. (N.S.). **27**(2016), 670-683.

[12] A. Tayebi and F. Eslami, *On a class of generalized Berwald manifolds*, Publ. Math. Debrecen, **105**(2024), 379-402.

[13] A. Tayebi and B. Najafi, *Shen's processes on Finslerian connection theorey*, Bull. Iran. Math. Soc. **36**(2010), 2198-2204.

[14] A. Tayebi and M. Rafie. Rad, *S-curvature of isotropic Berwald metrics*, Sci. China. Series A: Math. **51**(2008), 2198-2204.

[15] C. Vincze, *On a special type of generalized Berwald manifolds: semi-symmetric linear connections preserving the Finslerian length of tangent vectors*, European Journal of Math. **3**(2017), 1098-1171.

[16] C. Vincze, *On generalized Berwald manifolds with semi-symmetric compatible linear connections*, Publ. Math. Debrecen. **83**(2013), 741-755.

[17] M. Zohrehvand, *On the existence of 3-dimensional Berwald manifolds*, J. Finsler Geom. Appl. **2**(1) (2021), 86-95.