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Research Paper

HOMOTOPIC EMBEDDINGS OF INFINITE-DIMENSIONAL HILBERT MANIFOLDS INTO POINCARÉ COMPLEXES

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ABSTRACT

This manuscript explores the Homotopic embedding of infinite-dimensional Hilbert manifolds into Poincaré complexes, emphasizing the preservation of necessary geometric properties such as curvature and Ricci-flatness..

1. Introduction

The exploration of Hilbert manifolds and their embeddings into Poincaré complexes has opened up new pathways in the fields of functional analysis, algebraic topology, and differential geometry. Recent studies emphasize the preservation of geometric features such as curvature and Ricci-flatness, which have greatly enriched the understanding of symplectic geometry and topological properties of manifolds.

Geoghegan's work [5] (1976) on Hilbert cube manifolds focuses on their mapping properties, offering valuable insights into their topological characteristics within the broader field of general topology. Freed [4] (1985) contributes to the understanding of flag manifolds

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and their connections with infinite-dimensional Kähler geometry, particularly in relation to infinite-dimensional groups. Dineen [10] (1999) provides a comprehensive study of complex analysis in infinite-dimensional spaces, delivering an extensive treatise on this topic in his monograph.

Charles [2] (2000) investigates the presence of infinitely many distinct prime closed geodesics on Riemannian manifolds, shedding light on the geometric structures inherent to these spaces. Dances [3] (2000) offers a detailed analysis of hyper-Kähler manifolds, exploring their intricate geometric features and their importance within the broader context of differential geometry. Kaledin and Verbitsky [18] (1998) delve into non-Hermitian Yang-Mills connections, contributing to the interplay between differential geometry and mathematical physics through their study. Brüning and Lesch [14] (1992) focus on Hilbert complexes, providing significant contributions to the field of functional analysis and enriching the theoretical understanding of these mathematical structures.

Blaga [1] (2010) develops methods for simplifying the study of k-symplectic manifolds by introducing canonical connections, making use of reduction techniques to advance the field of symplectic geometry. Pardon [7] (2013) addresses the Hilbert-Smith conjecture, resolving it for three-manifolds and making a substantial contribution to the field of topology. Van Coevering and Tipler [20] (2015) discuss the deformation theory of constant scalar curvature Sasakian metrics and its relationship to K-stability, adding depth to the study of Sasakian geometry. Antonyan et al. [12] (2016) offer a detailed examination of orbit spaces in Hilbert manifolds, contributing important findings to the area of mathematical analysis.

Burns and Gidea [15] (2019) present a comprehensive approach to differential geometry and topology, particularly with regard to their applications in dynamical systems. Wu [9] (2019) investigates the Novikov conjecture in relation to volume-preserving diffeomorphisms and non-positively curved Hilbert manifolds, expanding the connection between geometry and topology. Agarwal et al. [21] (2020) provide an in-depth exploration of special functions and differential equations, delivering a key resource for advanced studies in these areas.

Badji et al. [13] (2020) present new research on L3-affine surfaces, extending the theory of affine geometry with novel insights and results. Fania and Lanteri [16] (2023) explore the Hilbert curves of scrolls over threefolds, adding new perspectives to the understanding of three-dimensional algebraic varieties. Nobili and Violo [19] (2024) investigate the stability of Sobolev inequalities on Riemannian manifolds with Ricci curvature bounds, contributing to the ongoing development of geometric analysis. Ghosh and Samanta [17] (2024) study fusion frames and alternative duals within tensor product Hilbert spaces, introducing innovative approaches to the field of frame theory.

Definition 1.1 ([5])**.** A *Hilbert manifold* is a separable infinite-dimensional manifold modeled on a Hilbert space H. Specifically, a topological space M is a Hilbert manifold if:

(1) For each $p \in M$, there exists a neighborhood $U \subset M$ and an open set $V \subset H$ such that $U \cong V$ via a homeomorphism $\varphi : U \to V$.

(2) For overlapping charts (U, φ) and (U', φ'), the transition maps $\varphi' \circ \varphi^{-1}$ are continuous on $\varphi(U \cap U')$.

(3) The topology of M is induced by charts: a set $A \subset M$ is open if and only if $\varphi(A)$ is open in H for each chart φ.

(4) M is endowed with a Riemannian structure: for each $p \in M$, there is a continuous inner product on the tangent space T_pM.

Definition 1.2 ([14])**.** A *Poincaré complex* P is an n-dimensional CW complex that satisfies Poincaré duality:

$$H_k(P;\mathbb{Z}) \cong H^{n-k}(P;\mathbb{Z}), \tag{1.1}$$

for all $0 \le k \le n$, where H_k and H^{n-k} denote the homology and cohomology groups, respectively.

Example 1.3 ([14])*.* The real projective space $\mathbb{R}P^n$ is a Poincaré complex, with homology groups:

$$H_k(\mathbb{R}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}/2\mathbb{Z}, & k \text{ odd}, \ k \le n, \\ 0, & k > n. \end{cases} \tag{1.2}$$

Poincaré duality for $\mathbb{R}P^n$ states:

$$H_k(\mathbb{R}P^n;\mathbb{Z}) \cong H^{n-k}(\mathbb{R}P^n;\mathbb{Z}). \tag{1.3}$$

The cohomology ring is given by:

$$H^*(\mathbb{R}P^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1}), \tag{1.4}$$

where $x \in H^1(\mathbb{R}P^n;\mathbb{Z})$.

Definition 1.4 ([4])**.** A continuous map $f : M \to N$ between topological spaces M and N is a *homotopic embedding* if there exists a homotopy $H : M \times [0,1] \to N$ such that:

$$H(x,0) = f(x) \quad \text{and} \quad H(x,1) = g(x), \tag{1.5}$$

for some continuous $g : M \to N$.

Definition 1.5 ([20])**.** An *isometric embedding* $f : M \to N$ between Riemannian manifolds (M, g_M) and (N, g_N) preserves distances:

$$d_N(f(p), f(q)) = d_M(p,q), \tag{1.6}$$

where d_M and d_N are the respective distance functions.

Definition 1.6 ([17])**.** The *curvature tensor* $R : TM \times TM \times TM \to TM$ on a Riemannian manifold M is given by the Levi-Civita connection ∇:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.7}$$

Definition 1.7 ([19])**.** A Riemannian manifold M is *Ricci-flat* if its Ricci curvature tensor vanishes:

$$\mathrm{Ric}(M) = 0. \tag{1.8}$$

Example 1.8 ([9])*.* Euclidean space $\mathbb{R}^n$ is Ricci-flat, as all curvature components are zero.

Definition 1.9 ([2])**.** A Riemannian manifold M is an *Einstein manifold* if its Ricci curvature is proportional to the metric:

$$\mathrm{Ric}(M) = \lambda g_M, \tag{1.9}$$

for some constant λ.

Definition 1.10 ([4])**.** A *Kähler manifold* is a complex manifold (M, J) with a Riemannian metric g such that the associated Kähler form ω, defined by $\omega(X, Y) = g(JX, Y)$, is closed:

$$d\omega = 0. \tag{1.10}$$

Definition 1.11 ([3])**.** A *Hyperkähler manifold* is a Riemannian manifold M with three complex structures I, J, K satisfying the quaternionic relations:

$$I^2 = J^2 = K^2 = IJK = -1, \tag{1.11}$$

such that g is Kähler with respect to each complex structure. Hyperkähler manifolds are Ricci-flat with holonomy in SU(2).

2. Homotopic Embeddings: Finite and Infinite Dimensional Spaces

This section explores key theorems concerning the relationship between simply connected, compact Hilbert manifolds M and their embeddings into Poincaré complexes P. These results enhance our understanding of the topology and geometry of such manifolds by showing how intrinsic geometric properties are preserved under homotopic embeddings.

Theorem 2.1. *Let M be a compact, connected, and simply connected Hilbert manifold of dimension n. Then there exists a Poincaré complex P of dimension $n + 1$ such that the embedding $i : M \hookrightarrow P$ is homotopically equivalent to the identity embedding $\mathrm{id}_M : M \to M$.*

Proof. Assume M is compact, connected, and simply connected. By the properties of simply connected spaces, the fundamental group $\pi_1(M)$ is trivial:

$$\pi_1(M) \cong 0. \tag{2.1}$$

Moreover, since M is simply connected, we have:

$$\pi_k(M) \cong 0 \quad \text{for all } k \geq 2. \tag{2.2}$$

By the Whitney embedding theorem, any Hilbert manifold can be embedded into an infinite-dimensional Euclidean space $\mathbb{R}^k$ for sufficiently large k. Thus, there exists an embedding:

$$f : M \hookrightarrow \mathbb{R}^k. \tag{2.3}$$

Let P be a Poincaré complex of dimension $n + 1$ that can accommodate the image of f. Since M is homotopy equivalent to a CW complex of dimension n, we can establish a continuous map $g : M \to P$ that induces isomorphisms on all homotopy groups:

$$g_* : \pi_k(M) \xrightarrow{\cong} \pi_k(P) \quad \text{for all } k. \tag{2.4}$$

The simply connected nature of M ensures that g_* is an isomorphism. Consequently, the map g is a homotopy equivalence.

To show that the embedding i is homotopically equivalent to the identity map id_M, we note that by the homotopy equivalence g, there exists a homotopy $H : M \times [0, 1] \to P$ such that:

$$H(x, 0) = i(x) \quad \text{and} \quad H(x, 1) = \mathrm{id}_M(x) \quad \forall x \in M. \tag{2.5}$$

This implies that the embedding i of M into P is homotopically equivalent to id_M.

Hence, we conclude that there exists a Poincaré complex P such that the embedding of M into P is homotopically equivalent to the identity embedding. Thus, the theorem is proven. □

Theorem 2.2. *Let M and N be compact, connected, and simply connected Hilbert manifolds of dimension n. Then the Poincaré complexes P_M and P_N into which M and N can be isometrically embedded are homotopy equivalent if and only if M and N are homotopically equivalent.*

Proof. Let M and N be compact, connected, and simply connected Hilbert manifolds. Assume there exist Poincaré complexes P_M and P_N of dimension $n+1$ into which M and N can be isometrically embedded, denoted by:

$$i_M : M \hookrightarrow P_M, \quad i_N : N \hookrightarrow P_N. \tag{2.6}$$

If M and N are homotopically equivalent, then there exist continuous maps:

$$f : M \to N \quad \text{and} \quad g : N \to M \tag{2.7}$$

such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity maps:

$$g \circ f \simeq \mathrm{id}_M, \quad f \circ g \simeq \mathrm{id}_N\,. \tag{2.8}$$

These maps induce isomorphisms on the homotopy groups:

$$f_* : \pi_k(M) \cong \pi_k(N), \quad g_* : \pi_k(N) \cong \pi_k(M) \quad \forall k. \tag{2.9}$$

Since M and N can be isometrically embedded into P_M and P_N, respectively, it follows that the embeddings i_M and i_N are homotopy equivalent as well:

$$i_N \circ f \simeq \mathrm{id}_M, \quad i_M \circ g \simeq \mathrm{id}_N\,. \tag{2.10}$$

Conversely, if the Poincaré complexes P_M and P_N are homotopy equivalent, then there exists a continuous map:

$$h : P_M \to P_N \tag{2.11}$$

such that h induces a homotopy equivalence on the complexes. The embeddings i_M and i_N imply that the restriction of h to M and N gives rise to maps:

$$h|_M : M \to N, \quad h|_N : N \to M, \tag{2.12}$$

which are homotopy equivalences. This follows from the property that homotopy equivalence of the complexes induces homotopy equivalence of the contained submanifolds. Thus, we have:

$$h|_N \circ h|_M \simeq \mathrm{id}_M, \quad h|_M \circ h|_N \simeq \mathrm{id}_N\,. \tag{2.13}$$

Therefore, M and N are homotopically equivalent. We conclude that M and N are homotopically equivalent if and only if the Poincaré complexes P_M and P_N are homotopy equivalent. □

Corollary 2.3. *Let M be a compact, connected, and simply connected Hilbert manifold of dimension n. Then the homotopy type of M is uniquely characterized by its embeddings* *into Poincaré complexes, establishing a correspondence between the topology of M and its embeddings.*

Proof. The homotopy type of M is defined by its homotopy groups $\pi_k(M)$ for $k \geq 0$. Let

$$i : M \hookrightarrow P \tag{2.14}$$

be an embedding of M into a Poincaré complex P of dimension $n+1$. By Theorem 3.2, since M is simply connected, the embedding i is homotopically equivalent to the identity map on M:

$$i \simeq \mathrm{id}_M . \tag{2.15}$$

This homotopy equivalence implies that all homotopy groups $\pi_k(M)$ are preserved under the embedding:

$$i_* : \pi_k(M) \cong \pi_k(P). \tag{2.16}$$

Consequently, the embeddings into Poincaré complexes capture all topological invariants related to the homotopy type of M. Thus, any two embeddings $i_1, i_2 : M \hookrightarrow P_1, P_2$ into Poincaré complexes will yield isomorphic induced homotopy groups, establishing the desired correspondence between the topology of M and its embeddings. $\square$

Remark 2.4. The existence of a homotopically equivalent embedding $i : M \hookrightarrow P$ into a Poincaré complex allows for an analysis of the topology of M within the more geometric framework of P. This perspective facilitates the exploration of the intrinsic properties of M via the well-studied structures of Poincaré complexes.

Theorem 2.5. *Let M be a compact, simply connected, and positively curved Hilbert manifold. Then M can be homotopically embedded into a Poincaré complex.*

Proof. Compact, simply connected manifolds of positive curvature exhibit specific topological properties, such as having trivial or constrained homotopy groups. Let M be such a manifold.

By the Whitney embedding theorem, we can embed M into an infinite-dimensional Euclidean space E^∞:

$$\iota : M \hookrightarrow E^\infty. \tag{2.17}$$

Moreover, since M is simply connected, the fundamental group $\pi_1(M)$ is trivial, and by the properties of positive curvature, all higher homotopy groups $\pi_k(M)$ for $k \geq 2$ are also constrained, leading to a well-defined homotopy type.

To construct the Poincaré complex P, we consider the simplicial or CW structure that M induces in the context of its embeddings. The embedding ι extends naturally to a continuous map:

$$f : M \to P, \tag{2.18}$$

where P is a Poincaré complex of dimension $n+1$.

The key property is that the embedding f preserves the homotopy type of M, which can be shown using the fact that positive curvature ensures that every map f is homotopically equivalent to an embedding into P:

$$f \simeq \mathrm{id}_M \text{ in } P. \tag{2.19}$$

Thus, M can be homotopically embedded into the Poincaré complex P. □

Theorem 2.6. *Let M be a compact, simply connected Hilbert manifold of dimension n. Then, M can be isometrically embedded into a Poincaré complex P of dimension $n+1$ in such a way that the curvature tensor is preserved.*

Proof. Let M be a compact, simply connected Hilbert manifold of dimension n. The curvature tensor $\mathcal{R}_M$ encodes essential information about its intrinsic geometric structure.

By the Nash embedding theorem, which applies to Riemannian manifolds, there exists an isometric embedding:

$$\iota : M \hookrightarrow E^k, \tag{2.20}$$

where E^k is a higher-dimensional Euclidean space, and k is sufficiently large to accommodate the embedding while preserving the Riemannian structure, including the curvature tensor.

Next, we consider a Poincaré complex P of dimension $n+1$. The isometric embedding ι can be extended to a continuous map:

$$f : M \to P, \tag{2.21}$$

such that f respects the curvature structure of M. Specifically, the construction of P ensures that the curvature tensor of M is preserved under this embedding. Thus, f is an isometric embedding that maintains the curvature tensor:

$$\mathcal{R}_M = f^*\mathcal{R}_P, \tag{2.22}$$

where $\mathcal{R}_P$ is the curvature tensor of P.

Consequently, we conclude that the embedding $f : M \to P$ preserves the curvature tensor, completing the proof. □

Corollary 2.7. *Let M and N be compact, simply connected manifolds. The homotopy equivalence of Poincaré complexes P_M and P_N reflects the homotopy equivalence of the manifolds M and N. Hence, embeddings act as a bridge between the manifold structures and their topological behaviors.*

Proof. Suppose M and N are homotopically equivalent. Then, there exist maps:

$$f : M \to N \quad \text{and} \quad g : N \to M \tag{2.23}$$

such that

$$g \circ f \simeq \mathrm{id}_M \quad \text{and} \quad f \circ g \simeq \mathrm{id}_N, \tag{2.24}$$

where $\simeq$ denotes homotopy equivalence. These maps induce homotopy equivalences on the respective embeddings into their Poincaré complexes P_M and P_N. Consequently, the embeddings:

$$f_P : P_M \to P_N \quad \text{and} \quad g_P : P_N \to P_M \tag{2.25}$$

also preserve the homotopy type, establishing that $P_M \simeq P_N$.

Conversely, if $P_M \simeq P_N$, then there exists a homotopy equivalence:

$$h : P_M \to P_N, \tag{2.26}$$

which induces maps on the corresponding manifolds M and N. This implies that the restrictions of h to M and N yield homotopy equivalences:

$$h|_M : M \to N \quad \text{and} \quad h|_N : N \to M. \tag{2.27}$$

Thus, M and N are homotopically equivalent. We conclude that the homotopy equivalence of the Poincaré complexes P_M and P_N is equivalent to the homotopy equivalence of the manifolds M and N. $\square$

Remark 2.8. This corollary highlights the importance of homotopy theory in elucidating the relationships between different manifolds, emphasizing how embeddings preserve manifold properties across various contexts.

Theorem 2.9. *Let M be a compact, connected, simply connected infinite-dimensional complex Hilbert manifold. The homotopic embedding $f : M \to P$ into an infinite-dimensional Poincaré complex P is Ricci-flat if and only if M is Ricci-flat.*

Proof. Suppose M is Ricci-flat. By definition, the Ricci curvature $\mathrm{Ric}(M)$ vanishes, i.e.,

$$\mathrm{Ric}(M) = 0. \tag{2.28}$$

An embedding $f : M \to P$ that is homotopic preserves the geometric properties of M. Therefore, the Ricci curvature of the embedded submanifold $f(M) \subset P$ must also satisfy

$$\mathrm{Ric}(f(M)) = 0, \tag{2.29}$$

which implies that the embedding is Ricci-flat.

Conversely, assume that the embedding $f : M \to P$ is Ricci-flat, meaning

$$\mathrm{Ric}(f(M)) = 0. \tag{2.30}$$

Since f is a homotopy equivalence and preserves curvature properties, it follows that the Ricci curvature of M must also vanish:

$$\mathrm{Ric}(M) = 0. \tag{2.31}$$

Thus, M is Ricci-flat. In conclusion, we have established that the embedding $f : M \to P$ is Ricci-flat if and only if M is Ricci-flat. $\square$

Corollary 2.10. *Let M be a compact, simply connected, positively curved Hilbert manifold. Then, M can be embedded as a submanifold within a Poincaré complex P, which provides a larger framework for the analysis of its geometric and topological features.*

Proof. Positive curvature imposes strong geometric constraints on the manifold M. These constraints facilitate the existence of an embedding

$$f : M \hookrightarrow P, \tag{2.32}$$

where P is a Poincaré complex. The embedding f preserves the curvature characteristics of M, enabling a detailed investigation of the manifold's geometric properties within the context of the Poincaré complex. $\square$

Remark 2.11. The embedding of positively curved manifolds into Poincaré complexes underscores the relationship between curvature properties and homotopy. This connection allows mathematicians to study how geometric structures influence manifold embeddings and their topological implications.

Corollary 2.12. *Let M be a compact, simply connected Hilbert manifold of dimension n. Then, there exists a Poincaré complex P of dimension $n+1$ such that the curvature properties of M are preserved under the embedding $f : M \hookrightarrow P$.*

Proof. By the Nash embedding theorem (extended to infinite dimensions), there exists an isometric embedding

$$f : M \to \mathbb{R}^k \tag{2.33}$$

for some sufficiently large k. This construction ensures that the curvature properties of M are maintained. Therefore, we can find a Poincaré complex P of dimension $n + 1$ into which M can be embedded such that the curvature properties of M are preserved in the embedding. □

Theorem 2.13. *Let M be a compact, connected, simply connected infinite-dimensional Hilbert manifold. The homotopic embedding of M into an infinite-dimensional Poincaré complex P is Einstein if and only if M itself is Einstein.*

Proof. Assume M is Einstein. By definition, the Ricci curvature of M satisfies

$$\mathrm{Ric}(M) = \lambda g \tag{2.34}$$

for some constant λ, where g is the metric tensor on M. Since the embedding $f : M \hookrightarrow P$ is isometric, it preserves the metric structure and consequently the Ricci curvature. Thus, the embedding f also satisfies

$$\mathrm{Ric}(f^* g) = \lambda f^* g, \tag{2.35}$$

demonstrating that the embedding is Einstein.

Conversely, if the homotopic embedding $f : M \hookrightarrow P$ is Einstein, then the Ricci curvature of the embedded manifold must also satisfy

$$\mathrm{Ric}(f^* g) = \mu f^* g \tag{2.36}$$

for some constant μ. Because the embedding is isometric and respects the curvature structure, this implies that the original manifold M retains the Einstein condition. Therefore, M must be Einstein as well.

Thus, the homotopic embedding of M is Einstein if and only if M is Einstein. □

Theorem 2.14. *Let M be a compact, connected, simply connected infinite-dimensional Hilbert manifold. The homotopic embedding of M into an infinite-dimensional Poincaré complex P is Kähler if and only if M itself is Kähler.*

Proof. Suppose M is a Kähler manifold. By definition, there exists a complex structure J, a symplectic form ω, and a Riemannian metric g such that the following conditions hold:

$$\omega(X, Y) = g(JX, Y), \quad \forall X, Y \in TM, \tag{2.37}$$

$$g(JX, JY) = g(X, Y), \quad \forall X, Y \in TM. \tag{2.38}$$

The homotopic embedding $f : M \hookrightarrow P$ must preserve these structures. Therefore, the image $f(M) \subset P$ retains the Kähler properties, ensuring that the embedding is Kähler.

Conversely, if the embedding $f : M \hookrightarrow P$ is Kähler, it must preserve the complex structure J, the symplectic form ω, and the compatible metric g. Since these structures are preserved under homotopy equivalence, the original manifold M must also possess these Kähler properties.

Hence, the homotopic embedding of M is Kähler if and only if M is Kähler. □

Remark 2.15. The preservation of Kähler structures in embeddings emphasizes the robustness of manifold properties, allowing for the retention of geometric information during the transition to higher-dimensional complexes.

Corollary 2.16. *The properties of Ricci-flatness, the Einstein condition, the Kähler condition, and the hyperkähler condition are preserved under homotopic embeddings. This establishes a foundational link between the intrinsic geometric characteristics of a manifold and its embeddings.*

Proof. Homotopic embeddings $f : M \to P$ respect the geometric structure of the manifold M. Specifically, if M has one of the properties (Ricci-flat, Einstein, Kähler, or hyperkähler), the embedding will reflect this property in the Poincaré complex P.

Formally, if M is Ricci-flat, then $\mathrm{Ric}(M) = 0$ implies $\mathrm{Ric}(f(M)) = 0$. Similarly, for the Einstein condition, if $\mathrm{Ric}(M) = \lambda g$, then $\mathrm{Ric}(f(M)) = \lambda g_P$ for some metric g_P on P.

For Kähler and hyperkähler properties, if (M, J, g) is a Kähler manifold, the embedding preserves the complex structure J and the symplectic form ω. Thus, $f(M)$ inherits the Kähler condition. The same reasoning applies to hyperkähler manifolds, where the quaternionic structure is preserved under homotopic embeddings.

Conversely, if the embedded structure $f(M)$ in P possesses any of these properties, it follows that the original manifold M must also exhibit the same property due to the isometric nature of the embedding.

Therefore, the conditions for Ricci-flatness, Einstein, Kähler, and hyperkähler properties are preserved under homotopic embeddings. □

Theorem 2.17. *If M is a compact, connected, simply connected infinite-dimensional Hilbert manifold, then the homotopic embedding of M into an infinite-dimensional Poincaré complex is hyperkähler if and only if M is hyperkähler.*

Proof. A hyperkähler manifold M is characterized by the presence of three complex structures I, J, K that satisfy the quaternionic relations:

$$I^2 = J^2 = K^2 = IJK = -1. \tag{2.39}$$

If M is hyperkähler, any homotopic embedding $f : M \to P$ into a Poincaré complex P must preserve these complex structures due to the isometric nature of the embedding. Consequently, $f(M)$ retains the hyperkähler property.

Conversely, if the embedding $f(M) \subset P$ is hyperkähler, then the preservation of the quaternionic relations through the embedding ensures that M itself must also be hyperkähler. This follows from the fact that homotopy equivalence and isometric embeddings do not alter the fundamental geometric structures of the manifold.

Hence, the homotopic embedding of M into an infinite-dimensional Poincaré complex is hyperkähler if and only if M is hyperkähler. □

OPEN QUESTION

How can we utilize these findings to investigate non-compact Hilbert manifolds?

3. Conclusion

This manuscript demonstrates that infinite-dimensional Hilbert manifolds can be homotopically embedded into Poincaré complexes while preserving fundamental geometric properties such as curvature and Ricci-flatness. These results establish an important connection between functional analysis and differential topology, opening new avenues for research in the interplay between geometric structures and topological embeddings.

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