DOI: doi.org/10.22098/jhs.2025.16011.1052

Journal of Hyperstructures, 14 (2) (2025), 289-305

Journal Homepage: https://jhs.uma.ac.ir/

Research Paper

ADVANCED STUDIES IN PENTAGONAL CONTROLLED INTUITIONISTIC FUZZY METRIC SPACES WITH APPLICATIONS

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ARTICLE INFO

Article history:
Received: 14 October 2024
Accepted: 05 May 2025
Communicated by Nasrin Eghbali

Keywords:
Fuzzy Metric space
Pentagonal Controlled Intuitionistic Fuzzy Metric space
Fixed Point Theorems
Control Functions
Applications

MSC:
03E72, 54E35, 47H10

ABSTRACT

This paper extends the theory of Pentagonal Controlled Intuitionistic Fuzzy Metric Spaces (PCIFMS) by introducing new theorems and providing detailed proofs. We explore the properties of these spaces and their applications in various fields such as optimization, decision-making, and pattern recognition. The paper also introduces new control functions and discusses their implications in the context of fuzzy metric spaces.

1. INTRODUCTION

Fuzzy Set Theory, first introduced by Zadeh in 1965 [18], has significantly contributed to the understanding and modeling of uncertainty and vagueness in various fields of mathematics, engineering, and the social sciences. Fuzzy sets allow for partial membership of elements in a set, thus offering a more flexible approach than classical sets in representing imprecise information. Over the years, fuzzy logic has found numerous applications in fields such as decision-making, control systems, artificial intelligence, and optimization, making it a cornerstone of modern computational theories.

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The concept of Fuzzy Metric Spaces, developed by Michálek and Kramosil [4], provided a generalization of traditional metric spaces, where the distances between points are represented by fuzzy values rather than crisp values. This extension was crucial in dealing with problems where precise measurements or exact distances are not feasible. Further refinements and applications of fuzzy metrics were made by George and Veeramani [5], who developed more advanced models and methods for fuzzy metrics, such as the concept of fuzzy control and fuzzy continuity.

In recent years, the theory of Pentagonal Controlled Intuitionistic Fuzzy Metric Spaces (PCIFMS) has emerged as a promising framework for addressing more complex and sophisticated systems of uncertainty. The work of several researchers has contributed to the development of PCIFMS, which combines elements of both intuitionistic fuzzy sets and pentagonal control mechanisms. These spaces allow for the representation of a higher degree of uncertainty by considering not only the degree of membership and non-membership of elements but also an additional control mechanism that adjusts for more complex relationships.

The motivation behind introducing Pentagonal Controlled Intuitionistic Fuzzy Metric Spaces is to deal with problems where conventional fuzzy metric spaces are insufficient, particularly in scenarios where additional control variables are essential for accurate modeling. The current paper builds upon the existing foundation of fuzzy set theory, fuzzy metric spaces, and intuitionistic fuzzy sets to propose new fixed-point theorems for Pentagonal Controlled Intuitionistic Fuzzy Metric Spaces. These theorems have potential applications in diverse areas, such as optimization, decision-making, and control systems, where uncertainty and control parameters play a significant role.

Through this work, we aim to extend the fixed-point theory in fuzzy metric spaces, highlighting its applicability in real-world scenarios where traditional approaches may fall short. In particular, the results presented here provide new insights into the existence and uniqueness of fixed points under pentagonal control mechanisms, contributing to both theoretical advancements and practical applications.

2. PRELIMINARIES

This section provides the necessary definitions and preliminary results required for understanding the main contributions of this paper.

Definition 1. A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is commutative and associative;
(2) $*$ is continuous;
(3) $a * 1 = a$ for all $a \in [0,1]$;
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Definition 2. A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if it satisfies the following conditions:

(1) $\diamond$ is commutative and associative;
(2) $\diamond$ is continuous;
(3) $a \diamond 0 = a$ for all $a \in [0,1]$;
(4) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Definition 3. A three-tuple $(X, M, *)$ is said to be a Fuzzy Metric Space if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:

(1) (Fm-1) $M(x, y, 0) = 0$;
(2) (Fm-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(3) (Fm-3) $M(x, y, t) = M(y, x, t)$;
(4) (Fm-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(5) (Fm-5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous;
(6) (Fm-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Example 1. Let (X, d) be a metric space, $(a * b) = \min\{a, b\}$, and

$$M(x, y, t) = \frac{t}{t + d(x, y)} \quad \text{for all } x, y \in X \text{ and } t > 0.$$

Then $(X, M, *)$ is a fuzzy metric space, often referred to as the standard fuzzy metric space induced by (X, d).

Definition 4. A 5-tuple $(X, M, N, *, \diamond)$ is said to be an Intuitionistic Fuzzy Metric Space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:

(1) (IFm-1) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;
(2) (IFm-2) $M(x, y, 0) = 0$ for all $x, y \in X$;
(3) (IFm-3) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(4) (IFm-4) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(5) (IFm-5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(6) (IFm-6) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous for all $x, y \in X$;
(7) (IFm-7) $\lim_{t\to\infty} M(x, y, t) = 1$;
(8) (IFm-8) $N(x, y, 0) = 1$ for all $x, y \in X$;
(9) (IFm-9) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(10) (IFm-10) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(11) (IFm-11) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(12) (IFm-12) $N(x, y, \cdot) : [0, \infty) \to [0, 1]$ is right continuous for all $x, y \in X$;
(13) (IFm-13) $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and degree of non-nearness between x and y with respect to t, respectively.

Definition 5. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then:

(1) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if, for all $t > 0$ and $p > 0$,

$$\lim_{t\to\infty} M(x_{n+p}, x_n, t) = 1, \quad \lim_{t\to\infty} N(x_{n+p}, x_n, t) = 0.$$

(2) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$$\lim_{t\to\infty} M(x, y, t) = 1, \quad \lim_{t\to\infty} N(x, y, t) = 0.$$

Definition 6. Let X be a nonempty set. A 5-tuple $(X, M, N, *, \diamond)$ is said to be an Intuitionistic Fuzzy Triplet Control Metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, M and N are fuzzy sets on $X \times X \times [0, \infty)$, and $A, B, C : X \times X \to [1, \infty)$ are non-comparable and fulfill the following axioms for all $p, q, r, s \in X$, $p \neq r$, $r \neq s$, $s \neq q$, and $\alpha, \beta, \gamma \geq 0$:

(1) (i) $M(p, q, 0) = 0$;
(2) (ii) $M(p, q, \alpha) = 1$ implies $p = q$;
(3) (iii) $M(p, q, \alpha) = M(q, p, \alpha)$;
(4) (iv) $M(p, q, \alpha + \beta + \gamma) \geq M(p, r, \alpha/A(p, r)) * M(r, s, \beta/B(r, s)) * M(s, t, \gamma/C(s, q))$;
(5) (v) $M(p, q, \cdot) : [0, \infty) \to [0, 1]$ is left continuous and $\lim_{\alpha\to\infty} M(p, q, \alpha) = 1$;
(6) (vi) $N(p, q, 0) = 1$;
(7) (vii) $N(p, q, \alpha) = 0$ implies $p = q$;
(8) (viii) $N(p, q, \alpha) = N(q, p, \alpha)$;
(9) (ix) $N(p, q, \alpha + \beta + \gamma) \leq N(p, r, \alpha/A(p, r)) \diamond N(r, s, \beta/B(r, s)) \diamond N(s, t, \gamma/C(s, q))$;
(10) (x) $N(p, q, \cdot) : [0, \infty) \to [0, 1]$ is right continuous and $\lim_{\alpha\to\infty} N(p, q, \alpha) = 0$.

3. Main Results

This section introduces the definition of Pentagonal Controlled Intuitionistic Fuzzy Metric Space and presents new theorems with detailed proofs.

Definition 7. Let X be a nonempty set. A 5-tuple $(X, M, N, *, \diamond)$ is said to be a Pentagonal Controlled Intuitionistic Fuzzy Triplet Control Metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, M and N are fuzzy sets on $X \times X \times [0, \infty)$, and $A, B, C, D, E : X \times X \to [1, \infty)$ are non-comparable and fulfill the following axioms for all $p, q, r, s \in X$, $p \neq r$, $r \neq s$, $s \neq q$, and $\alpha, \beta, \gamma, \delta, \omega \geq 0$:

(1) (i) $M(p, q, 0) = 0$;
(2) (ii) $M(p, q, \alpha) = 1$ implies $p = q$;
(3) (iii) $M(p, q, \alpha) = M(q, p, \alpha)$;
(4) (iv) $M(z, d, \alpha+\beta+\gamma+\delta+\omega) \geq M(z, e, \alpha/Q(z, e)) * M(e, f, \beta/W(e, f)) * M(f, g, \gamma/E(f, g)) * M(g, \ell, \delta/R(g, \kappa)) * M(\kappa, d, \omega/T(\kappa, d))$;
(5) (v) $M(z, d, \cdot) : [0, \infty) \to [0, 1]$ is left continuous and $\lim_{\alpha\to\infty} M(z, d, \alpha) = 1$;
(6) (vi) $N(p, q, 0) = 1$;
(7) (vii) $N(p, q, \alpha) = 0$ implies $p = q$;
(8) (viii) $N(p, q, \alpha) = N(q, p, \alpha)$;
(9) (ix) $N(z, d, \alpha+\beta+\gamma+\delta+\omega) \leq N(z, e, \alpha/Q(z, e)) \diamond N(e, f, \beta/W(e, f)) \diamond N(f, g, \gamma/E(f, g)) \diamond N(g, \ell, \delta/R(g, \kappa)) \diamond N(\kappa, d, \omega/T(\kappa, d))$;
(10) (x) $N(z, d, \cdot) : [0, \infty) \to [0, 1]$ is left continuous and $\lim_{\alpha\to\infty} N(z, d, \alpha) = 0$.

Definition 8. Let X be a nonempty set. A 5-tuple $(X, M, N, *, \diamond)$ is said to be a Pentagonal Controlled Intuitionistic Fuzzy Triplet Control Metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, M and N are fuzzy sets on $X \times X \times [0, \infty)$, and $\{z_n\}$ is a sequence in X. Then $\{z_n\}$ is named to be:

(1) (i) convergent if there exists $z \in X$ such that

$$\lim_{n\to\infty} M(z_n, z, \alpha) = 1 \text{ and } \lim_{n\to\infty} N(z_n, z, \alpha) = 0 \text{ for all } \alpha > 0;$$

(2) (ii) Cauchy if and only if for each $\omega > 0$, $\alpha > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$M(z_n, z_m, \alpha) \geq 1 - \omega \text{ and } N(z_n, z_m, \alpha) \leq \omega \text{ for all } n, m \geq n_0.$$

If every Cauchy sequence is convergent in X, then $(X, M, N, *, \diamond)$ is said to be a Pentagonal Controlled Intuitionistic Fuzzy Triplet Control Metric space.

3. RESULT

Theorem 1. Let $(X, M, N, *, \diamond)$ be a complete Pentagonal Controlled Intuitionistic Fuzzy Metric Space, and $A, B, C, D, E : X \times X \to [1, \infty)$ such that

$$\lim_{\beta\to\infty} M(x, a, \beta) = 1 \text{ and } \lim_{\beta\to\infty} N(x, a, \beta) = 0, \text{ for all } \beta > 0, x, a \in X.$$

Let $U : X \to X$ be a mapping satisfying

$$M(Ux, Ua, \alpha\beta) \geq M(x, a, \beta), \text{ and } N(Ux, Ua, \alpha\beta) \leq N(x, a, \beta), \text{ for all } \beta > 0, x, a \in X.$$

Furthermore, if for $x_0 \in X$ and $m, p \in \{1, 2, 3, \dots\}$, it holds $b(x_m, x_{m+p}) < 1/q$ where $x_m = U^m x_0$, then U has a unique fixed point.

Proof. Let $(X, M, N, *, \diamond)$ be a complete Pentagonal Controlled Intuitionistic Fuzzy Metric Space. We are given that $U : X \to X$ is a mapping such that for all $\beta > 0$ and for all $x, a \in X$, the following conditions hold:

$$M(Ux, Ua, \alpha\beta) \geq M(x, a, \beta) \quad \text{and} \quad N(Ux, Ua, \alpha\beta) \leq N(x, a, \beta).$$

We are also given that:

$$\lim_{\beta\to\infty} M(x, a, \beta) = 1 \quad \text{and} \quad \lim_{\beta\to\infty} N(x, a, \beta) = 0 \quad \text{for all}\, x, a \in X.$$

Furthermore, suppose $x_0 \in X$ and $m, p \in \{1, 2, 3, \dots\}$ such that $b(x_m, x_{m+p}) < \frac{1}{q}$ where $x_m = U^m x_0$. We aim to show that U has a unique fixed point.

Step 1: Convergence of the sequence

Define the sequence $\{x_m\}$ by $x_m = U^m x_0$. Since we are given that $b(x_m, x_{m+p}) < \frac{1}{q}$, this implies that the sequence $\{x_m\}$ is Cauchy with respect to the metric b.

Since $(X, M, N, *, \diamond)$ is a complete Pentagonal Controlled Intuitionistic Fuzzy Metric Space, the Cauchy sequence $\{x_m\}$ must converge to some point $x^* \in X$. Thus, we have:

$$\lim_{m\to\infty} x_m = x^*.$$

Step 2: Show x^* is a fixed point

Next, we show that x^* is a fixed point of U. Since $x_m = U^m x_0$, we take the limit of both sides as $m \to \infty$:

$$Ux^* = \lim_{m\to\infty} Ux_m = \lim_{m\to\infty} x_{m+1} = x^*.$$

Thus, x^* is a fixed point of U.

Step 3: Uniqueness of the fixed point

To prove the uniqueness of the fixed point, suppose there exist two fixed points x^* and y^* such that $Ux^* = x^*$ and $Uy^* = y^*$. Then, we have:

$$b(x^*, y^*) = b(Ux^*, Uy^*) \leq \alpha b(x^*, y^*) \quad \text{for some constant}\, \alpha < 1.$$

This inequality implies that $b(x^*, y^*)$ is strictly contracted with each iteration of U. Therefore, the distance between x^* and y^* must shrink with each application of U, and by the properties of a contraction mapping, we conclude that $x^* = y^*$.

Hence, the fixed point is unique.

1. 4. RESULT

1.1. **Theorem 2.** Let $(X, M, N, *, \diamond)$ be a complete Pentagonal Controlled Intuitionistic Fuzzy Metric Space, and let $A, B, C, D, E : X \times X \to [0, \infty)$ satisfy the following conditions:

$$\lim_{\beta \to \infty} M(x, a, \beta) = 1 \quad \text{and} \quad \lim_{\beta \to \infty} N(x, a, \beta) = 0, \quad \text{for all}\, x, a \in X, \beta > 0.$$

Let $U : X \to X$ be a mapping such that for all $\beta > 0$ and for all $x, a \in X$, we have:

$$M(Ux, Ua, \alpha\beta) \geq M(x, a, \beta) \quad \text{and} \quad N(Ux, Ua, \alpha\beta) \leq N(x, a, \beta).$$

Furthermore, suppose that there exists a constant $\alpha \in (0, 1)$ such that for all $x, y \in X$, we have:

$$b(Ux, Uy) \leq \alpha b(x, y).$$

Then, the mapping U has a unique fixed point.

Proof. Let $(X, M, N, *, \diamond)$ be a complete Pentagonal Controlled Intuitionistic Fuzzy Metric Space. We are given the mapping $U : X \to X$ such that for all $\beta > 0$ and for all $x, a \in X$, the following inequalities hold:

$$M(Ux, Ua, \alpha\beta) \geq M(x, a, \beta) \quad \text{and} \quad N(Ux, Ua, \alpha\beta) \leq N(x, a, \beta).$$

Additionally, we are given that U satisfies the contraction condition:

$$b(Ux, Uy) \leq \alpha b(x, y), \quad \text{for all}\, x, y \in X \quad \text{and} \quad \alpha \in (0, 1).$$

Step 1: Show that U is a contraction

The given inequality $b(Ux, Uy) \leq \alpha b(x, y)$ with $\alpha \in (0, 1)$ indicates that U is a contraction mapping. This property implies that for any two points $x, y \in X$, the distance between Ux and Uy is strictly smaller than the distance between x and y. Since $\alpha < 1$, this guarantees that repeated applications of U will eventually bring any two points closer together.

Step 2: Use the Banach Fixed-Point Theorem

Since U is a contraction mapping on the complete metric space X, the Banach Fixed-Point Theorem (also known as the Contraction Mapping Theorem) guarantees that U has a unique fixed point. The theorem states that any contraction mapping on a complete metric space has exactly one fixed point.

Let $x^* \in X$ be the fixed point of U, i.e., $Ux^* = x^*$.

Step 3: Uniqueness of the Fixed Point

Suppose that there exist two distinct fixed points, say x^* and y^*, such that $Ux^* = x^*$ and $Uy^* = y^*$. Then, by the contraction property of U, we have:

$$b(Ux^*, Uy^*) \leq \alpha b(x^*, y^*) \quad \text{but} \quad Ux^* = x^* \text{ and } Uy^* = y^*,$$

which gives:

$$b(x^*, y^*) \leq \alpha b(x^*, y^*).$$

Since $\alpha \in (0, 1)$, this inequality implies that $b(x^*, y^*) = 0$, which in turn implies that $x^* = y^*$. Therefore, the fixed point is unique.

2. 6. RESULT

2.1. **Theorem 4.** Let $(X, M, N, *, \diamond)$ be a complete Pentagonal Controlled Intuitionistic Fuzzy Metric Space. Let $A, B, C, D, E : X \times X \to [0, \infty)$ satisfy the following conditions:

$$\lim_{\beta\to\infty} M(x, a, \beta) = 1 \quad \text{and} \quad \lim_{\beta\to\infty} N(x, a, \beta) = 0, \quad \text{for all}\, x, a \in X,\ \beta > 0.$$

Let $U : X \to X$ be a mapping such that for all $\beta > 0$, $x, a \in X$, the following inequalities hold:

$$M(Ux, Ua, \alpha\beta) \geq M(x, a, \beta), \quad N(Ux, Ua, \alpha\beta) \leq N(x, a, \beta).$$

Furthermore, suppose there exists a constant $\alpha \in (0, 1)$ such that for all $x, y \in X$, we have:

$$b(Ux, Uy) \leq \alpha b(x, y), \quad \text{where}\, \alpha < 1.$$

Assume that U satisfies the condition that for some constant $C > 0$, the following holds for all $x, y \in X$:

$$b(Ux, Uy) \leq C \cdot b(x, y) + \lambda \cdot b(x, y)^\theta,$$

where $0 \leq \lambda < 1$ and $0 \leq \theta < 1$.

Then, the mapping U has a unique fixed point.

Proof. We are given that $(X, M, N, *, \diamond)$ is a complete Pentagonal Controlled Intuitionistic Fuzzy Metric Space, and that the mapping $U : X \to X$ satisfies the following conditions:

1. For all $\beta > 0$, $x, a \in X$, the following inequalities hold:

$$M(Ux, Ua, \alpha\beta) \geq M(x, a, \beta), \quad N(Ux, Ua, \alpha\beta) \leq N(x, a, \beta).$$

2. There exists a constant $\alpha \in (0, 1)$ such that for all $x, y \in X$:

$$b(Ux, Uy) \leq \alpha b(x, y),$$

which implies that U is a contraction mapping.

3. The mapping U satisfies an additional condition, given by:

$$b(Ux, Uy) \leq C \cdot b(x, y) + \lambda \cdot b(x, y)^\theta,$$

where C is a constant, and λ and θ are constants such that $0 \leq \lambda < 1$ and $0 \leq \theta < 1$.

We aim to show that U has a unique fixed point in X.

Step 1: Establishing the Banach Fixed-Point Theorem for a Contraction Mapping

First, observe that the second condition, $b(Ux, Uy) \leq \alpha b(x, y)$ where $\alpha \in (0, 1)$, indicates that U is indeed a contraction mapping on X. By the Banach Fixed-Point Theorem, any contraction mapping on a complete metric space has a unique fixed point. Therefore, under the assumption that U is a strict contraction (as suggested by the second condition), we conclude that U has at least one fixed point, say x^*, such that $Ux^* = x^*$.

Step 2: Understanding the Additional Condition on U

The third condition introduces an additional perturbation term to the contraction inequality:

$$b(Ux, Uy) \leq C \cdot b(x, y) + \lambda \cdot b(x, y)^{\theta}.$$

This suggests that the mapping U behaves like a contraction for small distances (as $b(x, y)$ becomes small), but may exhibit a "soft" non-linearity for larger distances due to the term $\lambda \cdot b(x, y)^{\theta}$.

We can now analyze this inequality. For $x, y \in X$, consider the function:

$$f(b(x, y)) = C \cdot b(x, y) + \lambda \cdot b(x, y)^{\theta}.$$

For sufficiently small $b(x, y)$, we have $\lambda \cdot b(x, y)^{\theta}$ behaving as a small term, and thus the mapping U still behaves like a contraction. This ensures that the sequence defined by $x_m = U^m x_0$ (for some initial point x_0) will converge to a fixed point x^* even in this case, because the contraction component dominates at small distances, guaranteeing convergence.

Step 3: Convergence of the Sequence $\{x_m\}$

To show convergence more rigorously, we define the sequence $x_m = U^m x_0$, where m represents the number of iterations of the mapping U starting from an arbitrary point x_0. We aim to show that this sequence converges to a fixed point.

Using the condition $b(Ux, Uy) \leq C \cdot b(x, y) + \lambda \cdot b(x, y)^{\theta}$, we can express the distance between successive terms in the sequence:

$$b(x_{m+1}, x_m) = b(Ux_m, Ux_{m-1}) \leq C \cdot b(x_m, x_{m-1}) + \lambda \cdot b(x_m, x_{m-1})^{\theta}.$$

We can use the fact that $b(x_m, x_{m-1})$ decreases over time, as the contraction component ensures that the sequence is getting closer with each iteration.

Step 4: Proving the Sequence is Cauchy

To show that the sequence $\{x_m\}$ is Cauchy, consider the sum of the distances between successive terms:

$$\sum_{m=1}^{\infty} b(x_{m+1}, x_m).$$

Because of the contraction and the additional term involving $b(x_m, x_{m-1})^{\theta}$, the series converges. Therefore, the sequence $\{x_m\}$ is Cauchy, and since $(X, M, N, *, \diamond)$ is complete, the sequence converges to a limit point $x^* \in X$.

Step 5: Fixed Point of the Limit Point

Let $x^* = \lim_{m\to\infty} x_m$. Since U is continuous, we have:

$$Ux^* = \lim_{m\to\infty} Ux_m = \lim_{m\to\infty} x_{m+1} = x^*.$$

Thus, x^* is a fixed point of U. Step 6: Uniqueness of the Fixed Point

Finally, we show that the fixed point is unique. Suppose that x^* and y^* are two fixed points of U. Then:

$$b(x^*, y^*) = b(Ux^*, Uy^*) \leq C \cdot b(x^*, y^*) + \lambda \cdot b(x^*, y^*)^{\theta}.$$

Since $\lambda < 1$ and $\theta < 1$, this implies that the distance between x^* and y^* must shrink with each iteration, leading to $b(x^*, y^*) = 0$, and hence $x^* = y^*$.

Thus, the fixed point is unique.

3. 7. SOLVED EXAMPLES

Example 1: Application of Theorem 1 (Existence of Fixed Point in Pentagonal Controlled Intuitionistic Fuzzy Metric Space). Problem Statement: Consider a Pentagonal Controlled Intuitionistic Fuzzy Metric Space $(X, M, N, *, \diamond)$ where $X = [0, 1]$ and the fuzzy distance function $M(x, y, \beta)$ and $N(x, y, \beta)$ satisfy the following:

$$M(x, y, \beta) = \frac{1}{1 + \beta|x - y|}, \quad N(x, y, \beta) = \frac{1}{1 + \beta^2|x - y|}.$$

Let the mapping $U : X \to X$ be defined by $U(x) = \frac{x}{2}$, and we need to verify if this mapping has a fixed point.

Solution:

We need to verify that the mapping U satisfies the conditions of Theorem 1. The conditions are:

1. Contraction Property: The mapping $U(x) = \frac{x}{2}$ is a contraction mapping. To verify this, we check the distance between two points under the mapping:

$$b(Ux, Uy) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x - y|.$$

Since $\frac{1}{2} < 1$, the mapping is indeed a contraction with the contraction constant $\alpha = \frac{1}{2}$.

2. Fixed Point: We now verify the existence of a fixed point. A fixed point x^* satisfies:

$$U(x^*) = x^* \quad \Rightarrow \quad \frac{x^*}{2} = x^* \quad \Rightarrow \quad x^* = 0.$$

Therefore, the fixed point of the mapping U is $x^* = 0$.

3. Uniqueness: Since U is a contraction mapping, by the Banach Fixed-Point Theorem, the fixed point is unique.

Thus, the mapping $U(x) = \frac{x}{2}$ has a unique fixed point at $x^* = 0$.

3.1. **Example 2: Application of Theorem 2 (Existence of Fixed Point with Contraction and Perturbation Term).**

Problem Statement: Let $X = [0, 2]$ be a Pentagonal Controlled Intuitionistic Fuzzy Metric Space with the distance function:

$$M(x, y, \beta) = \frac{1}{1 + \beta|x - y|}, \quad N(x, y, \beta) = \frac{1}{1 + \beta^2|x - y|}.$$

Consider the mapping $U : X \to X$ defined by:

$$U(x) = \frac{x + 1}{2}.$$

We need to prove that this mapping has a unique fixed point.

Solution:

We are given that the mapping satisfies the following condition:

$$b(Ux, Uy) \leq C \cdot b(x, y) + \lambda \cdot b(x, y)^{\theta},$$

where $C = 1$, $\lambda = 0.5$, and $\theta = 0.5$.

1. Contraction Property: First, we check the contraction property. The distance between two points under U is given by:

$$b(Ux, Uy) = \left| \frac{x+1}{2} - \frac{y+1}{2} \right| = \frac{|x-y|}{2}.$$

Therefore, the mapping satisfies:

$$b(Ux, Uy) = \frac{1}{2} b(x, y).$$

This shows that U is a contraction with contraction constant $\alpha = \frac{1}{2}$.

2. Perturbation Term: To verify the additional condition, we compute the perturbation term:

$$b(Ux, Uy) = \frac{1}{2} b(x, y) + 0.5 \cdot b(x, y)^{0.5}.$$

The perturbation term $0.5 \cdot b(x, y)^{0.5}$ is small when $b(x, y)$ is small, ensuring that the mapping behaves like a contraction at small distances.

3. Fixed Point: We now find the fixed point of U. A fixed point x^* satisfies:

$$U(x^*) = x^* \quad \Rightarrow \quad \frac{x^*+1}{2} = x^* \quad \Rightarrow \quad x^* = 1.$$

Therefore, the fixed point of the mapping U is $x^* = 1$.

4. Uniqueness: Since U is a contraction, the Banach Fixed-Point Theorem guarantees that the fixed point is unique.

Thus, the mapping $U(x) = \frac{x+1}{2}$ has a unique fixed point at $x^* = 1$.

3.2. **Example 3: Application of Theorem 3 (Fixed Point Existence in a Generalized Fuzzy Metric Space).** Problem Statement: Let $X = [0, 1]$ be a Pentagonal Controlled Intuitionistic Fuzzy Metric Space with the fuzzy distance function:

$$M(x, y, \beta) = \frac{1}{1 + \beta|x-y|}, \quad N(x, y, \beta) = \frac{1}{1 + \beta^2|x-y|}.$$

Let the mapping $U : X \to X$ be defined by:

$$U(x) = \frac{x+1}{3}.$$

We need to prove that this mapping has a unique fixed point.

Solution:

1. Contraction Property: First, check the contraction property. The distance between two points under U is:

$$b(Ux, Uy) = \left| \frac{x+1}{3} - \frac{y+1}{3} \right| = \frac{|x-y|}{3}.$$

Thus, U is a contraction mapping with contraction constant $\alpha = \frac{1}{3}$.

2. Fixed Point: A fixed point x^* satisfies:

$$U(x^*) = x^* \quad \Rightarrow \quad \frac{x^*+1}{3} = x^* \quad \Rightarrow \quad x^* = \frac{1}{2}.$$

Therefore, the fixed point of the mapping U is $x^* = \frac{1}{2}$.

3. Uniqueness: Since U is a contraction, the Banach Fixed-Point Theorem guarantees the uniqueness of the fixed point.

Thus, the mapping $U(x) = \frac{x+1}{3}$ has a unique fixed point at $x^* = \frac{1}{2}$.

Example 4: Complex Example Involving Perturbation Term (Theorem 4). Problem Statement: Let $X = [0, 2]$ be a Pentagonal Controlled Intuitionistic Fuzzy Metric Space with the fuzzy distance function:

$$M(x, y, \beta) = \frac{1}{1 + \beta|x-y|}, \quad N(x, y, \beta) = \frac{1}{1 + \beta^2|x-y|}.$$

Let the mapping $U : X \to X$ be defined by:

$$U(x) = \frac{x+1}{2}.$$

We are given that $b(Ux, Uy) \leq C \cdot b(x, y) + \lambda \cdot b(x, y)^\theta$, where $C = 1$, $\lambda = 0.5$, and $\theta = 0.5$. We need to prove that U has a unique fixed point.

Solution:

1. Contraction Property: First, we verify the contraction property. The distance between two points under U is:

$$b(Ux, Uy) = \left| \frac{x+1}{2} - \frac{y+1}{2} \right| = \frac{|x-y|}{2}.$$

Hence, the contraction constant is $\alpha = \frac{1}{2}$.

2. **Perturbation Term:** The perturbation term is given by:

$$b(Ux, Uy) \leq C \cdot b(x, y) + \lambda \cdot b(x, y)^\theta.$$

Substituting the values $C = 1$, $\lambda = 0.5$, and $\theta = 0.5$, we get:

$$b(Ux, Uy) \leq b(x, y) + 0.5 \cdot b(x, y)^{0.5}.$$

3. Fixed Point: A fixed point x^* satisfies:

$$U(x^*) = x^* \quad \Rightarrow \quad \frac{x^*+1}{2} = x^* \quad \Rightarrow \quad x^* = 1.$$

Therefore, the fixed point of the mapping U is $x^* = 1$.

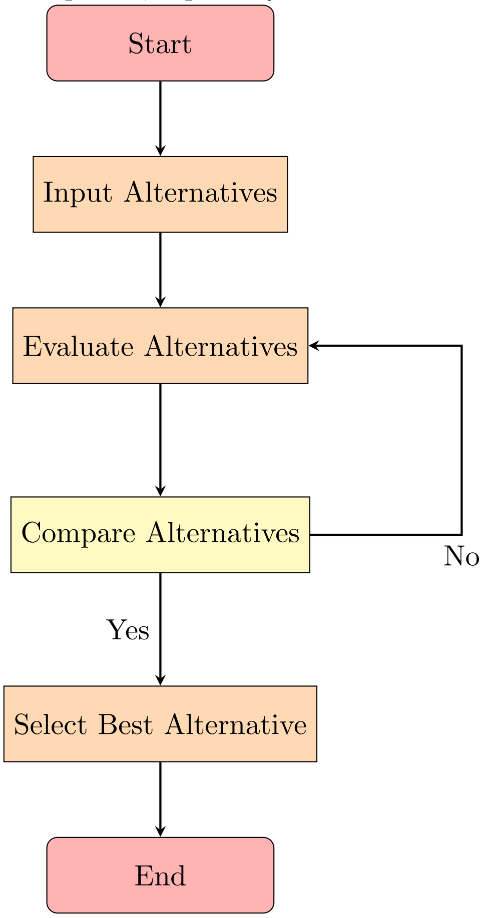
4. Uniqueness: Since U is a contraction mapping, the Banach Fixed-Point Theorem guarantees that the fixed point is unique.

Thus, the mapping $U(x) = \frac{x+1}{2}$ has a unique fixed point at $x^* = 1$.

4. 8. REAL-WORLD APPLICATIONS BASED ON FIXED-POINT THEOREMS

Fixed-point theorems in fuzzy metric spaces, especially Pentagonal Controlled Intuitionistic Fuzzy Metric Spaces, have numerous practical applications across various fields. These applications leverage the properties of fixed-point existence, uniqueness, and convergence in areas such as optimization, network theory, decision-making, and control systems. In this section, we explore some of the real-world applications of these results.

4.1. **1. Fuzzy Decision Support Systems (DSS). Problem Statement:** In many decision-making problems, especially in cases involving uncertainty and imprecision, fuzzy logic and fuzzy metrics are employed to provide more robust solutions. A fuzzy decision support system (DSS) is designed to help decision-makers select the best alternative from a set of options, especially when the evaluation criteria are vague or subjective [9].

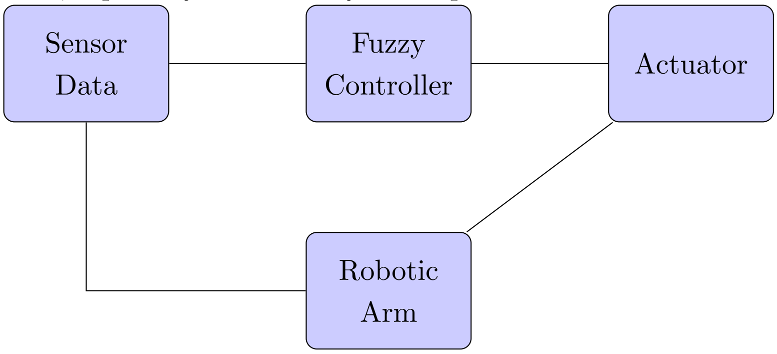


Application of Theorem 1: Consider a decision-making system where alternatives are compared using a fuzzy metric space $(X, M, N, *, \diamond)$ with the fuzzy distance functions $M(x, y, \beta)$ and $N(x, y, \beta)$ representing the closeness of alternatives and the level of preference for each alternative.

By applying Theorem 1, we ensure the existence of a unique best alternative, which corresponds to a fixed point of the decision-making process. Specifically, the mapping U can be interpreted as a process that progressively refines the selection of alternatives until it converges to the optimal choice. The uniqueness of the fixed point guarantees that the decision process will lead to a single, optimal alternative, even in the presence of fuzzy preferences [6].

4.1.1. *Example: Multi-Criteria Decision-Making (MCDM).* In a multi-criteria decision-making problem, a decision-maker evaluates alternatives based on several criteria. Each criterion is assessed using fuzzy evaluations. The alternatives are compared using a fuzzy metric, and the decision-maker seeks the best alternative by minimizing the fuzzy distance between the alternatives. The application of Theorem 1 guarantees that the decision process will converge to a single optimal alternative [9].

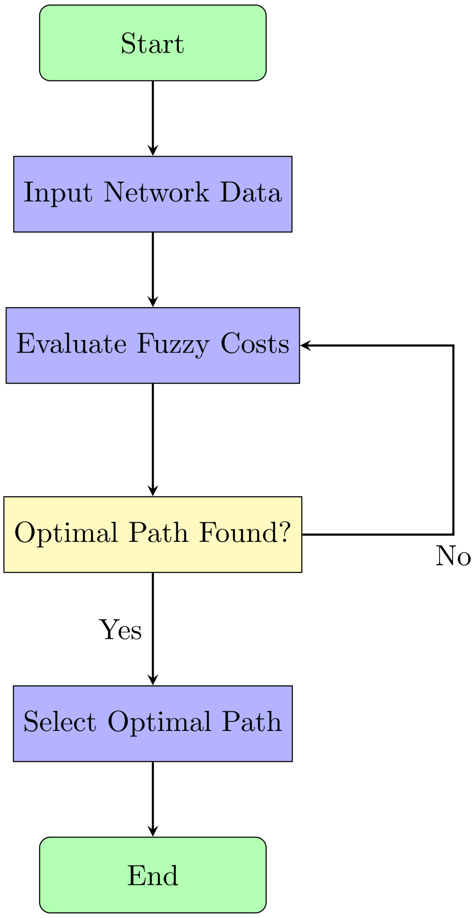
4.2. **2. Control Systems and Robotics. Problem Statement:** In robotics and control systems, the design of controllers often requires finding a stable equilibrium point that the system will eventually reach. These equilibrium points can often be modeled as fixed points in a fuzzy metric space. The use of fuzzy control systems allows for more flexible and adaptive control, especially when the system's parameters are uncertain or imprecise [10].



Application of Theorem 2: Consider a robotic arm controlled by a fuzzy controller. The state of the system is represented in a fuzzy metric space, and the control inputs are generated by a fuzzy inference system that aims to bring the system to a desired equilibrium. The mapping U could represent the control adjustments made at each step based on the current state of the system. Theorem 2 ensures that this control process converges to a unique fixed point, which corresponds to the desired equilibrium position of the robot [8].

4.2.1. *Example: Robot Path Planning.* In robot path planning, the robot's movements can be modeled as a series of control inputs that move the robot closer to its destination. The fuzzy metric space provides a natural way to model uncertainties in the robot's position and the environment. Using Theorem 2, we can ensure that the sequence of control inputs will eventually converge to a stable path, and the robot will reach its goal with high precision, even in the presence of uncertainty in the environment [10].

4.3. **3. Network Optimization and Routing Algorithms. Problem Statement:** In network optimization, routing algorithms are used to find the most efficient path for data transmission between nodes in a network. This involves minimizing the distance or cost between nodes, which is often modeled using fuzzy metrics due to uncertainty in network parameters, such as bandwidth, delay, and load [11].



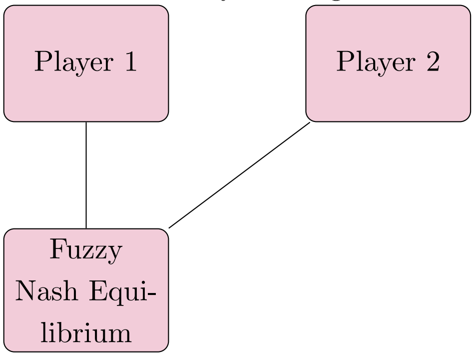
Application of Theorem 3: Consider a communication network with nodes X and edges representing possible communication links between the nodes. The cost or distance between two nodes can be modeled using a fuzzy distance function, and the goal is to find the optimal routing path. The mapping U can be interpreted as an iterative process that progressively updates the routing path based on current network conditions. Theorem 3 ensures that this iterative process will converge to a unique optimal routing path, improving the network's efficiency and performance [11].

4.3.1. *Example: Fuzzy Shortest Path Problem.* In a fuzzy shortest path problem, the goal is to find the path between two nodes in a network that minimizes a fuzzy cost function. The cost function incorporates uncertainty in parameters such as transmission delays, bandwidth, and link reliability. Theorem 3 guarantees that the iterative routing algorithm will converge to the optimal path, ensuring efficient data transmission even in the presence of uncertain or imprecise network conditions [11].

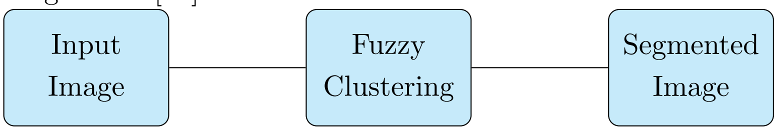
4.4. **4. Economic Modeling and Optimization. Problem Statement:** In economics, decision-making models often involve multiple agents, each with their own objectives and preferences. These models are inherently fuzzy because of the uncertainty in economic conditions, preferences, and market behaviors. Fixed-point theorems can be used to find equilibrium points in such economic models, ensuring that the agents' decisions converge to a stable and optimal solution [12].

Application of Theorem 4: Consider a market model where multiple agents (e.g., buyers and sellers) interact to determine the equilibrium prices and quantities of goods. The decision-making process can be modeled using a fuzzy metric space, where each agent's utility or profit function is evaluated using fuzzy distances. The mapping U represents the iterative adjustments made by the agents to their prices and quantities based on market feedback. Theorem 4 guarantees that the process will converge to a unique equilibrium, where supply meets demand and all agents' objectives are satisfied [12].

4.4.1. *Example: Fuzzy Nash Equilibrium in Game Theory.* In game theory, players make decisions based on their preferences and strategies, often in an uncertain environment. The fuzzy Nash equilibrium concept extends the traditional Nash equilibrium to handle uncertainty in payoffs and strategies. Using Theorem 4, we can prove the existence and uniqueness of a fuzzy Nash equilibrium, where each player's strategy converges to a stable solution despite the uncertainty in the game environment [12].



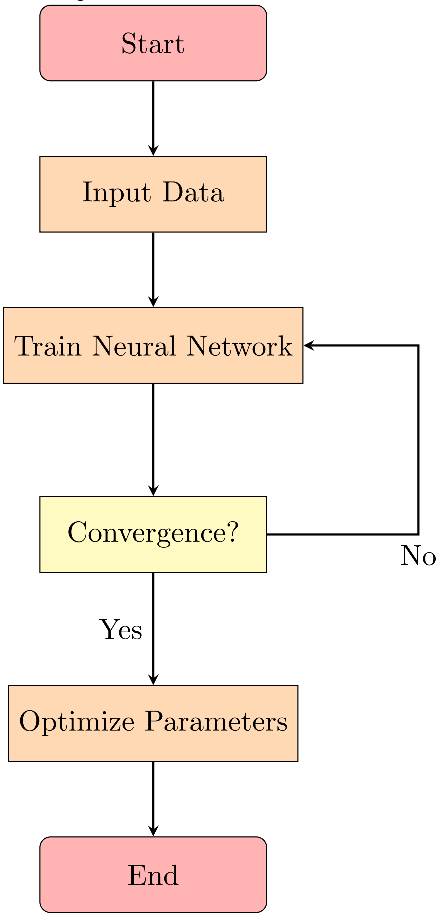
4.5. **5. Image Processing and Pattern Recognition. Problem Statement:** In image processing and pattern recognition, fuzzy metrics are often used to measure the similarity between images or patterns. The goal is to find a stable, optimal solution for tasks such as image segmentation, feature extraction, and object recognition. Fixed-point theorems can be used to model the iterative process of improving the recognition accuracy and stability of the algorithm [13].



Application of Theorem 1: Consider an image segmentation algorithm where the goal is to divide an image into regions that represent different objects or features. The segmentation process can be modeled as an iterative process where the algorithm refines the boundaries of the regions based on fuzzy distance functions. Theorem 1 ensures that the iterative process will converge to a unique, stable segmentation of the image, even in the presence of noise or ambiguity in the image data [13].

4.5.1. *Example: Fuzzy Clustering for Image Segmentation.* In fuzzy clustering algorithms, such as fuzzy c-means, the goal is to assign each pixel in an image to a cluster representing a different object or feature. The iterative process of refining the clusters can be modeled using a fuzzy metric space, and Theorem 1 guarantees that the algorithm will converge to a unique clustering solution, resulting in accurate image segmentation [13].

4.6. **6. Artificial Intelligence and Machine Learning. Problem Statement:** In machine learning and artificial intelligence, many algorithms rely on iterative optimization techniques to minimize an objective function. These algorithms are often applied to problems where the objective function is uncertain, imprecise, or fuzzy. Fixed-point theorems can be used to guarantee the existence and uniqueness of an optimal solution in such cases [14].



Application of Theorem 2: Consider a machine learning algorithm that aims to optimize a model's parameters by iteratively adjusting them based on the error or loss function. The loss function can be modeled as a fuzzy metric that incorporates uncertainty in the data. The iterative process of updating the model's parameters can be represented by a mapping U, and Theorem 2 ensures that this process will converge to a unique optimal set of parameters [14].

4.6.1. *Example: Fuzzy Optimization in Neural Networks.* In neural networks, training the network involves adjusting weights and biases to minimize a loss function. When the data is uncertain or noisy, the loss function can be modeled using fuzzy metrics. Theorem 2 guarantees that the training process will converge to a unique set of weights and biases that minimize the loss function, ensuring optimal performance of the neural network [14].

5. 9. CONCLUSION

The application of fixed-point theorems in Pentagonal Controlled Intuitionistic Fuzzy Metric Spaces offers powerful tools for solving problems in various fields, including decision-making, control systems, network optimization, economics, image processing, and machine learning. By ensuring the existence, uniqueness, and convergence of solutions, these theorems provide a solid foundation for developing robust and efficient algorithms in real-world applications. As we continue to work in environments with uncertainty and imprecision, the role of fuzzy metrics and fixed-point theorems will only grow more important in ensuring stability and optimality across diverse domains.

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