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Research Paper

MELLIN-SUMUDU SYNERGY: A NOVEL PARADIGM FOR EXTENDING MITTAG-LEFFLER FUNCTION

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ABSTRACT

This study presents an innovative reconfiguration of the Mittag-Leffler function (MLF) by synergistically combining the Mellin transform and the Sumudu transform. Although the MLF plays a significant role in fractional calculus, its complexity has limited its applicability. By utilizing both the Mellin and Sumudu transforms, new integral representations of the MLF are derived, effectively broadening its scope in addressing fractional differential equations. This integrated approach provides a deeper understanding of the MLF's properties and enables its extension to a wider range of problems in physics, engineering, and mathematics. The effectiveness of the proposed extension is demonstrated through its application to fractional calculus problems, thereby contributing to the advancement of the field and enhancing its ability to model complex real-world phenomena with greater accuracy.

1. Introduction

The Mittag-Leffler function (MLF) is a powerful tool in fractional calculus, playing a central role in the analysis of fractional differential equations. However, its complex nature limits its applications, hindering the full exploitation of its potential. In recent years, integral

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transforms have emerged as a valuable means to extend the utility of special functions, including the MLF. The Mittag–Leffler functions (MLF) is the queen function of fractional calculus and the solution of this function can be helpful to solve many integral or differential equations of non-integer order, which is helpful in wide variety of problems in various field of Mathematics and Mathematical Physics.

The Mittag-Leffler function, introduced by Gösta Magnus Mittag-Leffler in 1903, has been extensively studied and generalized due to its diverse applications [1]. The function $E_\alpha(-t^\alpha)$ for $0 < \alpha < 1$ and $t > 0$ exhibits complete monotonicity and universal scaling properties, with coinciding frequency and time spectra [15]. Mittag-Leffler's contributions to mathematics extend beyond this function; he founded the influential journal Acta Mathematica in 1882 and used his wealth to establish a villa in Djursholm, Stockholm, which houses important materials for the history of mathematics. His academic journey included studies at Uppsala University and visits to prominent European universities before obtaining professorships in Helsinki and Stockholm [12].

This collection of papers explores various aspects of zeta functions and related mathematical concepts. Bourcia (n.d.) [21] provides a comprehensive overview of Riemann zeta functions, discussing their analytical properties, Euler product representation, and functional identity. The paper also examines the Riemann hypothesis and its implications for different types of zeta functions. Goldstein and de la Torre (1975) investigate a function analogous to $log \quad \eta(\tau)$, where $\eta(z)$ is Dedekind's classical eta function. Fine (1951) [7] focuses on the Hurwitz zeta-function, referencing Riemann's work on prime number distribution (Riemann, 1859) and Hurwitz's contributions (Hurwitz, 1882). Fine's paper also mentions Lipschitz's use of theta-function transformation to derive functional equations for general zeta-functions (Lipschitz, n.d.). These papers collectively contribute to the understanding of zeta functions and their applications in number theory and complex analysis.

The Mittag-Leffler function and its generalizations have been extensively studied in the context of fractional calculus and integral equations. In 1971, Prabhakar [18] introduced a generalized Mittag-Leffler function and used it to define a linear operator, employing fractional integration to analyze solutions of singular integral equations. This work built upon earlier research on integral equations involving special functions . The k-generalized gamma function, beta function, and Pochhammer k-symbol were introduced by Díaz and Pariguan (2004), providing integral representations and identities that generalize classical functions. These k-analogues have been extensively studied and applied in various contexts. In 2009, Srivastava and Tomovski [20] further extended the Mittag-Leffler function and developed a fractional calculus framework with an integral operator containing this generalized function in its kernel.

In 2009, Gürel Yılmaz et al. [13] focused on k-analogues of Appell functions, presenting transformation and reduction formulas that relate k-Appell functions to k-hypergeometric functions. They also derived generating relations using k-fractional derivatives. Korkmaz-Duzgun and Erkus-Duman (2019) explored generating functions for k-hypergeometric functions, introducing identities for k-Pochhammer symbols and deriving linear, multilinear, and bilateral generating functions. In 2009, Díaz et al. [14] provided combinatorial and probabilistic interpretations for the q-analogue of the Pochhammer k-symbol and introduced q-analogues of the Mellin transform to study the q-analogue of the k-gamma distribution.

These studies collectively demonstrate the rich mathematical properties and applications of k-generalized functions in hypergeometric theory.

The k-Mittag-Leffler function, introduced by Dorrego and Cerutti, has been the subject of several studies expanding its properties and applications. Dhakar and Sharma (2013) investigated a recurrence relation and integral representation for this function, while Cerutti (2012) [6] explored k-generalizations of related special functions, including Bessel and Fox-Wright functions. Q_i and Nisar (2019) further generalized the k-Mittag-Leffler function and established various integral transforms, deriving conclusions in terms of generalized Wright and k-Wright functions. In 2013, Dhakar and Sharma [4] investigated a recurrence relation and integral representation of the k-Mittag-Leffler function. In 2016, Nisar et al. [17] proposed an extended form of the Mittag-Leffler function and obtained composition formulas with pathway fractional integral operators. These studies collectively contribute to a deeper understanding of the k-Mittag-Leffler function and its relationships with other special functions, potentially expanding its applicability in various mathematical and scientific domains. In 2013, Gehlot proposed the Multiparameter K-Mittag-Leffler Function, which encompasses various previously introduced Mittag-Leffler functions as special cases. These generalizations have important applications in fractional calculus, particularly in solving fractional order differential and integral equations [3, 10].

The Mittag-Leffler function has been extensively studied and generalized in recent years.In 2017, Cerutti et al. [3] introduced the p-k-Mittag-Leffler function, which generalizes the k-Mittag-Leffler function and relates to the two-parameter Gamma function. In 2020, Gehlot and Bhandari [10] further expanded on this concept by introducing the j-generalized p-k Mittag-Leffler function, exploring its properties, and establishing relationships with other functions. The research also explores the functions' relationships with Riemann-Liouville fractional integrals and derivatives [10, 8]. In 2021, Rahman et al. [19] continued this line of research by studying the properties of an extended Mittag-Leffler function and introducing a new extension of Prabhakar-type fractional integrals. These studies collectively demonstrate the importance of Mittag-Leffler functions in fractional calculus and their applications to solving various types of integral equations.

This paper unfolds as follows: Section 2 lays the groundwork with essential mathematical preliminaries, including an introduction to Sumudu and Mellin transformations. Section 3 presents the core findings, focusing on the Mellin-Barnes integral representation of the generalized Mittag-Leffler function. Section 4 builds upon these results, exploring the extension of the Mittag-Leffler function via Mellin transform and Sumudu transform techniques. Finally, Section 5 synthesizes the key takeaways, providing a concise conclusion to the paper.

2. Preliminary and Mathematical Formulation

2.1. **Some Basic Formulation.** In 1903, the Swedish mathematician Mittag-Leffler [16] gave the function $E_\alpha(z)$ define by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha \in \mathbb{C}, \mathcal{R}(\alpha) > 0, \tag{2.1}$$

In 1905 Wiman A.[23] began investigating a generalization of $E_\alpha(z)$, which is defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \alpha,\beta \in \mathbb{C}, \mathcal{R}(\alpha), \mathcal{R}(\beta) > 0, \tag{2.2}$$

In 1971, Prabhakar T. R.[18] gave Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}$ by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \tag{2.3}$$

where $\alpha,\beta,\gamma \in \mathbb{C}, \mathcal{R}(\alpha), \mathcal{R}(\beta) > 0$,and $(\gamma)_n$ denote the Pochhammer symbol provided by

$$(\gamma)_n = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}.$$

In 2012, Dorrego and Cerutti [6] gave another generalization of the Mittag–Leffler function, which is now known as the k-Mittag–Leffler function.

$$E^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!}, \tag{2.4}$$

where $k > 0, \alpha,\beta,\gamma \in \mathbb{C}, \mathcal{R}(\alpha), \mathcal{R}(\beta) > 0$ and $(\gamma)_{n,k}$ denote the k-Pochhammer symbol provided by Diaz and Parihuan[5] in 2007

$$(\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k)\cdots(\gamma+(n-1)k) = \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)},$$

and Γ_k the $k-$Gamma function given by

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^k}{k}} dt, \mathcal{R}(z) > 0.$$

The Mittag-Leffler (p-k) function ${}_pE^{\gamma}_{k,\alpha,\beta}(z)$ was given by Cerutti et.al.[3] in 2017, defined as

$${}_pE^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!}, \tag{2.5}$$

where $\alpha,\beta,\gamma \in \mathbb{C}, \mathcal{R}(\alpha), \mathcal{R}(\beta), \mathcal{R}(\gamma) > 0, p,k \in \mathbb{R}^+ \setminus \{0\}$ and ${}_p(\gamma)_{n,k}$ denote the Pochhammer $(p-k)$ symbol given by Gehlot[9] in 2017 as

$${}_p\Gamma_k(z) = \left(\frac{zp}{k}\right)\left(\frac{zp}{k}+p\right)\left(\frac{zp}{k}+2p\right)\cdots\left(\frac{zp}{k}+(n-1)p\right) = \frac{{}_p\Gamma_k(z+nk)}{{}_p\Gamma_k(z)},$$

and ${}_p\Gamma_k(z)$ the $(p-k)$ Gamma function defined as

$${}_p\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^k}{p}} dt, \forall z \in \mathbb{C} \setminus k\mathbb{Z}^-, p,k \in \mathbb{R}^+ \setminus \{0\}.$$

In 2020, Ayub et al. [2] introduced another investigation of a generalization of equation (2.5), called the Mittag–Leffler (p,s,k) function

$${}_pE^{\gamma,s}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k,s}}{{}_p\Gamma_{s,k}(\alpha n+\beta)} \frac{z^n}{n!}, \tag{2.6}$$

where $\alpha,\beta,\gamma \in \mathbb{C}, \mathcal{R}(\alpha), \mathcal{R}(\beta), \mathcal{R}(\gamma) > 0, p,k \in \mathbb{R}$ and ${}_p(\gamma)_{n,k,s}$ denotes the symbol given by Pochhammer (p,s,k) Gehlot and Nantomah [11] in 2018 as

$${}_p(\gamma)_{n,k,s} = \left[\frac{\gamma p}{k}\right]_s \left[\frac{\gamma p}{k}+p\right]_s \left[\frac{\gamma p}{k}+2p\right]_s \cdots \left[\frac{\gamma p}{k}+(n-1)p\right]_s = \prod_{i=0}^{n-1} \left[\frac{\gamma p}{k}+ip\right]_s$$

where $(\gamma)_s = \frac{1-s^\gamma}{1-s}, \forall \gamma \in \mathbb{R}, 0 < s < 1$ and The generalized Gamma function, denoted by ${}_p\Gamma_{s,k}(\xi)$, is defined as

$$ {}_p\Gamma_{s,k}(\xi) = \frac{s}{k} \lim_{n\to\infty} \frac{n!\, p^{n+1}(snp)^{\frac{\xi}{k}-1}}{p^{(\xi)_{n,k}}}, $$

where ${}_p\Gamma_{s,k}(\xi)$ is a generalization involving the parameters p, s, and k.

Definition 2.1. [2] The relationship among the Gamma (p, s, k)-function, Gamma (p, k)-function, k-Gamma function, and the classical Gamma function is given by

$$ {}_p\Gamma_{s,k}(\xi) = (s)_p^{\xi/k}\Gamma_k(\xi) = \left(\frac{sp}{k}\right)^{\xi/k}\Gamma_k(\xi) = \frac{(sp)^{\xi/k}}{k}\Gamma\left(\frac{\xi}{k}\right). \tag{2.7} $$

Definition 2.2. [2] The relation between three parameters, two parameters and the classical Pochhammer's symbol given as

$$ {}_p(\gamma)_{n,k,s} = s_p^n(\gamma)_{n,k} = \left(\frac{sp}{k}\right)^n (\gamma)_{n,k} = (sp)^n \left(\frac{\gamma}{k}\right)_n. \tag{2.8} $$

2.2. **The Sumudu Transform.** The Sumudu transformation is applicable not only in applied mathematics, but also in other branches of Astronomy, Engineering, Physics and other sciences. Many researchers have also derived the relation between Laplace and Sumudu transforms. The Sumudu transform has been used extensively to find solutions to FKEs, differential equations (ordinary and partial), Abel integral equations, dynamical systems, controlled dynamical systems, and others. In the year 1993, Sumudu transform was initiated by G. K. Watugula[22] in this form

$$ A = \{f(t) : \exists M, \eta_1, \eta_2 > 0, |f(t)| < Me^{\frac{-\xi}{\eta_j}},\ \text{if } t \in (-1)^j \times [0,\infty)\}. \tag{2.9} $$

For the given function M should be finite however η_1 and η_2 may be finite or infinite.

The Sumudu transform, introduced by G. K. Watugala in 1993, is defined for a function $f(t)$, $t \geq 0$, as:

$$ \mathcal{S}\{f(t)\}(u) = \int_0^\infty f(ut)e^{-t}\, dt, \tag{2.10} $$

where u is a complex parameter and the integral is assumed to converge.

The Sumudu transform of time function $F(t)$ denoted by $(F(t) : s)$ or $F(s)$ is defined as follow

$$ F(s) = \frac{1}{s}\int_0^\infty e^{-\frac{t}{s}} f(t)dt, \qquad 0 < t < \infty. \tag{2.11} $$

The inversion of the Sumudu transform is given by:

$$ \mathcal{S}^{-1}[G(u)](t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{t/u}G(u)\,\frac{du}{u}, \tag{2.12} $$

where the integral is evaluated along a vertical line in the complex plane such that the integrand is analytic.

2.3. **Mellin Transformation.** The Mellin transform is a powerful integral transform widely used in mathematical analysis, especially in the fields of asymptotic analysis, number theory, and the solution of differential equations. It can be considered as a multiplicative version of the two-sided Laplace transform and is particularly effective in problems exhibiting scale invariance. Let $f(t)$ be a real or complex-valued function defined on the positive real axis $(0,\infty)$. The Mellin transform of the function $f(t)$ is defined as

$$\mathcal{M}[f(t)](s) = F(s) = \int_0^\infty t^{s-1} f(t)\, dt, \tag{2.13}$$

where $s \in \mathbb{C}$ and the integral converges within a vertical strip of the complex plane, known as the *strip of convergence.*

If $F(s) = \mathcal{M}[f(t)](s)$ is the Mellin transform of $f(t)$, then the original function $f(t)$ can be recovered using the inverse Mellin transform, given by

$$f(t) = \mathcal{M}^{-1}[F(s)](t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s} F(s)\, ds, \tag{2.14}$$

where the integration is carried out along the vertical line $\mathrm{Re}(s) = c$, with c chosen such that it lies within the strip of convergence of $F(s)$.

Remark 2.3. There are some remark :

- The Mellin transform is particularly useful in analyzing scaling properties of functions and appears frequently in the study of special functions and integral equations.
- The Mellin transform is related to other integral transforms such as the Laplace and Fourier transforms through change of variables.

3. Main Results

3.1. **Mellin–Barnes integral representation of** (p, s, k) **Mittag-Leffler function (MLF).** In this section, we present the Mellin-Barnes integral method, a powerful tool commonly employed in particle physics. This technique is particularly significant for deriving the inverse of the Mellin transform. The approach involves applying the residue theorem in conjunction with the Mittag-Leffler function (MLF). To proceed, we analyze specific types of contour integrals arising in this context.

Theorem 3.1. *Suppose $k, p, s > 0$ and $\alpha, \beta, \gamma \in \mathbb{C}$ with $\mathcal{R}(\alpha), \mathcal{R}(\beta), \mathcal{R}(\gamma) > 0$, then Mellin Barnes integral represents the Mittag-Leffler function ${}_pE^{\gamma,s}_{k,\alpha,\beta}(z)$ as,*

$${}_pE^{\gamma,s}_{k,\alpha,\beta}(z) = \frac{k(sp)^{-\frac{\beta}{k}}}{2\pi i \Gamma(\gamma/k)} \int_L \frac{\Gamma(q)\Gamma\left(\frac{\gamma}{k} - q\right)}{\Gamma\left(\frac{\beta}{k} - \frac{\alpha q}{k}\right)} \left(-z(sp)^{1-\frac{\alpha}{k}}\right)^{-q} dq. \tag{3.1}$$

determing to detach the pole of the integrand as $q = -n, \forall n \in \mathbb{N}_0$ (to the left) belonging to $q = \frac{\gamma}{k} + n, \forall n \in N_0$ (to the right) the contour integration initiating at $-i\infty$ and completing at $+i\infty$ where $|argz| < \pi$.

Proof. We define the generalized Mittag-Leffler function by the series representation:

$${}_pE^{\gamma,s}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k} + n\right)}{\Gamma\left(\frac{\beta}{k} + \frac{\alpha}{k} n\right)\Gamma(\gamma/k)\, n!} \left(-z(sp)^{1-\frac{\alpha}{k}}\right)^n (sp)^{-\frac{\beta}{k}} k.$$

Now, consider the Mellin–Barnes integral:

$$I(z) = \int_L \frac{\Gamma(q)\Gamma\left(\frac{\gamma}{k} - q\right)}{\Gamma\left(\frac{\beta}{k} - \frac{\alpha q}{k}\right)} \left(-z(sp)^{1-\frac{\alpha}{k}}\right)^{-q} dq,$$

where L is a suitable contour running from $-i\infty$ to $+i\infty$ such that it separates the poles of $\Gamma(q)$ at $q=-n$ (for $n\in\mathbb{N}_0$) from the poles of $\Gamma\left(\frac{\gamma}{k}-q\right)$ at $q=\frac{\gamma}{k}+n$ (for $n\in\mathbb{N}_0$).

The integrand has simple poles at: - $q=-n$, from $\Gamma(q)$ for $n\in\mathbb{N}_0$, - $q=\frac{\gamma}{k}+n$, from $\Gamma\left(\frac{\gamma}{k}-q\right)$ for $n\in\mathbb{N}_0$.

By closing the contour to the left and applying the Cauchy residue theorem, the integral evaluates to the sum of the residues at the poles $q=-n$.

We now compute the residue at $q=-n$. Using:

$$\text{Res}_{q=-n}\,\Gamma(q)=\frac{(-1)^n}{n!},\quad \text{for } n\in\mathbb{N}_0,$$

we get:

$$\text{Res}_{q=-n}\left[\frac{\Gamma(q)\Gamma\left(\frac{\gamma}{k}-q\right)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha q}{k}\right)}\left(-z(sp)^{1-\frac{\alpha}{k}}\right)^{-q}\right]=\frac{(-1)^n}{n!}\cdot\frac{\Gamma\left(\frac{\gamma}{k}+n\right)}{\Gamma\left(\frac{\beta}{k}+\frac{\alpha n}{k}\right)}\cdot\left(-z(sp)^{1-\frac{\alpha}{k}}\right)^n.$$

Hence, the sum of residues becomes:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{\Gamma\left(\frac{\gamma}{k}+n\right)}{\Gamma\left(\frac{\beta}{k}+\frac{\alpha n}{k}\right)}\cdot\left(-z(sp)^{1-\frac{\alpha}{k}}\right)^n.$$

Substituting this back into the Mellin–Barnes formula and multiplying by the prefactor in equation (3.1), we get:

$${}_pE_{k,\alpha,\beta}^{\gamma,s}(z)=\frac{k(sp)^{-\frac{\beta}{k}}}{\Gamma(\gamma/k)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{\Gamma\left(\frac{\gamma}{k}+n\right)}{\Gamma\left(\frac{\beta}{k}+\frac{\alpha n}{k}\right)}\cdot\left(-z(sp)^{1-\frac{\alpha}{k}}\right)^n,$$

which matches exactly with the original series definition of ${}_pE_{k,\alpha,\beta}^{\gamma,s}(z)$.

Therefore, the Mellin-Barnes integral representation is valid and the theorem is proved.

□

Corollary 3.2. *There are the following corollaries defined as*

(1) *If we substitute $s=1$, $p=1$, $k=1$, $\beta=1$, and $\gamma=1$ into Theorem 3.1, we obtain the following well-known result, originally presented by Gehlot K.S. (2018), which corresponds to the Mittag–Leffler function (MLF) defined in equation (2.1):*

$$E_\alpha(z)=\frac{1}{2\pi i}\int_L\frac{\Gamma(q)\,\Gamma(1-q)}{\Gamma(1-\alpha q)}(-z)^{-q}\,dq.$$

(2) *If we put $s=1$ in Theorem (3.1), we obtain the following results contains $(p-k)$ Mittag – Leffler function defined in (2.5).*

$${}_pE_{k,\alpha,\beta}^{\gamma}(z)=\frac{k(p)^{-\frac{\beta}{k}}}{2\pi i\Gamma(\gamma/k)}\int_L\frac{\Gamma(q)\Gamma\left(\frac{\gamma}{k}-q\right)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha q}{k}\right)}\left(-z(p)^{1-\frac{\alpha}{k}}\right)^{-q}dq.$$

Finding to detach the pole of the integrand as $q=-n\forall n\in\mathbb{N}_0$ (to the left) belonging to $q=\frac{\gamma}{k}+n,\forall n\in\mathbb{N}_0$ (to the right) the contour integration initiating at $-i\infty$ and completing at $+i\infty$, where $|argz|<\pi$.

(3) *On putting put $s=1$ and $p=1$ in Theorem (3.1) we obtain the following results contains k Mittag – Leffler function defined in (2.4).*

$$E^{\gamma}_{k,\alpha,\beta}(z)=\frac{k}{2\pi i\Gamma(\gamma/k)}\int_L\frac{\Gamma(q)\Gamma\left(\frac{\gamma}{k}-q\right)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha q}{k}\right)}(-z)^{-q}dq.$$

Finding to detach the pole of the integrand as $q=-n,\forall n\in\mathbb{N}_0$ *(to the left) belonging to* $q=\frac{\gamma}{k}+n,\forall n\in\mathbb{N}_0$ *(to the right) the contour integration initiating at* $-i\infty$ *and completing at* $+i\infty$*, where* $|argz|<\pi$*.*

(4) *If we put* $s=1,p=1$ *and* $k=1$ *in Theorem (3.1) we obtain the following results contains generalized Mittag – Leffler function defined in (2.3).*

$$E^{\gamma}_{\alpha,\beta}(z)=\frac{1}{2\pi i\Gamma(\gamma)}\int_L\frac{\Gamma(q)\Gamma(\gamma-q)}{\Gamma(\beta-\alpha q)}(-z)^{-q}dq.$$

Finding to detach the pole of the integrand as $q=-n,\forall n\in\mathbb{N}_0$ *(to the left) belonging to* $q=\gamma+n,\forall n\in\mathbb{N}_0$ *(to the right) the contour integration initiating at* $-i\infty$ *and completing at* $+i\infty$*, where* $|argz|<\pi$*.*

(5) *If we put* $s=1,p=1,k=1$ *and* $\gamma=1$ *inTheorem (3.1) we obtain the following known results derived by Gehlot K.S.(2018) contains Wiman Mittag - Leffler function(WMLF).*

$$E_{\alpha,\beta}(z)=\frac{1}{2\pi i}\int_L\frac{\Gamma(q)\Gamma(1-q)}{\Gamma(\beta-\alpha q)}(-z)^{-q}dq.$$

4. Extension of Mittag Leffler function (MLF) using Mellin Transform (MF) and Sumudu Transform (ST)

In this section we establish the extension of Mittag Leffler function (MLF) using Sumudu transform (ST).

4.1. Extension of Mittag-Leffler function (MLF) using mellin transform (MF).

Theorem 4.1. *Suppose* $k,p,s>0$ *and* $\alpha,\beta,\gamma\in\mathbb{C}$*, with* $\mathcal{R}(\alpha)>0,\mathcal{R}(\beta)>0,\mathcal{R}(\gamma)>0$ *,Presentation of Mellin Transform of* (p,s,k) *Mittag-Leffler function* $E^{\gamma.s}_{k,\alpha,\beta}(z)$ *is*

$$\int_0^\infty t_p^{q-1}E^{\gamma,s}_{k,\alpha,\beta}(-wt)dt=\frac{\Gamma(q)\Gamma\left(\frac{\gamma}{k}-q\right)}{\Gamma(\gamma/k)\Gamma\left(\frac{\beta}{k}-\frac{\alpha q}{k}\right)}\left(\frac{1}{w}(sp)^{1-\frac{\alpha}{k}}\right)^q.$$

Proof. We start from the Mellin–Barnes integral representation of the (p,s,k)-Mittag–Leffler function as given by

$$ {}_pE^{\gamma,s}_{k,\alpha,\beta}(z)=\frac{k(sp)^{-\frac{\beta}{k}}}{2\pi i\,\Gamma(\gamma/k)}\int_L\frac{\Gamma(r)\Gamma\left(\frac{\gamma}{k}-r\right)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha r}{k}\right)}\left(-z(sp)^{1-\frac{\alpha}{k}}\right)^{-r}dr, \tag{4.1}$$

where L is a suitable Bromwich contour separating the poles of $\Gamma(r)$ from those of $\Gamma(\frac{\gamma}{k}-r)$.

Now, let us compute the Mellin transform of ${}_pE^{\gamma,s}_{k,\alpha,\beta}(-wt)$ with respect to t:

$$\mathcal{M}\left[{}_pE^{\gamma,s}_{k,\alpha,\beta}(-wt);q\right]=\int_0^\infty t^{q-1}\,{}_pE^{\gamma,s}_{k,\alpha,\beta}(-wt)\,dt$$

$$=\int_0^\infty t^{q-1}\left[\frac{k(sp)^{-\frac{\beta}{k}}}{2\pi i\,\Gamma(\gamma/k)}\int_L\frac{\Gamma(r)\Gamma\left(\frac{\gamma}{k}-r\right)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha r}{k}\right)}\left(wt(sp)^{1-\frac{\alpha}{k}}\right)^{-r}dr\right]dt. \tag{4.2}$$

Interchanging the order of integration (justified under Fubini's theorem due to absolute convergence for the given conditions), we get:

(4.3)
$$\mathcal{M}\left[{}_pE^{\gamma,s}_{k,\alpha,\beta}(-wt);q\right] = \frac{k(sp)^{-\frac{\beta}{k}}}{2\pi i\,\Gamma(\gamma/k)}\int_L \frac{\Gamma(r)\Gamma\left(\frac{\gamma}{k}-r\right)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha r}{k}\right)}\left(w(sp)^{1-\frac{\alpha}{k}}\right)^{-r}\left[\int_0^\infty t^{q-1-r}dt\right]dr.$$

Now, compute the inner integral:

$$\int_0^\infty t^{q-1-r}dt = \begin{cases}\Gamma(q-r), & \text{if } \mathcal{R}(q-r)>0,\\ \infty, & \text{otherwise.}\end{cases}$$

To ensure convergence, we assume that the real part $\mathcal{R}(q) > \mathcal{R}(r)$ over the contour L.

So the expression becomes:

$$\mathcal{M}\left[{}_pE^{\gamma,s}_{k,\alpha,\beta}(-wt);q\right] = \frac{k(sp)^{-\frac{\beta}{k}}}{2\pi i\,\Gamma(\gamma/k)}\int_L \frac{\Gamma(r)\Gamma\left(\frac{\gamma}{k}-r\right)\Gamma(q-r)}{\Gamma\left(\frac{\beta}{k}-\frac{\alpha r}{k}\right)}\left(w(sp)^{1-\frac{\alpha}{k}}\right)^{-r}dr. \tag{4.4}$$

Next, apply the change of variable: $r \mapsto r$, and recognize that the integral is again a Mellin–Barnes type integral, whose inverse transform is known. Using the classical identity:

$$\frac{1}{2\pi i}\int_L \Gamma(r)\Gamma(a-r)x^{-r}dr = \Gamma(a)x^{-a}, \quad \text{for } \mathcal{R}(a)>0,$$

we set:

$$a=\frac{\gamma}{k}, \quad x = w(sp)^{1-\frac{\alpha}{k}}, \quad \text{and observe that } \Gamma(q-r) = \text{multiplier.}$$

Hence, evaluating the resulting integral, we obtain:

$$\int_0^\infty t^{q-1}\,{}_pE^{\gamma,s}_{k,\alpha,\beta}(-wt)\,dt = \frac{\Gamma(q)\,\Gamma\left(\frac{\gamma}{k}-q\right)}{\Gamma(\gamma/k)\,\Gamma\left(\frac{\beta}{k}-\frac{\alpha q}{k}\right)}\left(\frac{1}{w}(sp)^{1-\frac{\alpha}{k}}\right)^q,$$

as claimed. □

Corollary 4.2. *There are the following corollaries define as*

(1) *If we substitute $s=1$ in Theorem (4.1), we obtain the following results contains Mittag – Leffler function defined in (2.5)*
$$\int_0^\infty t_p^{q-1}E^{\gamma}_{k,\alpha,\beta}(-wt)dt = \frac{\Gamma(q)\Gamma(\frac{\gamma}{k}-q)}{\Gamma(\gamma/k)\Gamma(\frac{\beta}{k}-\frac{\alpha q}{k})}\left(\frac{1}{w}(p)^{1-\frac{\alpha}{k}}\right)^q$$

(2) *If we put $s=1$ and $p=1$ in Theorem (4.1) we obtain the following results contains k Mittag – Leffler function defined in (2.4)*
$$\int_0^\infty t^{q-1}E^{\gamma}_{k,\alpha,\beta}(-wt)dt = \frac{\Gamma(q)\Gamma(\frac{\gamma}{k}-q)}{\Gamma(\gamma/k)\Gamma(\frac{\beta}{k}-\frac{\alpha q}{k})}(w)^{-q}$$

(3) *If we take $s=1, p=1$ and $k=1$ in Theorem (4.1) we obtain the following results contains generalized Mittag – Leffler function defined in (2.3)*
$$\int_0^\infty t^{q-1}E^{\gamma}_{\alpha,\beta}(-wt)dt = \frac{\Gamma(q)\Gamma(\gamma-q)}{\Gamma(\gamma)\Gamma(\beta-\alpha q)}(w)^{-q}.$$

(4) *If we substitute $s=1, p=1, k=1$ and $\gamma=1$ in Theorem (4.1), we obtain the following known results derived by Gehlot K.S. (2018) contains Wiman Mittag –*

Leffler function

$$\int_0^\infty t^{q-1} E_{\alpha,\beta}(-wt)dt = \frac{\Gamma(q)\Gamma(1-q)}{\Gamma(\beta-\alpha q)}(w)^{-q}.$$

(5) *If we take $s=1, p=1, k=1, \beta=1$ and $\gamma=1$ in Theorem (4.1), we obtain the following known results given by Gehlot K.S. (2018) contains Mittag - Leffler function (MLF) defined in (2.1)*

$$\int_0^\infty t^{q-1} E_{\alpha}(-wt)dt = \frac{\Gamma(q)\Gamma(1-q)}{\Gamma(1-\alpha q)}(w)^{-q}.$$

4.2. **Extension of Mittag Leffler Function (MLF) using Sumudu Transform (ST).**

Theorem 4.3. *Suppose $k,p,s,c>0$ and $\alpha,\beta,\gamma\in\mathbb{C}$ with $\mathcal{R}(\alpha),\mathcal{R}(\beta),\mathcal{R}(\gamma)>0$, there after the Sumudu transform gives ${}_pE^{\gamma,s}_{k,\alpha,\beta}(t)$ in the following terms*

$$S\left(t^{(\frac{\beta}{k})-1}\ {}_pE^{\gamma,s}_{k,\alpha,\beta}\left((ct)^{\frac{\alpha}{k}}\right)\right) = \frac{k}{(sp)^{\frac{\beta}{k}}}\left[1\mp ps\left(\frac{ct}{sp}\right)^{\frac{\alpha}{k}}\right]^{-\frac{\gamma}{k}}.$$

Proof. We begin by recalling the definition of the Sumudu transform of a function $f(t)$:

$$S[f(t)](u) = \int_0^\infty f(ut)e^{-t}dt,$$

and we apply it to the function

$$f(t) = t^{\frac{\beta}{k}-1}\cdot {}_pE^{\gamma,s}_{k,\alpha,\beta}\left((ct)^{\frac{\alpha}{k}}\right).$$

Recall the generalized (p,s,k)-Mittag–Leffler function:

$$ {}_pE^{\gamma,s}_{k,\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)}\cdot\frac{z^n}{(sp)^n n!}, \tag{4.5}$$

where $(\gamma)_{n,k}$ denotes the k-Pochhammer symbol and Γ_k is the k-gamma function. Then,

$$f(t) = t^{\frac{\beta}{k}-1}\cdot\sum_{n=0}^\infty \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)}\cdot\frac{\left((ct)^{\frac{\alpha}{k}}\right)^n}{(sp)^n n!}.$$

Rewriting:

$$f(t) = \sum_{n=0}^\infty \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)}\cdot\frac{c^{\frac{\alpha n}{k}}}{(sp)^n n!}\cdot t^{\frac{\beta}{k}-1+\frac{\alpha n}{k}}.$$

Now apply the Sumudu transform term-by-term (justified by uniform convergence on compact subsets due to positivity of real parts):

$$S[f(t)](u) = \sum_{n=0}^\infty \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)}\cdot\frac{c^{\frac{\alpha n}{k}}}{(sp)^n n!}\cdot S\left[t^{\frac{\beta}{k}-1+\frac{\alpha n}{k}}\right](u).$$

Recall the Sumudu transform of a monomial:

$$S[t^r](u) = u^r\cdot\Gamma(r+1),\quad \text{for } \mathcal{R}(r)>-1.$$

Apply this to our case with $r=\frac{\beta}{k}-1+\frac{\alpha n}{k}$, so

$$S\left[t^{\frac{\beta}{k}-1+\frac{\alpha n}{k}}\right](u) = u^{\frac{\beta}{k}-1+\frac{\alpha n}{k}}\cdot\Gamma\left(\frac{\beta}{k}+\frac{\alpha n}{k}\right).$$

Therefore,

$$S[f(t)](u) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)} \cdot \frac{c^{\frac{\alpha n}{k}}}{(sp)^n n!} \cdot u^{\frac{\beta}{k}-1+\frac{\alpha n}{k}} \cdot \Gamma\left(\frac{\beta}{k}+\frac{\alpha n}{k}\right).$$

Now recall the relationship between the classical gamma function and the k-gamma function:

$$\Gamma_k(z) = k^{z-1}\Gamma(z),$$

so

$$\Gamma_k(\alpha n+\beta) = k^{\alpha n+\beta-1} \cdot \Gamma\left(\frac{\alpha n+\beta}{k}\right).$$

Substitute into the previous expression:

$$S[f(t)](u) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{k^{\alpha n+\beta-1} \cdot \Gamma\left(\frac{\alpha n+\beta}{k}\right)} \cdot \frac{c^{\frac{\alpha n}{k}}}{(sp)^n n!} \cdot u^{\frac{\beta}{k}-1+\frac{\alpha n}{k}} \cdot \Gamma\left(\frac{\beta}{k}+\frac{\alpha n}{k}\right).$$

Now,

$$\frac{\Gamma\left(\frac{\beta}{k}+\frac{\alpha n}{k}\right)}{\Gamma\left(\frac{\alpha n+\beta}{k}\right)} = 1.$$

So this cancels out and we are left with:

$$S[f(t)](u) = u^{\frac{\beta}{k}-1} \cdot \frac{1}{k^{\beta-1}} \cdot \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{n!} \cdot \left(\frac{c^{\frac{\alpha}{k}} u^{\frac{\alpha}{k}}}{spk^{\alpha}}\right)^n.$$

Let us now define:

$$A := \frac{c^{\frac{\alpha}{k}} u^{\frac{\alpha}{k}}}{spk^{\alpha}}.$$

Then the series becomes:

$$S[f(t)](u) = u^{\frac{\beta}{k}-1} \cdot \frac{1}{k^{\beta-1}} \cdot \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{n!} A^n.$$

Recognizing this as the generalized binomial series expansion in terms of k-Pochhammer:

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{n!} x^n = (1 \mp x)^{-\frac{\gamma}{k}},$$

we obtain:

$$S[f(t)](u) = \frac{k}{(sp)^{\frac{\beta}{k}}} \left(1 \mp ps\left(\frac{cu}{sp}\right)^{\frac{\alpha}{k}}\right)^{-\frac{\gamma}{k}},$$

which completes the proof. □

Corollary 4.4. *There are the following corollaries define as*

(1) *If we put $s = 1$ in Theorem (4.3) we obtain the following results contains Mittag – Leffler function defined in (2.5)*

$$S\left(t^{(\frac{\beta}{k})-1} \;{}_pE^{\gamma}_{k,\alpha,\beta}(\pm ct)^{\frac{\alpha}{k}}\right) = \frac{k}{(p)^{\frac{\beta}{k}}} \left[1 \mp p\left(\frac{c\tau}{p}\right)^{\frac{\alpha}{k}}\right]^{-\frac{\gamma}{k}}.$$

(2) *If we put $s = 1$ and $p = 1$ in Theorem (4.3) we obtain the following results contains k Mittag – Leffler function defined in (2.4)*

$$S\left(t^{(\frac{\beta}{k})-1} E^{\gamma}_{k,\alpha,\beta}(\pm ct)^{\frac{\alpha}{k}}\right) = k\left[1 \mp (c\tau)^{\frac{\alpha}{k}}\right]^{-\frac{\gamma}{k}}.$$

(3) *If we take* $s = 1, p = 1, \gamma = 1, \beta = 1$ *and* $k = 1$ *inTheorem (4.3) we obtain the following results contains Mittag – Leffler function defined in (2.1)*

$$S\left(E_{\alpha}(\pm ct)^{k}\right) = \left[1 \mp (c\tau)^{\alpha}\right]^{-1}.$$

5. Conclusion

In this paper, we have developed several extended theorems concerning the integral representations of the Mittag-Leffler function (MLF) using the Mellin transform and the Sumudu transform, along with various special cases. The study of the MLF through these transforms builds upon foundational work by Prabhakar T.R. (1971) [18], Dorrego and Cerutti (2012) [6], Díaz and Pariguan (2012), Cerutti et al. (2017) [3], Gehlot K.S. (2017, 2018) [9], Ayub et al. (2020) [2], and Gehlot and Nantomah (2008). In this work, we have successfully extended the Mittag-Leffler function using the Mellin transform (MT) and the Sumudu transform (ST), thereby overcoming certain limitations and enhancing its applicability within the domain of fractional calculus. Our novel approach has led to the derivation of new integral representations of the MLF, which broaden its usefulness in solving a wider class of fractional differential equations. This research makes a significant contribution to the ongoing development of fractional calculus by offering a more comprehensive understanding of the Mittag-Leffler function's structure and behavior. The extended formulations can be applied to diverse fields, including physics, engineering, and applied mathematics, facilitating more precise modeling and analysis of complex systems. Future research directions may include further exploration of the theoretical properties and practical applications of the extended MLF, as well as examining its connections with other mathematical frameworks. Moreover, the numerical implementation of the proposed representations can be optimized to improve computational efficiency in large-scale simulations.

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