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Research Paper

DEPTH OF AN IDEAL ON A PAIR OF MODULES

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ABSTRACT

Let R be a commutative Noetherian ring and I an ideal of R. Suppose that S is a Serre subcategory of the category of R-modules which satisfies the condition C_I. Let M be a ZD-module and N an R-module. As a generalization of the notion of S- $\operatorname{depth}(I, M)$, we define the S- depth of I on the pair (N, M) by S- $\operatorname{depth}(I, N, M) := S$- $\operatorname{depth}(\operatorname{Ann}_R(N/IN), M)$. We investigate the connections between S- $\operatorname{depth}(I, N, M)$, local cohomology modules, and Ext functors. In particular, when N is finitely generated, it is shown that S- $\operatorname{depth}(I, N, M) = \inf\{i : H^i_I(N, M) \notin S\} = \inf\{i : \operatorname{Ext}^i_R(N/IN, M) \notin S\}$. Moreover, various formulas are provided that relate this generalized S- depth to other notions of depth in the literature.

1. Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring with non-zero identity, I is an ideal of R, and M and N are two R-modules. We also consider S to be a Serre subcategory of the category of R-modules, i.e., S is closed under taking submodules, quotients, and extensions. One can see that the class of finitely generated modules, Artinian

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modules, minimax modules, weakly Laskerian modules, and Matlis reflexive modules are examples of Serre subcategories.

Aghapournahr and Melkersson [1] generalized the ordinary notion of regular sequences by introducing the concept of S-sequences with respect to a Serre subcategory S. An element $a \in R$ is called S-regular on an R-module M if $(0 :_M a) \in S$. A sequence $a_1, \dots, a_t$ is said to be an S-sequence on M if for each $i = 1, \dots, t$, the element a_i is S-regular on $M/(a_1, \dots, a_{i-1})M$. Also, when S satisfies a special condition called the condition C_I, they introduced the S-depth of an ideal I on M, denoted by $S\text{-}\operatorname{depth}(I, M)$. More precisely, a Serre subcategory S satisfies the condition C_I if, for any I-torsion R-module M, $(0 :_M I) \in S$ implies $M \in S$. Examples of Serre subcategories satisfying the condition C_I include the zero module, Artinian modules, I-cofinite Artinian modules, modules with finite support, and modules N with $\dim_R N \leq k$ for some non-negative integer k. If S satisfies the condition C_I, and M is a finitely generated R-module such that $M/IM \notin S$, then any maximal S-sequence in I has the same length. This common length is defined to be $S\text{-}\operatorname{depth}(I, M)$. By choosing appropriate Serre subcategories S, one recovers various depth concepts studied in the literature, including the ordinary depth, filter depth ($f\text{-}\operatorname{depth}(I, M)$), generalized depth ($g\text{-}\operatorname{depth}(I, M)$), etc.

The notion of ZD-modules (zero-divisor modules) was introduced by Evans in [5]. An R-module M is said to be a ZD-module if, for every submodule N of M, the set of zero-divisors of M/N is a finite union of associated prime ideals of M/N. As noted in [4, Example 2.2], the class of ZD-modules includes a wide variety of important modules: finitely generated modules, Laskerian and weakly Laskerian modules, linearly compact modules, Matlis reflexive modules, and minimax modules. Moreover, this class contains modules whose quotients have finite Goldie dimension, as well as those with finite support, particularly including all Artinian modules.

In [7], we extended the concept of $S\text{-}\operatorname{depth}(I, M)$ to the class of ZD-modules. More specifically, let S be a Serre subcategory satisfying the condition C_I, and let M be a ZD-module. Assuming that the ideal I contains a maximal S-sequence on M, we proved that all such maximal S-sequences in I have equal length. Moreover, we showed that if $M/IM \notin S$, then I indeed contains maximal S-sequences on M. Additionally, we generalized the notion of $S\text{-}\operatorname{depth}(I, M)$ to pairs of ideals. For two ideals I and J of R, and a ZD-module M, we defined $S\text{-}\operatorname{depth}(I, J, M)$ under the assumption that S satisfies the condition C_I; see [8] for details.

This paper aims to generalize the concept of S-depth to pairs of modules. Specifically, let S be a Serre subcategory satisfying the condition C_I, M be a ZD-module, and N an arbitrary R-module. We define the S-depth of the ideal I on N and M, denoted by $S\text{-}\operatorname{depth}(I, N, M)$, as $S\text{-}\operatorname{depth}(I, N, M) := S\text{-}\operatorname{depth}(\operatorname{Ann}_R(N/IN), M)$. It is immediate that if $N = R$, then this definition coincides with the $S\text{-}\operatorname{depth}(I, M)$.

In Section 2, we establish some fundamental properties of $S\text{-}\operatorname{depth}(I, N, M)$ and explore its connections with local cohomology and Ext functors. Assuming S satisfies the condition C_I, M is a ZD-module, and N is a finitely generated R-module, we prove in Theorems 2.7 and 2.9 that $S\text{-}\operatorname{depth}(I, N, M) = \inf\{i : H^i_I(N, M) \notin S\} = \inf\{i : \operatorname{Ext}^i_R(N/IN, M) \notin S\}$.

In Section 3, we explore the connections among various notions of depth. Let S be a Serre subcategory satisfying condition C_I, and let M and N be finitely generated R-modules.

Theorem 3.2 establishes that $S\text{-}\operatorname{depth}(I,N,M) = \inf\{\operatorname{depth}(IR_{\mathfrak{p}},N_{\mathfrak{p}},M_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N)$ and $R/\mathfrak{p} \notin S\}$. Furthermore, Theorem 3.4 states that if S is closed under taking injective hulls, M is a ZD-module, and N is finitely generated, then $S\text{-}\operatorname{depth}(I,N,M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N)$ and $R/\mathfrak{p} \notin S\}$.

2. S-depth of an ideal on a pair of modules and local cohomology

Recall that R is a Noetherian ring, I is an ideal of R, M and N are R-modules, and S is a Serre subcategory of the category of R-modules.

Definition 2.1. Suppose that S satisfies the condition C_I, M is a ZD-module, and N is an arbitrary R-module. The S-depth of ideal I on N and M, denoted by $S\text{-}\operatorname{depth}(I,N,M)$, is defined as $S\text{-}\operatorname{depth}(I,N,M) := S\text{-}\operatorname{depth}(\operatorname{Ann}_R(N/IN),M)$.

It is straightforward to see that when $N{=}R$, the equality $S\text{-depth}(I,N,M){=}S\text{-depth}(I,M)$ holds.

In the following results, we establish some fundamental properties of $S\text{-}\operatorname{depth}(I,N,M)$, which are helpful in the computation of this invariant.

Proposition 2.2. *Assume that S satisfies the condition C_I, M is a ZD-module, and J is an ideal of R. Then*

(i) $S\text{-}\operatorname{depth}(I,M) \le S\text{-}\operatorname{depth}(I,N,M)$.
(ii) *If* $I \subseteq J$, *then* $S\text{-}\operatorname{depth}(I,N,M) \le S\text{-}\operatorname{depth}(J,N,M)$.
(iii) $S\text{-}\operatorname{depth}(I+J,N,M) = S\text{-}\operatorname{depth}(I,\frac{N}{JN},M)$.
(iv) $S\text{-}\operatorname{depth}(\frac{I+J}{J},\frac{N}{JN},\frac{M}{JM}) = S\text{-}\operatorname{depth}(I,\frac{N}{JN},\frac{M}{JM})$.

Proof. Parts (i) and (ii) directly follow from [7, Proposition 3.1(i)]. For parts (iii) and (iv), observe that $\frac{\frac{N}{JN}}{(\frac{I+J}{J})(\frac{N}{JN})} \cong \frac{\frac{N}{JN}}{I(\frac{N}{JN})} \cong \frac{\frac{N}{JN}}{\frac{(I+J)N}{JN}} \cong \frac{N}{(I+J)N}$. Therefore, we have $\operatorname{Ann}_R(\frac{N}{(I+J)N}) = \operatorname{Ann}_R(\frac{\frac{N}{JN}}{I(\frac{N}{JN})})$ which proves part (iii). Furthermore, note that $\operatorname{Ann}_{\frac{R}{J}}(\frac{\frac{N}{JN}}{(\frac{I+J}{J})(\frac{N}{JN})})$ contains an S-sequence on $\frac{M}{JM}$, if and only if, $\operatorname{Ann}_R(\frac{\frac{N}{JN}}{(\frac{I+J}{J})(\frac{N}{JN})}) = \operatorname{Ann}_R(\frac{\frac{N}{JN}}{I(\frac{N}{JN})})$ contains an S-sequence on M. Thus, we obtain $S\text{-}\operatorname{depth}(\frac{I+J}{J},\frac{N}{JN},\frac{M}{JM}) = S\text{-}\operatorname{depth}(I,\frac{N}{JN},\frac{M}{JM})$ which is the same as part (iv). □

Proposition 2.3. *Suppose that S satisfies the condition C_I, M is a ZD-module, and $\underline{a} = a_1,\dots,a_t$ is an S-sequence on M. Then*

(i) *If* $\underline{a} \in \operatorname{Ann}_R(N/IN)$, *then* $S\text{-}\operatorname{depth}(I,N,\frac{M}{(\underline{a})M}) = S\text{-}\operatorname{depth}(I,N,M) - t$.
(ii) *If* $\underline{a} \in I$, *then*

$$\begin{aligned} S\text{-}\operatorname{depth}(\frac{I}{(\underline{a})},\frac{N}{(\underline{a})N},\frac{M}{(\underline{a})M}) &= S\text{-}\operatorname{depth}(I,\frac{N}{(\underline{a})N},\frac{M}{(\underline{a})M}) = S\text{-}\operatorname{depth}(I,N,\frac{M}{(\underline{a})M}) \\ &= S\text{-}\operatorname{depth}(I,N,M) - t. \end{aligned}$$

Proof. Part (i) follows from [7, Proposition 3.1(iv)]. Part (ii) follows from Proposition 2.2 and part (i). □

In the following result, we examine how the invariant $S\text{-}\operatorname{depth}(I,N,M)$ behaves with respect to exact sequences.

Proposition 2.4. *Suppose that S satisfies the condition C_I, and consider the exact sequence of ZD-modules* $0 \longrightarrow L \longrightarrow M \to U \longrightarrow 0$. *Then the following inequalities hold:*

(i) $S\text{-}\operatorname{depth}(I, N, M) \geq \min\{S\text{-}\operatorname{depth}(I, N, L), S\text{-}\operatorname{depth}(I, N, U)\}$.

(ii) $S\text{-}\operatorname{depth}(I, N, L) \geq \min\{S\text{-}\operatorname{depth}(I, N, M), S\text{-}\operatorname{depth}(I, N, U) + 1\}$.

(iii) $S\text{-}\operatorname{depth}(I, N, U) \geq \min\{S\text{-}\operatorname{depth}(I, N, L) - 1, S\text{-}\operatorname{depth}(I, N, M)\}$.

Proof. All parts follow from [7, Proposition 3.2]. □

The following lemma plays a key role in the subsequent results.

Lemma 2.5. *Suppose that S satisfies the condition C_I, M is a ZD-module, and N is a finitely generated R-module. Then* $S\text{-}\operatorname{depth}(I, N, M) = S\text{-}\operatorname{depth}(I + \operatorname{Ann}_R(N), M)$.

Proof. According to [9, Lemma 1.32], it follows that $\sqrt{\operatorname{Ann}_R(N/IN)} = \sqrt{I + \operatorname{Ann}_R(N)}$. Now, applying [7, Proposition 3.1(ii)], we get that

$$\begin{aligned} S\text{-}\operatorname{depth}(I, N, M) &= S\text{-}\operatorname{depth}(\operatorname{Ann}_R(N/IN), M) = S\text{-}\operatorname{depth}(\sqrt{\operatorname{Ann}_R(N/IN)}, M) \\ &= S\text{-}\operatorname{depth}(\sqrt{I + \operatorname{Ann}_R(N)}, M) = S\text{-}\operatorname{depth}(I + \operatorname{Ann}_R(N), M). \end{aligned}$$

□

Proposition 2.6. *Suppose that S satisfies the condition C_I, M is a ZD-module, and N is a finitely generated R-module. Then*

(i) $S\text{-}\operatorname{depth}(I, N, M) = S\text{-}\operatorname{depth}(\sqrt{I}, N, M)$.

(ii) *If J is an ideal of R, then* $S\text{-}\operatorname{depth}(IJ, N, M) = S\text{-}\operatorname{depth}(I \cap J, N, M)$.

Proof. It follows from [10, Exercise 2.25] that $\sqrt{I + \operatorname{Ann}_R(N)} = \sqrt{\sqrt{I} + \operatorname{Ann}_R(N)}$. Now, using Lemma 2.5 along with [7, Proposition 3.1(ii)], we obtain

$$\begin{aligned} S\text{-}\operatorname{depth}(I, N, M) &= S\text{-}\operatorname{depth}(I + \operatorname{Ann}_R(N), M) = S\text{-}\operatorname{depth}(\sqrt{I + \operatorname{Ann}_R(M)}, M) \\ &= S\text{-}\operatorname{depth}(\sqrt{\sqrt{I} + \operatorname{Ann}_R(M)}, M) = S\text{-}\operatorname{depth}(\sqrt{I} + \operatorname{Ann}_R(M), M) \\ &= S\text{-}\operatorname{depth}(\sqrt{I}, N, M), \end{aligned}$$

and the equality established in part (i) holds. Part (ii) follows directly from part (i). □

Now, we obtain some formulas on the relations between the S-depth of an ideal on a pair of modules, and local cohomology and Ext functors. The following theorem serves as a generalization of the results found in [2, Theorem 3.1], [3, Theorem 2.2], and [6, Remark 2.4].

Theorem 2.7. *Suppose that S satisfies the condition C_I, M is a ZD-module, and N is a finitely generated R-module. Then* $S\text{-}\operatorname{depth}(I, N, M) = \inf\{i : H_I^i(N, M) \notin S\}$.

Proof. Using Lemma 2.5, together with [7, Lemma 3.1] and [11, Corollary 2.14], we obtain that

$$\begin{aligned} S\text{-}\operatorname{depth}(I, N, M) &= S\text{-}\operatorname{depth}(I + \operatorname{Ann}_R(N), M) = \inf\{i : H^i_{I+\operatorname{Ann}_R(N)}(M) \notin S\} \\ &= \inf\{i : H_I^i(N, M) \notin S\}. \end{aligned}$$

□

Lemma 2.8. *Suppose that S satisfies the condition C_I, M is a ZD-module, and N is a finitely generated R-module. Then* $S\text{-}\operatorname{depth}(\operatorname{Ann}_R(N), M) = \inf\{i : \operatorname{Ext}^i_R(N, M) \notin S\}$.

Proof. From [7, Lemma 3.1] and [1, Theorem 2.9], it follows that

$$S\text{-}\operatorname{depth}(\operatorname{Ann}_R(N), M) = \inf\{i : \operatorname{Ext}^i_R(R/\operatorname{Ann}_R(N), M) \notin S\} = \inf\{i : \operatorname{Ext}^i_R(N, M) \notin S\}.$$

□

Theorem 2.9. *Suppose that S satisfies the condition C_I, M is a ZD-module, and N is a finitely generated R-module. Then* $S\text{-}\operatorname{depth}(I, N, M) = \inf\{i : \operatorname{Ext}^i_R(N/IN, M) \notin S\}$.

Proof. The statement is a direct consequence of Lemma 2.8. □

3. Some relations between different types of depths

In this section, we examine the relationships among various notions of depth. The following lemma serves as a fundamental component in the proof of the upcoming theorem.

Lemma 3.1. *Suppose that S satisfies the condition C_I, and M is a finitely generated R-module. Then* $S\text{-}\operatorname{depth}(I, M) = \inf\{\operatorname{depth}(IR_\mathfrak{p}, M_\mathfrak{p}) : \mathfrak{p} \in \mathrm{V}(I)$ *and* $R/\mathfrak{p} \notin S\}$.

Proof. The proof is analogous to the argument given in the proof of [1, Theorem 2.18(d)]. Put $t = S\text{-}\operatorname{depth}(I, M)$. It follows from [7, Lemma 3.1] that $\operatorname{Ext}^i_R(R/I, M) \in S$ for all $i < t$, and $\operatorname{Ext}^t_R(R/I, M) \notin S$. Suppose that $\mathfrak{p} \in \mathrm{V}(I)$ and $R/\mathfrak{p} \notin S$. By [1, Lemma 2.17], we have $\mathfrak{p} \notin \operatorname{Supp}_R(\operatorname{Ext}^i_R(R/I, M))$ for all $i < t$, and hence $\operatorname{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, M_\mathfrak{p}) = 0$ for all $i < t$. Now, it follows from [7, Lemma 3.1] that $\operatorname{depth}(IR_\mathfrak{p}, M_\mathfrak{p}) \geq t$, and so $t \leq \inf\{\operatorname{depth}(IR_\mathfrak{p}, M_\mathfrak{p}) : \mathfrak{p} \in \mathrm{V}(I)$ and $R/\mathfrak{p} \notin S\}$. On the other hand, since $\operatorname{Ext}^t_R(R/I, M) \notin S$, it follows from [1, Lemma 2.17] that there exists $\mathfrak{q} \in \operatorname{Supp}_R(\operatorname{Ext}^t_R(R/I, M))$ such that $R/\mathfrak{q} \notin S$. Therefore $\operatorname{Ext}^t_{R_\mathfrak{q}}(R_\mathfrak{q}/IR_\mathfrak{q}, M_\mathfrak{q})) \neq 0$, and hence $\mathfrak{q} \in \mathrm{V}(I)$. Now, by reusing [7, Lemma 3.1], we have $\operatorname{depth}(IR_\mathfrak{q}, M_\mathfrak{q}) = t$, and the claim follows. □

Theorem 3.2. *Suppose that S satisfies the condition C_I, and M and N are two finitely generated R-modules. Then* $S\text{-}\operatorname{depth}(I, N, M) = \inf\{\operatorname{depth}(IR_\mathfrak{p}, N_\mathfrak{p}, M_\mathfrak{p}) : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N)$ *and* $R/\mathfrak{p} \notin S\}$.

Proof. It is easy to see that, $(\operatorname{Ann}_R(N))R_\mathfrak{p} = \operatorname{Ann}_{R_\mathfrak{p}}(N_\mathfrak{p})$ for any prime ideal $\mathfrak{p}$ of R. Now, it follows from Lemma 2.5 and Lemma 3.1 that

$$\begin{aligned}
& S\text{-}\operatorname{depth}(I, N, M) = S\text{-}\operatorname{depth}(I + \operatorname{Ann}_R(N), M) \\
= \; & \inf\{\operatorname{depth}((I + \operatorname{Ann}_R(N))R_\mathfrak{p}, M_\mathfrak{p}) : \mathfrak{p} \in \mathrm{V}(I + \operatorname{Ann}_R(N)) \text{ and } R/\mathfrak{p} \notin S\} \\
= \; & \inf\{\operatorname{depth}(IR_\mathfrak{p} + (\operatorname{Ann}_R(N))R_\mathfrak{p}, M_\mathfrak{p}) : \mathfrak{p} \in \mathrm{V}(I) \cap \mathrm{V}(\operatorname{Ann}_R(N)) \text{ and } R/\mathfrak{p} \notin S\} \\
= \; & \inf\{\operatorname{depth}(IR_\mathfrak{p} + \operatorname{Ann}_{R_\mathfrak{p}}(N_\mathfrak{p}), M_\mathfrak{p}) : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N) \text{ and } R/\mathfrak{p} \notin S\} \\
= \; & \inf\{\operatorname{depth}(IR_\mathfrak{p}, N_\mathfrak{p}, M_\mathfrak{p}) : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N) \text{ and } R/\mathfrak{p} \notin S\}.
\end{aligned}$$

□

Proposition 3.3. *Suppose that S satisfies the condition C_I, and M and N are two finitely generated R-modules. Then*

(i) $S\text{-}\operatorname{depth}(I, N, M) = \inf\{S\text{-}\operatorname{depth}(\mathfrak{p}, N, M) : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N)$ *and* $R/\mathfrak{p} \notin S\}$.

(ii) *If L is a finitely generated R-module such that* $\operatorname{Supp}_R(L) \subseteq \operatorname{Supp}_R(N)$*, then* $S\text{-}\operatorname{depth}(I, N, M) \leq S\text{-}\operatorname{depth}(I, L, M)$.

(iii) *If J is an ideal of R, then*

$$S\text{-}\operatorname{depth}(I \cap J, N, M) = \min\{S\text{-}\operatorname{depth}(I, N, M), S\text{-}\operatorname{depth}(J, N, M)\}.$$

Proof. First, we prove the equality of part (i). By Proposition 2.2(ii), we have

$$S\text{-}\operatorname{depth}(I, N, M) \leq S\text{-}\operatorname{depth}(\mathfrak{p}, N, M)$$

for all $\mathfrak{p} \in \mathrm{V}(I)$. Therefore

$$S\text{-}\operatorname{depth}(I, N, M) \leq \inf\{S\text{-}\operatorname{depth}(\mathfrak{p}, N, M) : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N) \text{ and } R/\mathfrak{p} \notin S\}.$$

To prove the converse inequality, suppose that $\mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N)$ and $R/\mathfrak{p} \notin S$. According to Theorem 3.2, we have

$$\begin{aligned} S\text{-}\operatorname{depth}(\mathfrak{p}, N, M) &= \inf\{\operatorname{depth}(IR_\mathfrak{q}, N_\mathfrak{q}, M_\mathfrak{q}) : \mathfrak{q} \in \mathrm{V}(\mathfrak{p}) \cap \operatorname{Supp}_R(N) \text{ and } R/\mathfrak{q} \notin S\} \\ &\leq \operatorname{depth}(IR_\mathfrak{p}, N_\mathfrak{p}, M_\mathfrak{p}). \end{aligned}$$

By applying Theorem 3.2 once again, we obtain

$$\begin{aligned} & \inf\{S\text{-}\operatorname{depth}(\mathfrak{p}, N, M) : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N) \text{ and } R/\mathfrak{p} \notin S\} \\ \leq\ & \inf\{\operatorname{depth}(IR_\mathfrak{p}, N_\mathfrak{p}, M_\mathfrak{p}) : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N) \text{ and } R/\mathfrak{p} \notin S\} \\ =\ & S\text{-}\operatorname{depth}(I, N, M) \end{aligned}$$

and the equality of part (i) follows. Parts (ii) and (iii) directly follow from part (i). □

Aghapournahr and Melkersson [1] established that any Serre subcategory that is closed under taking injective hulls necessarily satisfies the condition C_I. Examples of such Serre subcategories include the class of zero modules, Artinian modules, modules with finite support, and the class of R-modules N with $\dim_R N \leq k$, where k is a non-negative integer. Moreover, the class of I-cofinite Artinian modules forms a Serre subcategory of the category of R-modules that satisfies the condition C_I; however, it does not remain closed under taking injective hulls.

Theorem 3.4. *Suppose that S is a Serre subcategory closed under taking injective hulls, M is a ZD-module, and N is a finitely generated R-module. Then*

$$S\text{-}\operatorname{depth}(I, N, M) = \inf\{\operatorname{depth} M_\mathfrak{p} : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N) \text{ and } R/\mathfrak{p} \notin S\}.$$

Proof. From Lemma 2.5 and [7, Theorem 3.1], it follows that

$$\begin{aligned} S\text{-}\operatorname{depth}(I, N, M) &= S\text{-}\operatorname{depth}(I + \operatorname{Ann}_R(N), M) \\ &= \inf\{\operatorname{depth} M_\mathfrak{p} : \mathfrak{p} \in \mathrm{V}(I + \operatorname{Ann}_R(N)) \text{ and } R/\mathfrak{p} \notin S\} \\ &= \inf\{\operatorname{depth} M_\mathfrak{p} : \mathfrak{p} \in \mathrm{V}(I) \cap \mathrm{V}(\operatorname{Ann}_R(N)) \text{ and } R/\mathfrak{p} \notin S\} \\ &= \inf\{\operatorname{depth} M_\mathfrak{p} : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N) \text{ and } R/\mathfrak{p} \notin S\}. \end{aligned}$$

□

Proposition 3.5. *Suppose that S is a Serre subcategory closed under taking injective hulls, M is a ZD-module, and N is a finitely generated R-module. Then*

(i) $S\text{-}\operatorname{depth}(I, N, M) = \inf\{S\text{-}\operatorname{depth}(\mathfrak{p}, N, M) : \mathfrak{p} \in \mathrm{V}(I) \cap \operatorname{Supp}_R(N)$ *and* $R/\mathfrak{p} \notin S\}$.

(ii) *If* L *is a finitely generated* R*-module such that* $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(N)$*, then* $S\text{-}\operatorname{depth}(I, N, M) \leq S\text{-}\operatorname{depth}(I, L, M)$.

(iii) *If* J *is an ideal of* R*, then*

$$S\text{-}\operatorname{depth}(I \cap J, N, M) = \min\{S\text{-}\operatorname{depth}(I, N, M), S\text{-}\operatorname{depth}(J, N, M)\}.$$

Proof. The proof, included here to assist the reader, follows a method similar to that employed in the proof of Proposition 3.3. First, we prove the equality of part (i). It follows from Proposition 2.2(ii) that $S\text{-}\operatorname{depth}(I, N, M) \leq S\text{-}\operatorname{depth}(\mathfrak{p}, N, M)$ for all $\mathfrak{p} \in \mathrm{V}(I)$. Therefore

$$S\text{-}\operatorname{depth}(I, N, M) \leq \inf\{S\text{-}\operatorname{depth}(\mathfrak{p}, N, M) : \mathfrak{p} \in \mathrm{V}(I) \cap \mathrm{Supp}_R(N) \text{ and } R/\mathfrak{p} \notin S\}.$$

To prove the converse inequality, suppose that $\mathfrak{p} \in \mathrm{V}(I) \cap \mathrm{Supp}_R(N)$ and $R/\mathfrak{p} \notin S$. According to Theorem 3.4, we have

$$S\text{-}\operatorname{depth}(\mathfrak{p}, N, M) = \inf\{\operatorname{depth} M_{\mathfrak{q}} : \mathfrak{q} \in \mathrm{V}(\mathfrak{p}) \cap \mathrm{Supp}_R(N) \text{ and } R/\mathfrak{q} \notin S\} \leq \operatorname{depth} M_{\mathfrak{p}}.$$

By applying Theorem 3.4 once again, we obtain

$$\begin{aligned} & \inf\{S\text{-}\operatorname{depth}(\mathfrak{p}, N, M) : \mathfrak{p} \in \mathrm{V}(I) \cap \mathrm{Supp}_R(N) \text{ and } R/\mathfrak{p} \notin S\} \\ \leq\ & \inf\{\operatorname{depth} M_{\mathfrak{p}} : \mathfrak{p} \in \mathrm{V}(I) \cap \mathrm{Supp}_R(N) \text{ and } R/\mathfrak{p} \notin S\} \\ =\ & S\text{-}\operatorname{depth}(I, N, M) \end{aligned}$$

and the equality of part (i) follows. Parts (ii) and (iii) directly follow from part (i). □

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