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Research Paper

FIXED POINT OF MULTI-VALUED ZAMFIRESCU OPERATOR AND CONVERGENCE RESULTS IN MODULAR METRIC SPACES ENDOWED WITH GRAPH

KIRAN DEWANGAN[1,*], JAY KUMAR DEWANGAN[2]

[1]Department of Mathematics, Government Dudhadhari Bajrang Girls Postgraduate Autonomous College, Raipur (Chhattisgarh)-492001, India, dewangan.kiran@gmail.com

[2]Department of Management, Anjaneya University, Raipur, (Chhattisgarh)-492001, India, jkdewangan@gmail.com

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ABSTRACT

This paper contains some convergence results and fixed point theorems of multi-valued Zamfirescu operator along with a numerical example in the framework of a complete modular metric space endowed with graph. An application of fixed point theory in the solution of a system of equations for multi-valued Zamfirescu operator is described here.

1. Introduction

Throughout the literature, some notations are used, which are as follows:

$$MS = Metric\ space$$

$$CMS = Complete\ metric\ space$$

$$MMS = Modular\ metric\ space$$

$$CMMS = Complete\ Modular\ metric\ space$$

*Address correspondence to K. Dewangan, Department of Mathematics, Government Dudhadhari Bajrang Girls Postgraduate Autonomous College, Raipur (Chhattisgarh)-492001, India, dewangan.kiran@gmail.com.

In mathematics, different mathematical problems like the system of equations, image recovery problem, signal processing, etc., can be solved by using fixed point theory. In solving non-linear equations, the "Banach contraction principle" was proved by Banach [5] in 1922. This principle is very useful for the solution of existence and uniqueness. In functional analysis, one of the most important theorems is the contraction principle, and this contraction principle is widely considered as the source of the fixed point theory in MS. This principle is as follows:

Theorem 1.1. [5] *Let $(\mathbb{S}, \varrho)$ be a CMS and $\Omega : \mathbb{S} \to \mathbb{S}$ a contraction, i.e., a mapping satisfying*

$$\varrho(\Omega\ell, \Omega\wp) \leq \sigma^{'} \varrho(\ell, \wp), \tag{1.1}$$

$\forall\ \ell, \wp \in \mathbb{S}$ and $0 \leq \sigma^{'} < 1$. Then Ω has a unique fixed point $v \in \mathbb{S}$, and the Picard iteration converges to v for any $\ell_0 \in \mathbb{S}$.

Definition 1.2. Let $\mathcal{K}$ be a non-empty subset of a CMS $(\mathbb{S}, \varrho)$. Let $\Omega : \mathcal{K} \to 2^{\mathcal{K}}$ be a multi-valued mapping. An element $\ell \in \mathcal{K}$ is said to be fixed point of multi-valued mapping Ω, if $\ell \in \Omega\ell$. .

Nadler [18] obtained multi-valued version of the "Banach contraction principle" in CMS-

Theorem 1.3. [18] *Let $(\mathbb{S}, \varrho)$ be a CMS. If $\Omega : \mathbb{S} \to CB(\mathbb{S})$ is a multi-valued contraction mapping, then Ω has a fixed point.*

One drawback of this principle is that it requires the contraction condition Ω to be continuous on $\mathbb{S}$. This raises the question: Are there contractive conditions that do not imply the continuity of Ω? To overcome these conditions, Kannan [13] proved a fixed point theorem that extend the Banach contraction principle to the mappings that need not be continuous by considering the condition instead of contractive condition (1.1) in 1968. This condition is, there exists a constant $\tau^{'} \in (0, \frac{1}{2})$ such that for all $\ell, \wp \in \mathbb{S}$

$$\varrho(\Omega\ell, \Omega\wp) \leq \tau^{'}[\varrho(\ell, \Omega\ell) + \varrho(\wp, \Omega\wp)]. \tag{1.2}$$

Theorem 1.4. [13] *Let $(\mathbb{S}, \varrho)$ be a CMS and let $\Omega : \mathbb{S} \to \mathbb{S}$ be a mapping such that there exists a constant $\tau^{'} \in (0, \frac{1}{2})$ such that for all $\ell, \wp \in \mathbb{S}$*

$$\varrho(\Omega\ell, \Omega\wp) \leq \tau^{'}[\varrho(\ell, \Omega\ell) + \varrho(\wp, \Omega\wp)].$$

Then Ω has a unique fixed point $v \in \Omega$, and for any $\ell \in \mathbb{S}$ the sequence of iterates $\{\Omega^n\ell\}$ converges to v and

$$\varrho(\Omega^{n+1}\ell, v) \leq \tau^{'}(\frac{\tau^{'}}{1-\tau^{'}})^n(\ell, v), n = 0, 1, 2, ...$$

Example 1.5. [13] Let $\mathbb{S} = \mathbb{R}$ with usual norm and $\Omega : \mathbb{R} \to \mathbb{R}$ given by

$$\Omega\ell = \begin{cases} 0, \ \ \ell \in (-\infty, 2]; \\ \frac{-1}{2}, \ \ \ell > 2. \end{cases}$$

Then Ω is not continuous in $\mathbb{R}$ and satisfies condition (1.2) with $\tau^{'} = \frac{1}{5}$.

Based on condition (1.2), Chatterjea [7] obtained a similar condition that there exists a constant $\vartheta^{'} \in (0, \frac{1}{2})$ such that for all $\ell, \wp \in \mathbb{S}$,

$$\varrho(\Omega\ell, \Omega\wp) \leq \vartheta^{'}[\varrho(\ell, \Omega\wp) + \varrho(\wp, \Omega\ell)]. \tag{1.3}$$

Theorem 1.6. [7] *If $(\mathbb{S}, \varrho)$ is a CMS and $\Omega : \mathbb{S} \to \mathbb{S}$ satisfies*

$$\varrho(\Omega\ell, \Omega\wp) \leq \vartheta^{'}[\varrho(\ell, \Omega\wp) + \varrho(\wp, \Omega\ell)], \text{ for all } \ell, \wp \in \mathbb{S}.$$

Then Ω has a unique fixed point.

In 1972, Zamfirescu [22] combined conditions (1.1), (1.2) and (1.3) and obtained a very nice and interesting fixed point theorem:

Theorem 1.7. [22] *Let $(\mathbb{S}, \varrho)$ be a CMS and $\Omega : \mathbb{S} \to \mathbb{S}$ be a mapping for which there are real numbers $0 \leq \sigma^{'} < 1$, $0 \leq \tau^{'}, \vartheta^{'} < \frac{1}{2}$ such that for all $\ell, \wp \in \mathbb{S}$, at-least one of the following is true:*

(z_1) $\varrho(\Omega\ell, \Omega\wp) \leq \sigma^{'}\varrho(\ell, \wp)$;
(z_2) $\varrho(\Omega\ell, \Omega\wp) \leq \tau^{'}[\varrho(\ell, \Omega\ell) + \varrho(\wp, \Omega\wp)]$;
(z_3) $\varrho(\Omega\ell, \Omega\wp) \leq \vartheta^{'}[\varrho(\ell, \Omega\wp) + \varrho(\wp, \Omega\ell)]$.

Then Ω has a fixed point.

An operator that satisfies conditions (z_1) to (z_3) is called the Zamfirescu operator or condition Z. Zamfirescu obtained the following result.

Theorem 1.8. *An operator satisfying condition Z has a unique fixed point that can be approximated using the Picard iteration scheme.*

To obtain a fixed point of a mapping, graph theory is widely used. Echenique [9] started the study of fixed point with the use of graph theory in 2005. After that, some fixed point results in graph theory were established by Kirk and Espinola [10].

The fixed point theorems of Zamfirescu operators have been primarily developed mostly in CMS as well as Banach space. Ali and Ali [3] obtained some convergence results in the Banach space and the approximate fixed point of the Zamfirescu operator by using their proposed iterative scheme. Babu and Prasad [4] proved that the Mann iteration converges faster than the Ishikawa iteration scheme for the class of Zamfirescu operator in Banach spaces. Later, Rhodes [20] proved that the claim obtained by Babu and Prasad is not always true. Berinde [6] extended the result of Rhodes from a uniformly convex Banach space to an arbitrary Banach space and proved convergence of the Mann iteration for the Zamfirescu operator class. In addition, George and Shaini [11] proved some convergence results in normed space by using the generalized Mann iteration scheme for the Zamfirescu operator and claimed that their results were generalized and improve the results obtained by Berinde et al.

Kritsana and Kaewkhao [12] studied the fixed point results of multi-valued Zamfirescu operator in CMS. Pathak et al. [2] obtained a common fixed point of Kannan-type mappings on MMS endowed with graph. Since, in the field of fixed point theory by combining graph theory, not lots of results are derived for Zamfirescu operator, so the motivation to write this article is to obtain convergence results for multi-valued Zamfirescu operator in MMS and CMMS endowed with a graph.

2. Preliminaries

This section contains some useful concepts and results to get desire results.

Definition 2.1. A graph $G$is a combination of two non-empty sets- set of vertices and set of edges and mathematically it is written as $G = (V(G), E(G))$. If ℓ and $\wp$ are vertices in G, then a path in G from ℓ to $\wp$ of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{\ell_i\}_{i=0}^{n}$ of $n+1$ vertices such that $\ell_0 = \ell, \ell_n = \wp$, $(\ell_{i-1}, \ell_i) \in E(G)$ for $i = 1, 2, ..., n$. A graph is called a connected graph, if there is an edge(path) between any two vertices of G. A directed graph, also called a digraph, is a graph in which the edges have a direction. A digraph is weakly connected if when considering it as an undirected graph, it is connected, i.e., for every pair of distinct vertices u and v, there exists an undirected path from u to v.

Definition 2.2. [21] Let $G = (V(G), E(G))$ be a directed graph. A graph G is called transitive if for any $\ell, \wp, \nu \in V(G)$ such that $(\ell, \wp) \in E(G)$ and $(\wp, \nu) \in E(G)$, then $(\ell, \nu) \in E(G)$.

Markin [17] introduced the concept of Hausdorff metric to approximate fixed points of multi-valued mappings. Let

$$CB(\mathcal{K}) = Collection\ of\ all\ non-empty\ closed\ bounded\ subsets\ of\ \mathcal{K},$$
$$P(\mathcal{K}) = Collection\ of\ all\ non-empty\ proximal\ bounded\ closed\ subsets\ of\ \mathcal{K},$$
$$H(A, B) = \max\{\sup d(\ell, B), \sup d(\wp, A)\},$$

where $A, B \in CB(\mathcal{K})$, $d(x, A) = \inf_{a \in A} d(\ell, a)$.

The notion of modular spaces was introduced by Nakano [19] as a generalization of metric spaces in 2008. Later, the notion of modular metric space was introduced in 2010 by Chitsyakov [8], generated by $F-$modular and developed the theory of modular metric spaces called the modular metric space.

Later Kowzslowski [15] introduced the formulation of a modular, which is given as follows:

Definition 2.3. [15] Let $\mathbb{S}$ be a vector space in $\mathbb{R}$ (or $\mathbb{C}$). A function $m^{'} : \mathbb{S} \to [0, \infty]$ is called modular if, it satisfies the following conditions: for $\ell, \wp \in \mathbb{S}$-

(i) $m^{'}(0) = 0$ or $m^{'}(\ell) = 0 \Leftrightarrow \ell = 0$;
(ii) $m^{'}(\sigma^{'}\ell) = m^{'}(\ell)\ \forall\ \sigma^{'} \in \mathbb{R}$ with $|\sigma^{'}| = 1$;
(iii) $m^{'}(\sigma^{'}\ell + \tau^{'}\wp) = m^{'}(\ell) + m^{'}(\wp)$ if $\sigma^{'}, \tau^{'} \geq 0$, $\sigma^{'} + \tau^{'} = 1$.

Definition 2.4. [15] A modular $m^{'}$ is called convex, if

$$m^{'}_{\chi+\nu}(\ell, \wp) \leq \frac{\chi}{\chi+\nu} m^{'}_{\chi}(\ell, z) + \frac{\nu}{\chi+\nu} m^{'}_{\nu}(z, \wp),$$

for all $\chi, \nu > 0$ and $\ell, \wp, \vartheta \in \mathbb{S}$.

Definition 2.5. [8] Let $\mathbb{S} \neq \emptyset$. A metric modular on $\mathbb{S}$ is a function $w : (0, \infty) \times \mathbb{S} \times \mathbb{S} \to [0, \infty]$ written as $(\chi, \ell, \wp) \to w_{\chi}(\ell, \wp)$ that satisfies the following axioms:

(i) $w_\chi(\ell,\wp)=0 \Leftrightarrow \ell=\wp\ \forall\ \ell,\wp\in\mathbb{S},\ \chi>0$;

(ii) $w_\chi(\ell,\wp)=w_\chi(\wp,\ell)\ \forall\ \ell,\wp\in\mathbb{S},\ \chi>0$;

(iii) $w_{\chi+\nu}(\ell,\wp)\le w_\chi(\ell,z)+w_\nu(z,\wp)\ \forall\ \ell,\wp\in\mathbb{S},\ \chi,\nu>0$;

Definition 2.6. [1] Modular w is said to be regular if the following weaker version of (i) is satisfied:

$$\ell=\wp \iff w_\chi(\ell,\wp)=0,$$

for some $\chi>0$.

Remark 2.7. [8] A convex modular satisfies

$$w_\chi(\ell,\wp)\le\frac{\nu}{w_\nu}(\ell,\wp)\le w_\nu(\ell,\wp),$$

for all $\ell,\wp\in\mathbb{S}$ and $0<\nu\le\chi$. In condition (iii) of the definition 2.5, a modular w satisfies

$$w_{\chi_2}(\ell,\wp)\le w_{\chi_1}(\ell,\wp),$$

for $\chi_1<\chi_2$ and $\ell,\wp\in\mathbb{S}$.

Definition 2.8. [8] If w is a metric modular on a non-empty set $\mathbb{S}$, then the modular space $\mathbb{S}_w$ can be equipped with a metric generated by w and given by

$$\varrho_w(\ell,\wp)=\inf\{\chi>0:w_\chi(\ell,\wp)\le\chi\}$$

for all $\ell,\wp\in\mathbb{S}_w$. The pair $(\mathbb{S}_w,\varrho_w)$ is called a modular metric space.

Definition 2.9. [8] Modular w is said to be strict if for given $\ell,\wp\in\mathbb{S}$ with $\ell\neq\wp$, $w_\chi(\ell,\wp)>0$ $\forall\ \chi>0$.

Definition 2.10. [16] Let $\mathbb{S}_w$ be a MMS, $\{\ell_n\}$ be a sequence in $\mathbb{S}_w$ and $\mathcal{K}$ be a subset of $\mathbb{S}_w$. Then

(i) $\{\ell_k\}$ is called a modular convergent sequence such that $\ell_k\to\ell$, $\ell\in\mathbb{S}_w$, if for $\chi>0$,

$$w_\chi(\ell_k,\ell)\to 0,\ n\to\infty.$$

(ii) $\{\ell_k\}$ is called a modular Cauchy sequence, if for $\chi>0$,

$$w_\chi(\ell_k,\ell_m)\to 0,\ k,m\to\infty.$$

(iii) $\mathcal{K}$ is called $w-$bounded, if

$$\delta_w(K)=\sup\{w_\chi(\ell,\wp):\ell,\wp\in\mathcal{K},\chi>0\}<\infty.$$

- A subset $\mathcal{K}$ of $\mathbb{S}_w$ is said to be $w-$compact if for any $\{\ell_k\}$ in $\mathcal{K}$ there exists a subsequence $\{\ell_{k_n}\}$ and $\ell\in\mathcal{K}$ such that $w_\chi(\ell_{k_n},\ell)\to 0$.
- w is said to satisfy the Fatou property iff for any sequence $\{\ell_k\}$ in $\mathcal{K}$ modular converges to ℓ,

$$w_\chi(\ell,\wp)\le\liminf w_\chi(\ell_k,\wp),$$

for any $\wp\in\mathcal{K}$.

Definition 2.11. [2] Let $\mathbb{S}_w$ be a MMS. Then w satisfies the Δ_2-type condition iff for any $\alpha>0$, there exists a constant $C>0$ such that

$$w_{\frac{\chi}{\alpha}}(\ell,\wp)\le Cw_\chi(\ell,\wp),$$

for any $\chi>0$, $\ell,\wp\in\mathbb{S}_w$ with $\ell\neq\wp$.

Definition 2.12. [2] Property (P). For any sequence $\{\ell_k\}$ in $\mathcal{K} \subset \mathbb{S}_w$, if $\ell_k \to \ell$ as $k \to \infty$ and $(\ell_k, \ell_{k+1}) \in E(G_w)$, then $(\ell_k, \ell) \in E(G_w)$, for all k.

Definition 2.13. [2] Let $\mathcal{K}$ be a non-empty convex subset of a MMS $\mathbb{S}_w$. Let $G_w = (V(G_w), E(G_w))$ be a directed graph such that $V(G_w) = \mathcal{K}$ and $E(G_w)$ contain all loops, i.e., $(\ell, \ell) \in E(G_w)$ for any $\ell \in V(G_w)$. A mapping $\Omega : \mathcal{K} \to \mathcal{K}$ is called an edge-preserving mapping (or G_w-edge-preserving mapping) if for all $\ell, \wp \in \mathcal{K}$,

$$(\ell, \wp) \in E(G_w) \Rightarrow (\Omega(\ell), \Omega(\wp)) \in E(G_w).$$

Following is the definition of the Zamfirescu operator in the setting of MMS.

Definition 2.14. Let $\mathbb{S}_w$ be a MMS under a metric modular w, where $\mathbb{S} \neq \emptyset$ and $\Omega : \mathbb{S}_w \to \mathbb{S}_w$ be a mapping. Then Ω is called Zamfirescu operator if for all $\ell, \wp \in \mathbb{S}_w$ and $\chi > 0$, there exists real numbers $\sigma', \tau', \vartheta'$ such that $0 \leq \sigma' < 1$, $0 < \tau', \vartheta' < \frac{1}{2}$ such that at-least one the following is true:

(z_i) $w_\chi(\Omega\ell, \Omega\wp) \leq \sigma' w_\chi(\ell, \wp)$;
(z_{ii}) $w_\chi(\Omega\ell, \Omega\wp) \leq \tau'[w_{2\chi}(\ell, \Omega\ell) + w_{2\chi}(\wp, \Omega\wp)]$;
(z_{iii}) $w_\chi(\Omega\ell, \Omega\wp) \leq \vartheta'[w_{2\chi}(\ell, \Omega\wp) + w_{2\chi}(\wp, \Omega\ell)]$.

Consider $\mathbb{S}_w$ be a MMS under a metric modular w. The Hausdorff modular metric is defined by

$$H_w(A, B) = \max\{\sup w_\chi(\ell, B), \sup w_\chi(\wp, A)\},$$

where $A, B \in CB(\mathcal{K})$, $w_\chi(\ell, A) = \inf_{a \in A} w_\chi(\ell, \wp)$.

Definition 2.15. [12] Let $\mathbb{S}$ be a MS under a metric ϱ, where $\mathbb{S} \neq \emptyset$ and $\Omega : \mathbb{S} \to CB(\mathbb{S})$ be a multi-valued mapping. Then Ω is called multi-valued Zamfirescu operator if for all $\ell, \wp \in \mathbb{S}$ and $\chi > 0$, there exists real numbers $\sigma', \tau', \vartheta'$ such that $0 \leq \sigma' < 1$, $0 \leq \tau', \vartheta' < \frac{1}{2}$ such that at-least one the following is true:

(z_i) $H(\Omega\ell, \Omega\wp) \leq \sigma' \varrho(\ell, \wp)$;
(z_{ii}) $H(\Omega\ell, \Omega\wp) \leq \tau'[\varrho(\ell, \Omega\ell) + \varrho(\wp, \Omega\wp)]$;
(z_{iii}) $H(\Omega\ell, \Omega\wp) \leq \vartheta'[\varrho(\ell, \Omega\wp) + \varrho(\wp, \Omega\ell)]$.

Following is the definition of the multi-valued Zamfirescu operator in the setting of MMS.

Definition 2.16. Let $\mathbb{S}_w$ be a MMS under a metric modular w, where $\mathbb{S} \neq \emptyset$ and $\Omega : \mathbb{S}_w \to CB(\mathbb{S}_w)$ be a multi-valued mapping. Then Ω is called multi-valued Zamfirescu operator if for all $\ell, \wp \in \mathbb{S}_w$ and $\chi > 0$, there exists real numbers $\sigma', \tau', \vartheta'$ such that $0 \leq \sigma' < 1$, $0 < \tau', \vartheta' < \frac{1}{2}$ such that at-least one the following is true:

(z_i) $H_w(\Omega\ell, \Omega\wp) \leq \sigma' w_\chi(\ell, \wp)$;
(z_{ii}) $H_w(\Omega\ell, \Omega\wp) \leq \tau'[w_{2\chi}(\ell, \Omega\ell) + w_{2\chi}(\wp, \Omega\wp)]$;
(z_{iii}) $H_w(\Omega\ell, \Omega\wp) \leq \vartheta'[w_{2\chi}(\ell, \Omega\wp) + w_{2\chi}(\wp, \Omega\ell)]$.

Definition 2.17. [14] Let $\mathcal{K}$ be a non-empty subset of a MMS space $\mathbb{S}$. A sequence $\{\ell_k\}$ in $\mathbb{S}$ is said to be Fejer monotone with respect to subset $\mathcal{K}$, if

$$w_\chi(\ell_{k+1}, p) \leq w_\chi(\ell_k, p), \ for \ all \ p \in \mathcal{K}, \ k \geq 1.$$

Proposition 2.18. [14] *Let $\mathcal{K}$ be a non-empty subset of MMS space $\mathbb{S}$. Suppose that $\{\ell_k\}$ is a Fejer monotone sequence with respect to $\mathcal{K}$. Then the followings are hold:*

(a) *Sequence* $\{\ell_k\}$ *is bounded.*
(b) *For every* $\ell \in K$, $\{w_\chi(\ell_k, \ell)\}$ *converges.*

Lemma 2.19. [1] *Let* $\mathbb{S}_w$ *be a MMS under a metric modular* w, *where* $\mathbb{S} \neq \emptyset$ *and* $\mathcal{K}$ *be a non-empty subset of* $\mathbb{S}_w$. *Let* $A, B \in CB(\mathcal{K})$, *then for each* $\epsilon > 0$ *and* $\ell \in A$, *there exists* $\wp \in B$ *such that*

$$w_\chi(\ell, \wp) \leq H_w(A, B) + \epsilon,$$

Moreover, if B *is* $w-$*compact and* w *satisfies the Fatou property, then for any* $\ell \in A$, *there exists* $\wp \in B$ *such that*

$$w_\chi(\ell, \wp) \leq H_w(A, B).$$

Lemma 2.20. [1] *Let* $\mathbb{S}_w$ *be a MMS under a metric modular* w. *Assume that* w *is a convex regular modular which satisfies the* Δ_2*-type condition. Let* $\{\ell_k\}$ *be a sequence in* $\mathbb{S}_w$ *such that*

$$w_\chi(\ell_k, \ell_{k+1}) \leq K\alpha^k, k = 1, 2, ...,$$

where K *is an arbitrary non-zero constant and* $\alpha \in (0, 1)$. *Then* $\{\ell_k\}$ *is Cauchy for* w.

3. Main Results

This section contains some convergence results and fixed point theorems related to multi-valued Zamfirescu operator in a complete modular metric space endowed with a directed graph.

Lemma 3.1. *Let* $\mathbb{S}_w \neq \emptyset$ *be a MMS under a metric modular* w, *where* w *is a convex regular modular which satisfies* Δ_2*-type condition and* $\mathcal{K}$ *be a non-empty* $w-$*bounded,* $w-$*complete subset of* $\mathbb{S}_w$. *Suppose that* $\mathbb{S}_w$ *is associated with a directed transitive graph* G_w *such that* $\mathcal{K} = V(G_w)$ *and* $E(G_w)$ *contain all loops. Suppose that* $\Omega : K \to CB(\mathcal{K})$ *is an edge-preserving Zamfirescu operator with* $\ell_0 \in \mathcal{K}$ *such that* $(\ell_0, \Omega\ell_0) \in E(G_w)$. *Then*

(a) $(\ell_k, \ell_{k+1}) \in E(G_w)$, *for* $k = 0, 1, 2...$
(b) $(\ell_k, \Omega\ell_k) \in E(G_w)$ *for* $k \geq 1$.

Proof. (1) By assumption $(\ell_0, \Omega\ell_0) \in E(G_w)$, and Ω is multi-valued, there exists $\ell_1 \in \Omega\ell_0$,i.e., $(\ell_0, \ell_1) \in E(G_w)$. Since Ω is an edge-preserving, $(\Omega\ell_0, \Omega\ell_1) \in E(G_w)$, i.e., there exists $\ell_2 \in \Omega\ell_1$ such that $(\ell_1, \ell_2) \in E(G_w)$. Continuing the process, $(\ell_k, \ell_{k+1}) \in E(G_w)$ for $k = 0, 1, 2....$

(2) By assumption $(\ell_0, \Omega\ell_0) \in E(G_w)$, and Ω is multi-valued, there exists $\ell_1 \in \Omega\ell_0$ such that $(\ell_0, \ell_1) \in E(G_w)$. Since Ω is edge-preserving, $(\Omega\ell_0, \Omega\ell_1) \in E(G_w)$. By transitivity of Ω, $(\ell_1, \Omega\ell_0) \in E(G_w)$. Now $(\ell_0, \Omega\ell_0) \in E(G_w)$, $(\Omega\ell_0, \Omega\ell_1) \in E(G_w)$, by transitivity of Ω, $(\ell_0, \Omega\ell_1) \in E(G_w)$. Again $(\ell_1, \Omega\ell_0) \in E(G_w)$, $(\Omega\ell_0, \Omega\ell_1) \in E(G_w)$, by transitivity of Ω, $(\ell_1, \Omega\ell_1) \in E(G_w)$. Continuing the process, $(\ell_k, \Omega\ell_k) \in E(G_w)$ for $k \geq 1$.

□

Lemma 3.2. *Let* $\mathbb{S}_w \neq \emptyset$ *be a MMS under a metric modular* w, *where* w *is a convex regular modular which satisfies* Δ_2*-type condition and* $\mathcal{K}$ *be a non-empty* $w-$*bounded,* $w-$*complete subset of* $\mathbb{S}_w$. *Suppose that* $\mathbb{S}_w$ *is associated with a directed transitive graph* G_w *such that* $\mathcal{K} = V(G_w)$ *and* $E(G_w)$ *contain all loops. Suppose that* $\Omega : \mathcal{K} \to CB(\mathcal{K})$ *is an edge-preserving* *multi-valued Zamfirescu operator with* $\ell_0 \in \mathcal{K}$ *such that* $(\ell_0, \Omega\ell_0) \in E(G_w)$. *Suppose that* $\{\ell_k\}$ *be any sequence in* $\mathcal{K}$ *and* $F(\Omega) \neq \emptyset$ *with* $r \in F(\Omega)$ *such that* $(\ell_0, r), (r, \ell_0) \in E(G_w)$, *then*

(a) (ℓ_k, r) *and* (r, ℓ_k) *are in* $E(G_w)$ *for* $k \geq 1$,
(b) $\lim_{k\to\infty} w_\chi(\ell_k, r)$ *exists.*

Proof. (a) By assumption $(\ell_0, r) \in E(G_w)$, and Ω is edge-preserving mapping, $(\Omega\ell_0, \Omega r) \in E(G_w)$, i.e., $(\ell_1, r) \in E(G_w)$. Again Ω is an edge-preserving mapping, $(\Omega\ell_1, \Omega r) \in E(G_w)$, i.e., $(\ell_2, r) \in E(G_w)$. Continuing the process, (ℓ_k, r) and (r, ℓ_k) are in $E(G_w)$ for $k \geq 1$.

(b) From part (a), $(\ell_{k+1}, r) \in E(G_w)$ and From Lemma 2.19,

$$w_\chi((\ell_{k+1}, r) \leq H_w(\Omega\ell_k, \Omega r) + \epsilon. \tag{3.1}$$

If (z_i) is satisfied then, from (3.1)

$$\begin{aligned} w_\chi(\ell_{k+1}, r) &\leq H_w(\Omega\ell_k, \Omega r) + \epsilon \\ &\leq \sigma^{'} w_\chi(\ell_k, r) + \epsilon \\ &\leq w_\chi(\ell_k, r). \end{aligned}$$

This implies that $\{\ell_k\}$ is Fejer monotone with respect to $F(\Omega)$. Hence from Proposition 2.18, $\{\ell_k\}$ is bounded and $\lim_{k\to\infty} w_\chi(\ell_k, r)$ exists.

If (z_{ii}) is satisfied then, from (3.1) and From Lemma 2.19,

$$\begin{aligned} w_\chi(\ell_{k+1}, r) &\leq \tau^{'}[w_{2\chi}(\ell_k, \Omega\ell_k) + w_{2\chi}(r, \Omega r)] + \epsilon \\ &\leq \tau^{'} w_{2\chi}(\ell_k, \ell_{k+1}) + \epsilon \\ &\leq \tau^{'}[w_\chi(\ell_k, r) + w_\chi(r, \ell_{k+1})] + \epsilon \\ (1-\tau^{'}) w_\chi(\ell_{k+1}, r) &\leq \tau^{'} w_\chi(\ell_k, r) + \epsilon \\ w_\chi(\ell_{k+1}, r) &\leq \frac{\tau^{'}}{1-\tau^{'}} w_\chi(\ell_k, r). \end{aligned}$$

Put $\beta = \frac{\tau^{'}}{1-\tau^{'}}$, then

$$w_\chi(\ell_{k+1}, r) \leq \beta w_\chi(\ell_k, r).$$

Doing the same procedure as above, $\lim_{k\to\infty} w_\chi(\ell_k, r)$ exists.

If (z_{iii}) is satisfied then, from (3.1) and From Lemma 2.19,

$$\begin{aligned} w_\chi((\ell_{k+1}, r) &\leq H_w(\Omega\ell_k, \Omega r) + \epsilon \\ &\quad \vartheta^{'}[w_{2\chi}(\ell_k, \Omega r) + w_{2\chi}(r, \Omega\ell_k)] + \epsilon \\ (1-\vartheta^{'}) w_\chi((\ell_{k+1}, r) &\leq \vartheta^{'} w_\chi(\ell_k, r) + \epsilon \\ w_\chi((\ell_{k+1}, r) &\leq \frac{\vartheta^{'}}{1-\vartheta^{'}} w_\chi(\ell_k, r). \end{aligned}$$

Put $\gamma = \frac{\vartheta^{'}}{1-\vartheta^{'}}$, then

$$w_\chi((\ell_{k+1}, r) \leq \gamma w_\chi(\ell_k, r).$$

Doing the same procedure as above, $\lim_{k\to\infty} w_\chi(\ell_k, r)$ exists.

□

Theorem 3.3. *Let $\mathbb{S}_w \neq \emptyset$ be a MMS under a metric modular w, where w is a convex regular modular which satisfies the Δ_2-type condition and $\mathcal{K}$ be a non-empty $w-$bounded, $w-$complete subset of $\mathbb{S}_w$. Suppose that $\mathbb{S}_w$ is associated with a directed transitive graph G_w such that $\mathcal{K} = V(G_w)$ and $E(G_w)$ contain all loops. Suppose that $\Omega : \mathcal{K} \to CB(\mathcal{K})$ is an edge-preserving multi-valued Zamfirescu operator and the triplet $(\mathbb{S}, w_\chi, G_w)$ has property (P). Then Ω has unique fixed point in $\mathbb{S}_w$. Moreover for any $\ell \in \mathbb{S}_w$, the sequence $\{\ell_k\}$ converges to a fixed point of Ω.*

Proof. Let $\{\ell_k\}$ be a sequence in $\mathcal{K}$. To prove that $\{\ell_k\}$ is a Cauchy sequence in $\mathcal{K}$, for this, consider the following cases:
Case I: when (z_i) is satisfied, then form Lemma 2.19,

$$\begin{aligned}
w_\chi(\ell_{k+1}, \ell_k) &\leq H_w(\Omega\ell_k, \Omega\ell_{k-1}) + {\sigma'}^{2n} \\
&\leq \sigma' w_\chi(\ell_k, \ell_{k-1}) + {\sigma'}^{2n} \\
&= \sigma' w_\chi(\Omega\ell_{k-1}, \Omega\ell_{k-2}) + {\sigma'}^{2n} \\
&\leq {\sigma'}^{2} w_\chi(\ell_{k-1}, \ell_{k-2}) + {\sigma'}^{2n} \\
&. \\
&. \\
&. \\
&\leq {\sigma'}^{n} w_\chi(\ell_1, \ell_0) + {\sigma'}^{2n} \\
&\leq {\sigma'}^{n} [w_\chi(\ell_1, \ell_0) + {\sigma'}^{n}].
\end{aligned}$$

Put $w_\chi(\ell_1, \ell_0) + {\sigma'}^{2} = \zeta$, then

$$w_\chi(\ell_{k+1}, \ell_k) \leq \zeta {\sigma'}^{n}.$$

From Lemma 2.20, $\{\ell_k\}$ is a Cauchy sequence in $\mathcal{K}$. By completeness of $\mathcal{K}$, there exists a point $\ell \in \mathcal{K}$ such that $\ell_k \to \ell$ as $k \to \infty$. Now

$$\begin{aligned}
w_\chi(\Omega\ell, \ell) &\leq w_{\frac{\chi}{2}}(\Omega\ell, \Omega\ell_k) + w_{\frac{\chi}{2}}(\Omega\ell_k, \ell) \\
&\leq \sigma' w_{\frac{\chi}{2}}(\ell, \ell_k) + w_{\frac{\chi}{2}}(\ell_{k+1}, \ell).
\end{aligned}$$

Hence $w_\chi(\Omega\ell, \ell) = 0$ as $k \to \infty$, i.e. $\ell \in \Omega\ell$. For the uniqueness, suppose that ν is another fixed point of Ω, i.e. $\nu \in \Omega\nu$. If (z_i) is satisfied, then from Lemma 2.19,

$$\begin{aligned}
w_\chi(p', \nu) &\leq H_w(\Omega p', \Omega\nu) + \epsilon \\
&\leq \sigma' w_\chi(p', \nu) + \epsilon \\
&\leq \sigma' w_\chi(p', \nu) \\
(1 - \sigma') w_\chi(p', \nu) &\leq 0.
\end{aligned}$$

Since $\sigma' < 1$, $w_\chi(p', \nu) = 0$.

When (z_{ii}) is satisfied, from Lemma 2.19,

$$\begin{aligned}
w_\chi(\ell_{k+1},\ell_k) &\leq H_w(\Omega\ell_k,\Omega\ell_{k-1})+\beta^{'2n}\\
&\leq \tau^{'}[w_{2\chi}(\ell_k,\Omega\ell_k)+w_{2\chi}(\ell_{k-1},\Omega\ell_{k-1})]+\beta^{'2n}\\
&\leq \tau^{'}[w_{2\chi}(\ell_k,\ell_{k+1})+w_{2\chi}(\ell_{k-1},\ell_k)]+\beta^{'2n}\\
(1-\tau^{'})w_\chi(\ell_{k+1},\ell_k) &\leq \tau^{'}w_\chi(\ell_{k-1},\ell_k)+\beta^{'2n}\\
&\leq \frac{\tau^{'}}{1-\tau^{'}}w_\chi(\ell_{k-1},\ell_k)+\beta^{'2n}.
\end{aligned}$$

Suppose that $\beta^{'}=\frac{\tau^{'}}{1-\tau^{'}}$. Then

$$\begin{aligned}
w_\chi(\ell_{k+1},\ell_k) &\leq \beta^{'}w_\chi(\ell_{k-1},\ell_k)+\beta^{'2n}\\
&\leq \beta^{'2}w_\chi(\ell_{k-2},\ell_{k-1})+\beta^{'2n}\\
&.\\
&.\\
&.\\
&\leq \beta^{'n}w_\chi(\ell_0,\ell_1)+\beta^{'2n}\\
&\leq \beta^{'n}[w_\chi(\ell_0,\ell_1)+\beta^{'n}]
\end{aligned}$$

Doing the same procedure as above, $\{\ell_k\}$ is a Cauchy sequence in $\mathcal{K}$. By completeness of $\mathcal{K}$, there exists a point $\ell\in\mathcal{K}$ such that $\ell_k\to\ell$ as $k\to\infty$. Now

$$\begin{aligned}
w_\chi(\Omega\ell,\ell) &\leq w_{\frac{\chi}{2}}(\Omega\ell,\Omega\ell_k)+w_{\frac{\chi}{2}}(\Omega\ell_k,\ell)\\
&\leq \tau^{'}[w_\chi(\Omega\ell,\ell)+w_\chi(\Omega\ell_k,\ell_k)]+w_{\frac{\chi}{2}}(\Omega\ell_k,\ell)\\
&\leq \tau^{'}[w_\chi(\Omega\ell,\ell)+w_{\frac{\chi}{2}}(\Omega\ell_k,\ell)+w_{\frac{\chi}{2}}(\ell_k,\ell)]+w_{\frac{\chi}{2}}(\Omega\ell_k,\ell)\\
&= \tau^{'}[w_\chi(\Omega\ell,\ell)+w_{\frac{\chi}{2}}(\ell_{k+1},\ell)+w_{\frac{\chi}{2}}(\ell_k,\ell)]+w_{\frac{\chi}{2}}(\ell_{k+1},\ell).
\end{aligned}$$

Taking $\lim_{k\to\infty}$,

$$\begin{aligned}
(1-\tau^{'})w_\chi(\Omega\ell,\ell) &\leq 0\\
&\Rightarrow w_\chi(\Omega\ell,\ell)=0.
\end{aligned}$$

For the uniqueness, suppose that ν is another fixed point of Ω, i.e. $\nu\in\Omega\nu$. Since

$$\begin{aligned}
w_\chi(p^{'},\nu) &\leq H_w(\Omega p^{'},\Omega\nu)+\epsilon\\
&\leq \tau^{'}[w_{2\chi}(\Omega p^{'},p^{'})+w_{2\chi}(\Omega\nu,\nu)]+\epsilon\\
&\leq \tau^{'}[w_\chi(\Omega p^{'},p^{'})+w_\chi(\Omega\nu,\nu)].
\end{aligned}$$

Taking $\lim\limits_{k\to\infty}$, $w_\chi(p^{'},\nu)=0$.

When (z_{iii}) is satisfied, Then from Lemma 2.19

$$\begin{aligned} w_\chi(\ell_{k+1},\ell_k) &\le H_w(\Omega\ell_k,\Omega\ell_{k-1}) + {\gamma^{'}}^{2n} \\ &\le \vartheta^{'}[w_{2\chi}(\ell_{k-1},\Omega\ell_k) + w_{2\chi}(\ell_k,\Omega\ell_{k-1})] + {\gamma^{'}}^{2n} \\ &\le \vartheta^{'}[w_\chi(\ell_{k-1},\ell_{k+1}) + w_\chi(\ell_k,\ell_k)] + {\gamma^{'}}^{2n} \\ &\le \vartheta^{'}[w_{\frac{\chi}{2}}(\ell_{k-1},\ell_k) + w_{\frac{\chi}{2}}(\ell_k,\ell_{k+1})] + {\gamma^{'}}^{2n} \\ (1-\vartheta^{'})w_\chi(\ell_{k+1},\ell_k) &\le \vartheta^{'} w_\chi(\ell_{k-1},\ell_k) + {\gamma^{'}}^{2n} \\ w_\chi(\ell_{k+1},\ell_k) &\le \frac{\vartheta^{'}}{1-\vartheta^{'}} w_\chi(\ell_{k-1},\ell_k) + {\gamma^{'}}^{2n}. \end{aligned}$$

Suppose that $\gamma^{'} = \frac{\vartheta^{'}}{1-\vartheta^{'}}$. Then

$$\begin{aligned} w_\chi(\ell_{k+1},\ell_k) &\le \gamma^{'} w_\chi(\ell_{k-1},\ell_k) + {\gamma^{'}}^{2n} \\ &\le {\gamma^{'}}^{2} w_\chi(\ell_{k-2},\ell_{k-1}) + {\gamma^{'}}^{2n} \\ &. \\ &. \\ &. \\ &\le {\gamma^{'}}^{n} w_\chi(\ell_0,\ell_1) + {\gamma^{'}}^{2n} \\ &\le {\gamma^{'}}^{n}[w_\chi(\ell_0,\ell_1) + {\gamma^{'}}^{n}] \end{aligned}$$

Doing the same procedure as above, $\{\ell_k\}$ is a Cauchy sequence in $\mathcal{K}$. By completeness of $\mathcal{K}$, there exists a point $\ell \in \mathcal{K}$ such that $\ell_k \to \ell$ as $k \to \infty$. Now

$$\begin{aligned} w_\chi(\Omega\ell,\ell) &\le w_{\frac{\chi}{2}}(\Omega\ell,\Omega\ell_k) + w_{\frac{\chi}{2}}(\Omega\ell_k,\ell) \\ &\le \vartheta^{'}[w_\chi(\Omega\ell,\ell_k) + w_\chi(\Omega\ell_k,\ell)] + w_{\frac{\chi}{2}}(\Omega\ell_k,\ell) \\ &\le \vartheta^{'}[w_\chi(\Omega\ell,\ell_k) + w_{\frac{\chi}{2}}(\ell_{k+1},\ell) + w_{\frac{\chi}{2}}(\ell_k,\ell)] + w_{\frac{\chi}{2}}(\ell_{k+1},\ell) \\ &= \vartheta^{'}[w_{\frac{\chi}{2}}(\Omega\ell,\ell) + w_{\frac{\chi}{2}}(\ell,\ell_k) + w_{\frac{\chi}{2}}(\ell_{k+1},\ell) + w_{\frac{\chi}{2}}(\ell_k,\ell)] + w_{\frac{\chi}{2}}(\ell_{k+1},\ell) \\ (1-\vartheta^{'})w_\chi(\Omega\ell,\ell) &\le \vartheta^{'}[w_{\frac{\chi}{2}}(\ell,\ell_k) + w_{\frac{\chi}{2}}(\ell_{k+1},\ell) + w_{\frac{\chi}{2}}(\ell_k,\ell) + w_{\frac{\chi}{2}}(\ell_{k+1},\ell)]. \end{aligned}$$

Taking $\lim_{k\to\infty}$,

$$\begin{aligned} (1-\vartheta^{'})w_\chi(\Omega\ell,\ell) &\le 0 \\ &\Rightarrow w_\chi(\Omega\ell,\ell) = 0. \end{aligned}$$

For the uniqueness, suppose that z is another fixed point of Ω, i.e. $\nu \in \Omega\nu$. From Lemma 2.19,

$$\begin{aligned} w_\chi(p^{'},\nu) &\le H_w(\Omega p^{'},\Omega\nu) + \epsilon \\ &\le \vartheta^{'}[w_{2\chi}(\Omega p^{'},\nu) + w_{2\chi}(\Omega\nu,p^{'})] + \epsilon. \end{aligned}$$

Taking $\lim_{k\to\infty}$, $w_\chi(p^{'},\nu) = 0$. □

Theorem 3.4. *Let $\mathbb{S}_w \neq \emptyset$ be a MMS under a metric modular w, where w is a convex regular modular which satisfies the Δ_2-type condition and $\mathcal{K}$ be a non-empty $w-$bounded,*

$w-$complete subset of $\mathbb{S}_w$. Suppose that $\mathbb{S}_w$ is associated with a directed transitive graph G_w such that $\mathcal{K} = V(G_w)$ and $E(G_w)$ contain all loops. Suppose that $\Omega : \mathcal{K} \to CB(\mathcal{K})$ is an edge-preserving multi-valued Zamfirescu operator with $(\ell_0, \Omega\ell_0) \in E(G_w)$ and the triplet $(\mathbb{S}, w_\chi, G_w)$ has property (P). Let $\mathbb{S}_\Omega = \{\ell_k \in \mathbb{S}_w : (\ell_k, \Omega\ell_k) \in E(G_w)\}$, for all $k \geq 0$. Then the following statements hold:

(a) *For any $\ell_k \in \mathbb{S}_\Omega$, for $k \geq 0$, $\Omega_{[\ell_k]_{G_w}}$ has a fixed point, where $\Omega_{[\ell_k]_{G_w}} = \{\wp \in \mathbb{S}_w :$ there is path between ℓ_k and $\wp\}$;*
(b) *If G is weakly connected, then Ω has a fixed point in G;*
(c) *If $F(\Omega) \subseteq E(G_w)$, then Ω has a fixed point;*
(d) *$F(\Omega) \neq \emptyset$ iff $\mathbb{S}_\Omega \neq \emptyset$.*

Proof. (a) Let $\ell_k \in \mathbb{S}_\Omega$, for $k \geq 0$. From Lemma 3.1, $(\ell_k, \Omega\ell_k) \in E(G_w)$ for $k \geq 0$. Since $\ell_k \in \mathbb{S}_\Omega$, there exists $z_k \in F(\Omega)$ such that $z_k \in \Omega\ell_k$. Since $\{\ell_k\}$ is Cauchy sequence in $\mathbb{S}$, there exists $z \in \mathbb{S}$ such that $\ell_k \to z$ as $k \to \infty$. By property (P), $(\ell_k, z) \in E(G_w)$, hence there is a path between ℓ_k and z. Therefore $z \in \Omega_{[\ell_k]_{G_w}}$.

(b) From Lemma 3.1, $(\ell_k, \Omega\ell_k) \in E(G_w)$ for $k \geq 1$, with the assumption that $(\ell_0, \Omega\ell_0) \in E(G_w)$, therefore $\mathbb{S}_\Omega \neq \emptyset$. Also G is weakly connected, $[\ell_k]_{G_w} = \mathbb{S}$. By part (a), Ω has a fixed point in G.

(c) Let $F(\Omega) \subseteq E(G_w)$. This implies that for any $z \in F(\Omega)$, $(z, \ell_k) \in E(G_w)$ (refer Lemma 3.2). So $\mathbb{S}_\Omega = \mathbb{S}$, so by part (b), Ω has a fixed point.

(d) Let $\mathbb{S}_\Omega \neq \emptyset$. Let $\ell_k \in \mathbb{S}$ with $(\ell_k, \Omega\ell_k) \in E(G_w)$ for $k \geq 1$ (Refer Lemma 3.1). Also form Lemma 3.1, $(\ell_k, \ell_{k+1}) \in E(G_w)$ and from Lemma 3.2, $\ell_k \to z$ for $z \in F(\Omega)$, this implies that $F(\Omega) \neq \emptyset$. Conversely, suppose that $F(\Omega) \neq \emptyset$ with $z \in F(\Omega)$, then from Lemma 3.2, $(\ell_k, z) \in E(G_w)$. This implies that $\mathbb{S}_\Omega \neq \emptyset$.

□

Theorem 3.5. *Let $\mathbb{S}_w \neq \emptyset$ be a CMMS under a metric modular w, where w is a convex regular modular which satisfies the Δ_2-type condition. Suppose that $\mathbb{S}_w$ is associated with a directed transitive graph G_w such that $\mathcal{K} = V(G_w)$ and $E(G_w)$ contain all loops. Suppose that $\Omega : \mathbb{S}_w \to CB(\mathbb{S}_w)$ is an edge-preserving multi-valued Zamfirescu operator such that $\Omega\ell$ is compact and w satisfies the Fatou property. Suppose triplet $(\mathbb{S}, w_\chi, G_w)$ has property (P). Then*

(a) *If $\{\ell_k\}$ is any sequence in $\mathbb{S}_w$, then $\{\ell_k\} \to \ell^* \in F(\Omega)$;*
(b) *$F(\Omega) \neq \emptyset$.*

Proof. (a) Let $\{\ell_k\}$ be a sequence in $\mathbb{S}_w$ with $\ell_0 \in \mathbb{S}_w$. Then there exists $\ell_1 \in \mathbb{S}_w$ such that $\ell_1 \in \Omega\ell_0$. Similarly, there exists $\ell_2 \in \mathbb{S}_w$ such that $\ell_2 \in \Omega\ell_1$. If Ω satisfies (z_i), then

$$\begin{aligned} w_\chi(\ell_1, \ell_2) &\leq H_w(\Omega\ell_0, \Omega\ell_1) \\ &\leq \sigma' w_\chi(\ell_0, \ell_1). \end{aligned}$$

If Ω satisfies (z_{ii}), then

$$\begin{aligned} w_\chi(\ell_1,\ell_2) &\leq H_w(\Omega\ell_0,\Omega\ell_1) \\ &\leq \tau^{'}[w_{2\chi}(\ell_0,\Omega\ell_0)+w_{2\chi}(\ell_1,\Omega\ell_1)] \\ &\leq \tau^{'}[w_\chi(\ell_0,\ell_1)+w_\chi(\ell_1,\ell_2)] \\ w_\chi(\ell_1,\ell_2) &\leq \frac{\tau^{'}}{1-\tau^{'}}w_\chi(\ell_0,\ell_1). \end{aligned}$$

If Ω satisfies (z_{iii}), then

$$\begin{aligned} w_\chi(\ell_1,\ell_2) &\leq H_w(\Omega\ell_0,\Omega\ell_1) \\ &\leq \vartheta^{'}[w_{2\chi}(\ell_0,\Omega\ell_1)+w_{2\chi}(\ell_1,\Omega\ell_0)] \\ &\leq \vartheta^{'}[w_\chi(\ell_0,\ell_2)+w_\chi(\ell_1,\ell_1)] \\ w_\chi(\ell_1,\ell_2) &\leq \frac{\vartheta^{'}}{1-\vartheta^{'}}w_\chi(\ell_0,\ell_1). \end{aligned}$$

Let $\alpha=\max\{\sigma^{'},\frac{\tau^{'}}{1-\tau^{'}},\frac{\vartheta^{'}}{1-\vartheta^{'}}\}$, then

$$\begin{aligned} w_\chi(\ell_1,\ell_2) &\leq \alpha w_\chi(\ell_0,\ell_1) \\ w_\chi(\ell_2,\ell_3) &\leq \alpha^2 w_\chi(\ell_1,\ell_2) \\ &. \\ &. \\ &. \\ w_\chi(\ell_k,\ell_{k+1}) &\leq \alpha^k w_\chi(\ell_0,\ell_1). \end{aligned}$$

Since $\alpha<1\Rightarrow\alpha^k<1$, hence

$$w_\chi(\ell_k,\ell_{k+1})\leq w_\chi(\ell_0,\ell_1).$$

This implies that $\{\ell_k\}$ is a Cauchy sequence in $\mathbb{S}_w$, which is complete, therefore there exists $\ell^*\in\mathbb{S}_w$ such that $\ell_k\to\ell^*$ as $k\to\infty$.

Next, to prove that $\ell^*\in\Omega\ell^*$, for this

$$\begin{aligned} w_\chi(\ell^*,\Omega\ell^*) &\leq w_{\frac{\chi}{2}}(\ell^*,\ell_{k+1})+w_{\frac{\chi}{2}}(\ell_{k+1},\Omega\ell^*) \\ &\leq w_\chi(\ell^*,\ell_{k+1})+w_\chi(\ell_{k+1},\Omega\ell^*) \\ &\leq w_\chi(\ell^*,\ell_{k+1})+H_w(\Omega\ell_k,\Omega\ell^*). \end{aligned}$$

If Ω satisfies (z_i), then

$$w_\chi(\ell^*,\Omega\ell^*)\leq w_\chi(\ell^*,\ell_{k+1})+\sigma^{'}w_\chi(\ell_k,\ell^*).$$

If Ω satisfies (z_{ii}), then

$$w_\chi(\ell^*,\Omega\ell^*)\leq w_\chi(\ell^*,\ell_{k+1})+\tau^{'}[w_\chi(\ell_k,\Omega\ell_k)+w_\chi(\ell^*,\Omega\ell^*)].$$

If Ω satisfies (z_{iii}), then

$$\begin{aligned} w_\chi(\ell^*, \Omega\ell^*) &\leq w_\chi(\ell^*, \ell_{k+1}) + \vartheta'[w_\chi(\ell_k, \Omega\ell^*) + w_\chi(\ell^*, \Omega\ell_k)] \\ &\leq w_\chi(\ell^*, \ell_{k+1}) + \vartheta'[w_\chi(\ell_k, \Omega\ell^*) + w_\chi(\ell^*, \ell_k) + w_\chi(\ell_k, \Omega\ell_k)] \\ &\leq w_\chi(\ell^*, \ell_{k+1}) + \vartheta'[w_\chi(\ell_k, \Omega\ell^*) + w_\chi(\ell^*, \ell_k) + w_\chi(\ell_k, \ell_{k+1})] \end{aligned}$$

Taking $k \to \infty$, $w_\chi(\ell^*, \Omega\ell^*) \to 0$. Since $\Omega\ell^*$ is compact, it is closed, hence $\ell^* \in \Omega\ell^*$.

(b) From (a), $F(\Omega) \neq \emptyset$.

□

4. Numerical example

Example 4.1. Let $\mathbb{S} = [0, 1] \subset \mathbb{R}$ and w_χ be a modular on $\mathbb{S}$ defined by

$$w_\chi(\ell, \wp) = \varrho(\ell, \wp)$$

and ϱ is usual metric defined on $\mathbb{R}$. Suppose that $\mathbb{S}$ is associated with a directed graph G_w such that $v(G_w) = \mathbb{S}$ and $(\ell, \wp) \in E(G_w)$ with $w_\chi(\ell, \wp) \leq 1$. Define a mapping $\Omega : \mathbb{S} \to \mathbb{S}$ by

$$\Omega\ell = \begin{cases} [0, \frac{\ell}{4}], \ \ell \in [0, 1); \\ 0, \ \ell = 1. \end{cases}$$

First, to show that Ω is edge-preserving mapping, for this, consider the following cases:
Case I: when $\ell = \wp = 1$, then $w_\chi(\Omega\ell, \Omega\wp) = 0 < 1$, hence $(\Omega\ell, \Omega\wp) \in E(G)$.
Case II: when $\ell, \wp \in [0, 1)$, then $H_w(\Omega\ell, \Omega\wp) = \frac{1}{4} w_\chi(\ell, \wp) < w_\chi(\ell, \wp)$, hence $(\Omega\ell, \Omega\wp) \in E(G)$.
Case III: when $\ell = 1$, and $\wp \in [0, 1)$, then $H_w(\Omega\ell, \Omega\wp) = \frac{|\wp|}{4} < 1$, hence $(\Omega\ell, \Omega\wp) \in E(G)$. Similar result can be obtained by considering $\wp = 1$, and $\ell \in [0, 1)$.

Now, to prove that Ω is multi-valued Zamfirescu operator, again consider the following cases:

Case I: when $\ell, \wp \in [0, 1)$. Then

$$\begin{aligned} H_w(\Omega\ell, \Omega\wp) &= H_w([0, \frac{\ell}{4}], [0, \frac{\wp}{4}]) \\ &= \frac{1}{4}|\ell - \wp| \\ H_w(\Omega\ell, \Omega\wp) &\leq \frac{1}{4} w_\chi(\ell, \wp). \end{aligned}$$

Hence Ω satisfies (z_i) for $\sigma' = \frac{1}{4}$. Suppose that

$$H_w(\Omega\ell, \Omega\wp) \leq \frac{1}{4}(|\ell| + |\wp|), \tag{4.1}$$

$$\begin{aligned} w_\chi(\Omega\ell, \ell) + w_\chi(\Omega\wp, \wp) &= \varrho(\Omega\ell, \ell) + \varrho(\Omega\wp, \wp) \\ &= \frac{3}{4}(|\ell| + |\wp|). \end{aligned}$$

Suppose that

$$w_\chi(\Omega\ell,\ell)+(\Omega\wp,\wp)=\frac{3}{4}(|\ell|+|\wp|), \tag{4.2}$$

By using (4.1) and (4.2)

$$H_w(\Omega\ell,\Omega\wp)\leq\frac{1}{3}[w_\chi(\Omega\ell,\ell)+(\Omega\wp,\wp)].$$

Hence Ω satisfies (z_{ii}) for $\tau^{'}=\frac{1}{3}$, and

$$\begin{aligned}w_\chi(\Omega\ell,\wp)+(\Omega\wp,\ell)&=\varrho(\Omega\ell,\wp)+\varrho(\Omega\wp,\ell)\\&=\frac{5}{4}(|\ell|+|\wp|).\end{aligned}$$

Suppose that

$$w_\chi(\Omega\ell,\wp)+w_\chi(\Omega\wp,\ell)=\frac{5}{4}(|\ell|+|\wp|), \tag{4.3}$$

By using (4.1) and (4.3)

$$H_w(\Omega\ell,\Omega\wp)\leq\frac{1}{5}[w_\chi(\Omega\ell,\wp)+(\Omega\wp,\ell)].$$

Hence Ω satisfies (z_{iii}) for $\vartheta^{'}=\frac{1}{5}$.
Therefore Ω is a Zamfirescu operator.

Case II: when $\ell=1=\wp$, then $w_\chi(\ell,\wp)=0$, and $w_\chi(\Omega\ell,\Omega\wp)=0\leq\sigma^{'}w_\chi(\ell,\wp)$ for any $\sigma^{'}\in(0,1)$. Hence Ω satisfies (z_i).

Now $\tau^{'}[w_\chi(\Omega\ell,\ell)+(\Omega\wp,\wp)]=\tau^{'}w_\chi(\ell,\wp)$, hence $w_\chi(\Omega\ell,\Omega\wp)\leq\tau^{'}[w_\chi(\Omega\ell,\ell)+(\Omega\wp,\wp)]$ for any $\tau^{'}\in(0,\frac{1}{2})$. Hence Ω satisfies (z_{ii}).

Also $\vartheta^{'}[w_\chi(\Omega\ell,\wp)+(\Omega\wp,\ell)]=\vartheta^{'}w_\chi(\ell,\wp)$, therefore $w_\chi(\Omega\ell,\Omega\wp)\leq\vartheta^{'}[w_\chi(\Omega\ell,\wp)+(\Omega\wp,\ell)]$ for any $\vartheta^{'}\in(0,\frac{1}{2})$. Hence Ω satisfies (z_{iii}).

Case III: when $\ell=1$, and $\wp\in[0,1)$. Then $H_w(\Omega\ell,\Omega\wp)=\frac{1}{4}$. Suppose that

$$\begin{aligned}H_w(\Omega\ell,\Omega\wp)&\leq\sigma^{'}w_\chi(\ell,\wp)\\\frac{1}{4}&\leq\sigma^{'}(1+|\wp|),\end{aligned}$$

which is always true for $\sigma^{'}\in(0,1)$ and $\wp\in[0,1)$.
Also suppose that

$$\begin{aligned}w_\chi(\Omega\ell,\Omega\wp)&\leq\tau^{'}[w_\chi(\Omega\ell,\wp)+w_\chi(\Omega\wp,\ell)]\\\frac{1}{4}&\leq\tau^{'}[1+\frac{3}{4}|\wp|],\end{aligned}$$

which is always true for $\tau^{'}\in(0,\frac{1}{2})$ and

$$\begin{aligned}H_w(\Omega\ell,\Omega\wp)&\leq\vartheta^{'}[w_\chi(\Omega\ell,\wp)+(\Omega\wp,\ell)]\\\frac{1}{4}&\leq\vartheta^{'}[1+\frac{5}{4}|\wp|],\end{aligned}$$

which is always true for $\vartheta' \in (0, \frac{1}{2})$. Hence Ω is a Zamfirescu operator.
Similar results can be obtained by considering $\wp = 1$, and $\ell \in [0, 1)$.

5. Application

Now consider a system of equations as follows:

Let $\mathbb{S} = \mathbb{R}^n$ and define $\varrho_H : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$ such that for $\ell, \wp \in \mathbb{S}$,

$$\varrho_H(\ell, \wp) = \max |\ell_j - \wp_j|. \tag{5.1}$$

Let $\Omega : \mathbb{S} \to \mathbb{S}$ be defined by

$$\Omega\ell = C\ell + b, \tag{5.2}$$

where $C = [c_{jk}]$ be a $n \times n$ matrix, b is the fixed vector of $\mathbb{S}$. Equation (5.2) can be written in component form as

$$\Omega\ell_j = \sum_{k=1}^{n} c_{jk}\ell_{jk} + \beta_j, \tag{5.3}$$

$b = (\beta_j)$, $j = 1, 2, ..., n$. Finding solution of system of equations (5.3) is equivalent to finding fixed points of Ω.

Theorem 5.1. *Let $\mathbb{S} = \mathbb{R}^n$ and define $\varrho_H : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$ such that for $\ell, \wp \in \mathbb{S}$,*

$$\varrho_H(\ell, \wp) = \max |\ell_j - \wp_j| \tag{5.4}$$

It is clear that $(\mathbb{S}, \varrho_H)$ is a complete metric space. Let $\Omega : \mathbb{S} \to CB(\mathbb{S})$ be a multi-valued mapping defined by (5.2) with the assumption that $|C| \leq \frac{1}{2}$. Let w be a modular on $\mathbb{S}$ defined by

$$w_\chi(\ell, \wp) = \varrho_H(\ell, \wp).$$

Then $\mathbb{S}$ is modular space under the modular w. Suppose that $\mathbb{S}$ is associated with a directed transitive graph G_w such that $V(G_w) = \mathbb{S}$ and $E(G_w)$ contains all the loops, i.e for $\ell, \wp \in V(G_w)$, $(\ell, \wp) \in E(G_w)$ with $w_\chi(\ell, \wp) \leq 1$. Suppose that w is convex regular modular which satisfies Δ_2−type condition and triplet $(\mathbb{S}, w_\chi, G_w)$ has property (P). Then Ω has a fixed point in $\mathbb{S}$.

Proof. First, Ω is an edge-preserving mapping, since

$$\begin{aligned} w_\chi(\Omega\ell, \Omega\wp) &= \varrho_H(\Omega\ell, \Omega\wp) \\ &= |C|\varrho_H(\ell, \wp) \\ &\leq \frac{1}{2} w_\chi(\ell, \wp) \\ &< w_\chi(\ell, \wp) \\ &\leq 1. \end{aligned}$$

Next, to prove that Ω is Zamfirescu operator, for this

$$H_w(\Omega\ell, \Omega\wp) = |C| w_\chi(\ell, \wp). \tag{5.5}$$

Now

$$\begin{aligned}\tau^{'}[w_{2\chi}(\Omega\ell,\ell)+w_{2\chi}(\Omega\wp,\wp)] &= \tau^{'}[|\ell-\Omega\ell|+|\wp-\Omega\wp|]\\ &= \tau^{'}|1-C||\ell-\wp|\\ &\leq (1+|C|)w_{\chi}(\ell,\wp).\end{aligned}$$

Suppose that

$$\tau^{'}[w_{2\chi}(\Omega\ell,\ell)+w_{2\chi}(\Omega\wp,\wp)] \leq (1+|C|)w_{\chi}(\ell,\wp). \tag{5.6}$$

Also

$$\begin{aligned}\vartheta^{'}[w_{\chi}(\Omega\ell,\wp)+w_{\chi}(\Omega\wp,\ell)] &= \vartheta^{'}[|\ell-\Omega\wp|+|\wp-\Omega\ell|]\\ &\vartheta^{'}|1-C||\ell-\wp|\\ &\leq \vartheta^{'}(1+|C|)w_{\chi}(\ell,\wp).\end{aligned}$$

Let

$$\vartheta^{'}[w_{2\chi}(\Omega\ell,\wp)+w_{2\chi}(\Omega\wp,\ell)] \leq \vartheta^{'}(1+|C|)w_{\chi}(\ell,\wp). \tag{5.7}$$

□

Choose $\eta=\max\{|C|,\tau^{'}(1+|C|),\vartheta^{'}(1+|C|)\}$, then $\eta\in(0,\frac{1}{2})$ and from equations (5.5), (5.6) and (5.7),

$$H_w(\Omega\ell,\Omega\wp)\leq\eta w_{\chi}(\ell,\wp),$$

$$H_w(\Omega\ell,\Omega\wp)\leq\eta[w_{2\chi}(\Omega\ell,\ell)+w_{2\chi}(\Omega\wp,\wp)],$$

$$H_w(\Omega\ell,\Omega\wp)\leq\eta[w_{2\chi}(\Omega\ell,\wp)+w_{2\chi}(\Omega\wp,\ell)]$$

Hence Ω is Zamfirescu operator. Hence from Theorem 3.3, Ω has a unique fixed point in $\mathbb{S}$, which is solution of a system of equations defined by (5.3).

Conflict of Interest

The authors declares that there is no conflict of interest.

6. Conclusions

Here some convergence results are obtained for multi-valued Zamfirescu operator and these results are justified by taking an example. Also an application of multi-valued Zamfirescu operator is discussed here.

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