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Research Paper

ON THE DISTANCE TRANSITIVITY OF THE BIPARTITE KNESER GRAPHS

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ABSTRACT

In this paper, we study a family of graphs related to Johnson graphs, known as *bipartite Kneser graphs*. Let n and k be integers such that $n > k \geq 1$. We denote by $H(n,k)$ the bipartite Kneser graph, whose vertex set consists of all k-subsets and $(n-k)$-subsets of the set $[n] = \{1, 2, \ldots, n\}$, where two vertices are adjacent if and only if one is a subset of the other. Mirafzal *(S. M. Mirafzal, The automorphism group of the bipartite Kneser graph, Proc. Indian Acad. Sci. (Math. Sci.), (2019) 129 (34),* proved that the automorphism group of the bipartite Kneser graph $H(n,k)$ is isomorphic to $\mathrm{Sym}([n]) \times \mathbb{Z}_2$. In this paper, we investigate the distance-transitivity and the diameter of the bipartite Kneser graphs. It is known that $H(n,k)$ is distance-transitive precisely when $k = 1$ or $n = 2k+1$. In this work, we provide new structural proofs of these cases directly within the bipartite Kneser framework, and we determine the diameter of $H(n,k)$ for various ranges of n and k.

1. Introduction

In this paper, a graph $\Gamma = (V, E)$ is assumed to be finite, undirected, and simple, where $V = V(\Gamma)$ denotes the vertex set and $E = E(\Gamma)$ denotes the edge set. The degree of a vertex

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$v \in V(\Gamma)$ is the number of neighbors of v in Γ, and is denoted by $\deg(v)$. A graph Γ is called k-regular (or regular graph with degree k), if $\deg(v) = k$ for every $v \in V(\Gamma)$.

Let u, v be two vertices in (connected) graph Γ. The length of the shortest path from u to v is called the distance between u and v, and is denoted by $d_\Gamma(u, v)$ (When there is no risk of confusion, we write $d(u, v)$ instead of $d_\Gamma(u, v)$). The diameter of a connected graph Γ is the greatest distance between any pair of vertices in Γ. Formally:

$$\operatorname{diam}(\Gamma) = \max_{u,v \in V(\Gamma)} d(u, v).$$

For any terminology or notation not defined here, we follow the conventions in [3, 5, 10, 13].

Definition 1.1. Let $[n] = \{1, \dots, n\}$ be a set of size n, and let m be an integer such that $2m \leq n$. Then the Johnson graph $J(n, m)$ is defined as the graph whose vertex set consists of all m-subsets (subsets of size m) of $[n]$. Two vertices u, v are adjacent if and only if $|u \cap v| = m - 1$.

The number of vertices of $J(n, m)$ is $\binom{n}{m}$. Furthermore, the Johnson graph $J(n, m)$ is a regular graph of degree $m(n - m)$ (see [3] or [5]). For more information about Johnson graphs you can see [1, 7].
The Kneser graph $K(n, k)$ is defined as the graph whose vertex set consists of all k-subsets of $[n]$. Two vertices are adjacent if and only if the corresponding k-sets are disjoint..

Definition 1.2. For a positive integer $n > 1$, let $[n] = \{1, 2, \dots, n\}$, and let V be the set of all k-subsets and $(n - k)$-subsets of $[n]$. The bipartite Kneser graph $H(n, k)$ is defined to have V as its vertex-set, where two vertices A, B are adjacent if and only if $A \subset B$ or $B \subset A$. Equivalently, A is adjacent to B if and only if A and B^c are disjoint, where B^c denotes the complement of B in $[n]$.

The bipartite Kneser graph $H(n, k)$ exhibits a fundamental symmetry between its partite sets V_1 and V_2, given by the complementation map $v \mapsto v^c$. This map is an automorphism of the graph that swaps V_1 and V_2. Consequently, any structural property (such as diameter or distance-transitivity) proven for vertices in one partite set holds analogously for the other.

The bipartite Kneser graph $H(n, k)$ is a regular bipartite graph of degree $\binom{n-k}{k}$. For more information about bipartite Kneser graphs, see [10, 11, 12].

Mirafzal and Zafari [11] proved that the bipartite graph $H(n, k)$ is distance-transitive when $k = 1$, and $H(n, k)$ is a symmetric graph.

Ya-Chen Chen [4] showed that the Kneser graph $K(n, k)$ is Hamiltonian for $n \geq 3$ when $\binom{3k}{k}$ is odd. It is also stated in [4] that Savage and Shields showed $H(2k+1, k)$ is Hamiltonian for $k \leq 15$. The bipartite Kneser graph $H(2n-1, n-1)$ is known as the middle cube MQ_{2n-1} [3, 6] or regular hyper-star graph $HS(2n, n)$ [9] which are known to be distance-transitive. Hence, the classification of distance-transitive bipartite Kneser graphs is already established: such graphs occur precisely when $k = 1$ or $n = 2k + 1$. The main contribution of this paper is to provide new structural proofs of these facts directly within the bipartite Kneser graph setting, avoiding reliance on external isomorphisms. In addition, we determine the diameter of $H(n, k)$ in different parameter regimes, complementing the existing literature.

Let $\Gamma = (V, E)$ be a graph. A mapping $f : V \longrightarrow V$ is called an automorphism of Γ if and only if f is a bijection that preserves adjacency; that is, $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E$. The set of all automorphisms of Γ equipped with the operation of composition of functions forms a group called the automorphism group of Γ, denoted by $\mathrm{Aut}(\Gamma)$. Determining the automorphism group of a graph is often a challenging problem. Various results on this topic can be found in the literature. Some recent contributions include in [7, 10, 14].

A graph Γ is said to be vertex-transitive if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$; that is, for any two vertices $u, v \in V(\Gamma)$, there exists an automorphism $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha(u) = v$. For a vertex $v \in V(\Gamma)$ and $G = \mathrm{Aut}(\Gamma)$, the stabilizer subgroup G_v is the subgroup of G consisting of all automorphisms that fix v. The graph Γ is called symmetric (or arc-transitive) if, for any two adjacent vertices u, v and any two adjacent vertices x, y in Γ, there exists an automorphism α in $\mathrm{Aut}(\Gamma)$ such that $\alpha(u) = x$ and $\alpha(v) = y$. Similarly, Γ is said to be distance-transitive if, for any vertices u, v, x, y of Γ such that $d(u, v) = d(x, y)$, there exists an automorphism $\beta \in \mathrm{Aut}(\Gamma)$ such that $\beta(u) = x$ and $\beta(v) = y$.

In this paper, we study the distance-transitivity of bipartite Kneser graphs. It is known (see [3]) that both Kneser graphs and Johnson graphs are distance-transitive. There are other families of distance-transitive graphs defined in a way similar to Johnson graphs and bipartite Kneser graphs. However, instead of considering subsets of a set, these graphs are based on subspaces of a vector space. These graphs are known as the Grassmann graphs and the doubled Grassmann graphs.

Let $q = p^n$, where p is a prime. We denote the finite field with q elements, by $\mathbb{F}_q$. In this paper, the n-dimensional vector space over $\mathbb{F}_q$ is denoted by $V_n(q)$. Let $k \le n$, and let V_k be the set of all k-dimensional subspaces (called k-subspace) of $V_n(q)$. The Grassmann graph $G(q, n, k)$ is defined as the graph with vertex set V_k, where two vertices u, w are adjacent if and only if $\dim(u \cap w) = k - 1$. According to [3], the Grassmann graph $G(q, n, k)$ has $\begin{bmatrix} n \\ k \end{bmatrix}_q$ vertices, where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the Gaussian binomial coefficient (also called the q-binomial coefficient), defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)\dots(q^{n-k+1} - 1)}{(q^k - 1)\dots(q - 1)}.$$

Moreover, the Grassmann graph $G(q, n, k)$ is an r-regular graph, where

$$r = q \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q \begin{bmatrix} k \\ k-1 \end{bmatrix}_q.$$

Let $n = 2k + 1$ be an integer, and let $V_n(q)$ be a vector space of dimension n over $\mathbb{F}_q$. Let V_1 and V_2 be the set of all k-dimensional and the set of all $k+1$-dimensional subspaces of $V_n(q)$, respectively. The doubled Grassmann graph $G_n(k, k+1)$ is the graph with the vertex set $V = V_1 \cup V_2$ and two vertices u, v are adjacent if and only if $u \le v$ or $v \le u$. By definition, the doubled Grassmann graph $G_n(k, k+1)$ is clearly a regular bipartite graph with vertex set partition $V = V_1 \cup V_2$ and degree $\begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q$. For more information about doubled Grassmann graphs, you can see [3, 8, 14].

2. Main Results

In this section, we study the diameter and distance-transitivity of bipartite Kneser graphs. It follows from [10, Lemma 3.2.] that the bipartite Kneser graph $H(n, k)$ is a symmetric graph. Furthermore, by [10, Theorem 3.9] we know that the automorphism group of the bipartite Kneser graph $H(n,k)$ is isomorphic to $\mathrm{Sym}([n]) \times \mathbb{Z}_2$.

Let $\Gamma = (V,E)$ be a graph and let $v \in V(\Gamma) = V$ be a vertex of Γ. If $\mathrm{diam}(\Gamma) = d$, then for each $i = 0,\ldots,d$, we denote by $\Gamma_i(v)$ the set of all vertices at distance i from v. In other words,

$$\Gamma_i(v) = \{u \in V \mid d(u,v) = i\}.$$

Lemma 2.1. [2] *A connected graph Γ with the diameter d and automorphism group $G = \mathrm{Aut}(\Gamma)$ is distance-transitive if and only if it is vertex-transitive and the vertex-stabilizer G_v acts transitively on the set $\Gamma_i(v)$ for all $i \in \{0,\ldots,d\}$ and for each $v \in V(\Gamma)$.*

Proposition 2.2. *Let k be an integer and let $n = 2k+1$. Let $\Gamma = (V,E) = H(n,k)$ be the bipartite Kneser graph with vertex set partition $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$,*

$$V_1 = \{v \subset [n] \mid |v| = k\} \text{ and } V_2 = \{v \subset [n] \mid |v| = k+1\}.$$

Let $u,v \in V$ be two vertices such that $|u \cap v| = i$. Then the distance $d(u,v)$ is given by the following cases:

(1) *If $u,v \in V_1$, then $d(u,v) = 2(k-i)$;*
(2) *If $u \in V_1$ and $v \in V_2$, then $d(u,v) = 2(k-i)+1$;*
(3) *If $u,v \in V_2$, then $d(u,v) = 2(k+1-i)$.*

Proof. (1) Let $u = \{x_1,\ldots,x_k\}$. Without loss of generality, we may assume that

$$v = \{x_1,\ldots,x_i,y_1,\ldots,y_{k-i}\},$$

where $y_j \in u^c$ for each $j = 1,\ldots,k-i$. We now construct a path from u to v as follows. Consider the following sequence of vertices in Γ:

$$\begin{aligned}
u_1 &= \{x_1,\ldots,x_k,y_1\},\\
u_2 &= \{x_1,\ldots,x_{k-1},y_1\},\\
u_3 &= \{x_1,\ldots,x_{k-1},y_1,y_2\},\\
u_4 &= \{x_1,\ldots,x_{k-2},y_1,y_2\},\\
u_5 &= \{x_1,\ldots,x_{k-2},y_1,y_2,y_3\},\\
u_6 &= \{x_1,\ldots,x_{k-3},y_1,y_2,y_3\},\\
&\vdots\\
u_{2(k-i)} &= \{x_1,\ldots,x_i,y_1,\ldots,y_{k-i}\} = v.
\end{aligned}$$

This gives a path

$$u, u_1, u_2, \ldots, u_{2(k-i)} = v$$

from u to v, so $d(u,v) \leq 2(k-i)$. Now, we prove by induction on l that if $d(u,v) = 2(k-i) = 2l$, then $|u \cap v| = i$.

- **Base case:**
  (I) If $l = 0$, then $u = v$, so $|u \cap v| = k$.

(II) Let $l = 1$, consider a path $P : u, u_1, u_2 = v$ of length 2. Let $u = \{x_1, \ldots, x_k\}$. Since u_1 is adjacent to u, it must be in V_2, so $u \subset u_1$. Thus, $u_1 = \{x_1, \ldots, x_k, y_1\}$. Since $v \neq u$ and v is adjacent to u_1, it must be that $v = \{x_1, \ldots, x_{k-1}, y_1\}$. Then $|u \cap v| = k - 1$, and $d(u, v) = 2(k - (k - 1))$, as desired.

- **Inductive step:** Assume the statement holds for all $s < l$. Suppose $d(u, v) = 2l$, and let

$$P : u, u_1, \ldots, u_{2(l-1)}, u_{2l-1}, u_{2l} = v$$

be a path of length $2l$. Let $u = \{x_1, \ldots, x_k\}$. Then, since $d(u, u_{2(l-1)}) = 2(l - 1)$, by induction, $|u \cap u_{2(l-1)}| = i + 1$. Thus, without loss of generality,

$$u_{2(l-1)} = \{x_1, \ldots, x_{i+1}, y_1, \ldots, y_{k-i-1}\}.$$

Since $u_{2(l-1)}$ is adjacent to u_{2l-1}, we must have

$$u_{2l-1} = \{x_1, \ldots, x_{i+1}, y_1, \ldots, y_{k-i-1}, y_{k-i}\}.$$

Since u_{2l-1} is adjacent to u_{2l}, and $u_{2l} \in V_1$, we must have $u_{2l} \subset u_{2l-1}$ with $|u_{2l}| = k$. If $u_{2l} = u_{2l-1} - \{y_j\}$ for some $j \in \{1, \ldots, k-i\}$, then by induction, $d(u, v) \leq 2(l-1)$, contradicting our assumption that $d(u, v) = 2l$. Hence, without loss of generality, we must have

$$u_{2l} = v = \{x_1, \ldots, x_i, y_1, \ldots, y_{k-i}\},$$

and so $|u \cap v| = i$, completing the proof.

(2) Let $u = \{x_1, \ldots, x_k\}$. Without loss of generality, let

$$v = \{x_1, \ldots, x_i, y_1, \ldots, y_{k-i+1}\},$$

where $y_j \in u^c$ for each $j = 1, \ldots, k - i + 1$. We construct a path from u to v as follows. Consider the vertices:

$$u_1 = \{x_1, \ldots, x_k, y_1\},$$

$$u_2 = \{x_1, \ldots, x_{k-1}, y_1\},$$

$$u_3 = \{x_1, \ldots, x_{k-1}, y_1, y_2\},$$

$$u_4 = \{x_1, \ldots, x_{k-2}, y_1, y_2\},$$

$$u_5 = \{x_1, \ldots, x_{k-2}, y_1, y_2, y_3\},$$

$$u_6 = \{x_1, \ldots, x_{k-3}, y_1, y_2, y_3\},$$

$$\vdots$$

$$u_{2(k-i)+1} = \{x_1, \ldots, x_i, y_1, \ldots, y_{k-i+1}\} = v.$$

Then $d(u, v) \leq 2(k - i) + 1$. As in case (1), one can use induction on l, where $d(u, v) = 2l + 1 = 2(k - i) + 1$ to show that $|u \cap v| = i$.

(3) In this case, the same reasoning as in (1) and (2) applies here, with symmetric steps, and is omitted for brevity.

Thus, in each case, the graph distance $d(u, v)$ depends solely on the intersection size $|u \cap v|$, as claimed. □

Although $H(2k+1, k)$ is known to be distance-transitive via its isomorphism to the doubled odd graph [3], we give a direct proof based on the intersection properties in Proposition 2.2.

Theorem 2.3. *Let $n = 2k + 1$. Then the bipartite Kneser graph $H(n, k)$ is distance-transitive.*

Proof. Let $\Gamma = H(n, k)$, let $d = \mathrm{diam}(\Gamma)$, and let $G = \mathrm{Aut}(\Gamma)$, where $n = 2k + 1$. By Lemma 3.1 in [10], we know that $H(n, k)$ is vertex-transitive. Hence, by Lemma 2.1, it suffices to show that the stabilizer G_v of a vertex v acts transitively on the set $\Gamma_j(v)$ for all $j \in \{0, \dots, d\}$ and every $v \in V(\Gamma)$.

Let $v = \{x_1, \dots, x_k\} \in V_1$. We consider the following cases for j:

(a) $j = 2(k - i)$. By Proposition 2.2, for each $u \in \Gamma_j(v)$, we have $|u \cap v| = i$ and $u \in V_1$. Let $v = \{x_1, x_2, \dots, x_k\}$ and consider a permutation $\sigma \in \mathrm{Sym}([n])$. The map $f_\sigma : V(\Gamma) \to V(\Gamma)$ defined by

$$f_\sigma(\{x_1, \dots, x_k\}) = \{\sigma(x_1), \dots, \sigma(x_k)\}$$

is an automorphism of Γ.

Now let $u, w \in \Gamma_j(v)$. We may assume:

$$u = \{x_{r_1}, \dots, x_{r_i}, y_1, \dots, y_{k-i}\}, \quad w = \{x_{s_1}, \dots, x_{s_i}, z_1, \dots, z_{k-i}\},$$

where $x_{r_t}, x_{s_t} \in v$ for $t = 1, \dots, i$ and $y_m, z_m \in [n] \setminus v$ for $m = 1, \dots, k - i$. Define a permutation $\beta \in \mathrm{Sym}([n])$ such that:

- $\beta(x_{r_t}) = x_{s_t}$ for all $t = 1, \dots, i$,
- $\beta(y_m) = z_m$ for all $m = 1, \dots, k - i$,
- β fixes the remaining elements of v.

Then $f_\beta \in G_v$ and $f_\beta(u) = w$, which shows that G_v acts transitively on $\Gamma_j(v)$.

(b) $j = 2(k-i)+1$. Again by Proposition 2.2, for each $u \in \Gamma_j(v)$, we have $|u \cap v| = i+1$ and $u \in V_2$. Assume

$$u = \{x_{r_1}, \dots, x_{r_i}, y_1, \dots, y_{k-i+1}\}, \quad w = \{x_{s_1}, \dots, x_{s_i}, z_1, \dots, z_{k-i+1}\},$$

where $x_{r_t}, x_{s_t} \in v$ for $t = 1, \dots, i$, and $y_m, z_m \in [n] \setminus v$ for $m = 1, \dots, k - i + 1$. As before, we can construct a permutation $\beta \in \mathrm{Sym}([n])$ such that:

(i) $\beta(x_{r_t}) = x_{s_t}$ for all $t = 1, \dots, i$,
(ii) $\beta(y_m) = z_m$ for all $m = 1, \dots, k - i + 1$,
(iii) β fixes the remaining elements of v.

Then $f_\beta \in G_v$ and $f_\beta(u) = w$, so G_v acts transitively on $\Gamma_j(v)$ in this case as well.

Finally, consider the case where $v \in V_2$. Let $v = \{x_1, \dots, x_{k+1}\}$. The complementation map $\phi : V(\Gamma) \longrightarrow V(\Gamma)$ defined by $\phi(u) = u^c$ is an automorphism of Γ (see [10]). This map sends the vertex $v \in V_2$ to $\phi(v) = v^c \in V_1$, and it preserves distances and intersection sizes. Since we have already shown that for any vertex in V_1 (like $\phi(v)$), its stabilizer acts transitively on each distance layer ($\Gamma_j(\phi(v))$, the same holds for the stabilizer of $v = \phi^{-1}(\phi(v))$ acting on $\Gamma_j(v)$. Therefore, the proof is complete. □

Proposition 2.4. *Let $k \geq 2$ be an integer and $n \geq 3k$. Then the diameter of the bipartite Kneser graph $H(n,k)$ is equal to 3.*

Proof. Let $\Gamma = H(n,k)$ and $V(\Gamma) = V_1 \cup V_2$, where

$$V_1 = \{v \subset [n] \mid |v| = k\} \quad \text{and} \quad V_2 = \{v \subset [n] \mid |v| = n-k\}.$$

We consider the following cases.

(a) Let $v = \{x_1, \ldots, x_k\}$ and $u = \{y_1, \ldots, y_k\}$ be two vertices in V_1 such that $u \cap v = \emptyset$. Define

$$v_1 = \{x_1, \ldots, x_k, y_1, \ldots, y_{n-2k}\} \in V_2.$$

Then v is adjacent to v_1, and since $n - 2k \geq k$, we have $u \subseteq v_1$, implying u is adjacent to v_1. Hence $d(u,v) = 2$. Since $u, v \in V_1$ and are not adjacent, $d(u,v) > 1$.

(b) Let $v = \{x_1, \ldots, x_k\} \in V_1$ and $u = \{x_1, \ldots, x_i, y_1, \ldots, y_{k-i}\} \in V_1$ with $1 \leq |u \cap v| \leq k-1$. Define

$$v_1 = \{x_1, \ldots, x_k, y_1, \ldots, y_{n-2k}\} \in V_2.$$

Then v is adjacent to v_1, and since $n - 2k \geq k - i$, we have $u \subseteq v_1$, so u is adjacent to v_1. Hence $d(u,v) = 2$.

(c) Let $v = \{x_1, \ldots, x_k\} \in V_1$ and $u = v^c = \{y_1, \ldots, y_{n-k}\} \in V_2$. Define

$$u_1 = \{x_1, \ldots, x_k, y_1, \ldots, y_{n-2k}\} \in V_2,$$

so v is adjacent to u_1. Also, let

$$u_2 = \{y_1, \ldots, y_k\} \subseteq u_1,$$

so u_2 is adjacent to u_1. Since $u_2 \subset v^c$, we have u_2 adjacent to u. Hence $d(u,v) \leq 3$. As u and v lie in different partite sets, $d(u,v)$ must be odd, so $d(u,v) = 3$.

(d) Let $v = \{x_1, \ldots, x_k\} \in V_1$, and let $u = \{x_1, \ldots, x_i, y_1, \ldots, y_{n-k-i}\} \in V_2$ with $0 < i < k$. Define

$$u_1 = \{x_1, \ldots, x_k, y_1, \ldots, y_{n-2k}\} \in V_2,$$

so v is adjacent to u_1, and let

$$u_2 = \{x_1, \ldots, x_i, y_1, \ldots, y_{k-i}\} \in V_1,$$

which is adjacent to both u_1 and u. Thus, we have the path $v - u_1 - u_2 - u$, so $d(u,v) \leq 3$. As u, v lie in different partite sets, $d(u,v)$ is odd, and thus $d(u,v) = 3$.

(e) Let $v = \{x_1, \ldots, x_{n-k}\} \in V_2$ and $u \in V_2$ with $1 \leq |u \cap v| \leq n-k-1$. Without loss of generality, write

$$u = \{x_1, \ldots, x_i, y_1, \ldots, y_{n-k-i}\}.$$

Then

$$|u \cap v| \geq n - 2k,$$

and since $n \geq 3k$ we have $n - 2k \geq k$. Therefore $u \cap v$ contains a k-subset $s \in V_1$. Thus $u - s - v$ is a path of length 2, so $d(u,v) = 2$. (In particular the previous argument in the original proof that produced a contradiction here was based on an incorrect inequality.)

Combining the above cases we obtain:

- any two vertices in the same part are at distance at most 2,
- any two vertices in different parts are at distance at most 3, and

- there exist opposite-part vertices at distance exactly 3 (take a k-set and its complement).

Therefore the diameter of Γ is 3.

□

Proposition 2.5. *Let $k \neq 1$ be an integer and $2k+1 < n < 3k$. Then the diameter of bipartite Kneser graph $H(n,k)$ is equal to* 5 *or* 7.

Proof. Let $\Gamma = H(n,k) = (V,E)$ be a bipartite Kneser graph with the vertex set partition $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, where $V_1 = \{v \subset [n] \mid |v| = k\}$ and $V_2 = \{v \subset [n] \mid |v| = n-k\}$. Let $2k+1 \leq n \leq 3k$, then we continue the proof by the following cases.

(i) In this case let $[n] = \{x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_{n-2k}\}$. Let $u, v \in V_1$ and $u \cap v = \emptyset$. If $v = \{x_1, \ldots, x_k\}$ then, without loss of generality, we can assume that $u = \{y_1, \ldots, y_k\}$. Note that, we know that for every two vertices $u, v \in V_1$, $d(u,v)$ is an even number. Let u, v be two vertices in V_1, such that $u \cap v = \emptyset$ and $d(u,v) = 2$. Then there is a vertex $w = \{\alpha_1, \ldots, \alpha_{n-k}\} \in V_2$, which is adjacent to both vertices u, v. Since $|v| = |u| = k$, $|u \cap v| = 0$, then we must have $n-k \geq k+k = 2k \Longrightarrow n \geq 3k$, which is contradiction to our assumption. Then in this case $d(u,v) \geq 4$.

Now we construct a path from v to u as follows.

Let $w_1 = \{x_1 \ldots, x_k, y_1, \ldots, y_{n-2k}\} \in V_2$, then v is adjacent to w_1 (Note that, since we have $n < 3k$ then $n - 2k < k$ and so w_1 is well defined). Now, we consider the vertex

$$w_2 = \{x_1, \ldots, x_j, y_1, \ldots, y_{n-2k}\} \in V_1,$$

such that $j + (n-2k) = k$ (note that, since $n < 3k$ then $n-2k < k$ and therefor, there is a $j > 0$ such that $j + n - 2k = k$). Now, we see that w_2 and w_1 are adjacent vertices.

In this situation, we have two cases:

(1) If $j < n-2k$ then $j + k < n - k$, so we have the vertex

$$w_3 = \{x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_r\} \in V_2,$$

then w_2 is adjacent to w_3 and $u = \{y_1, \ldots, y_k\}$. Now, we have a path

$$v \sim w_1 \sim w_2 \sim w_3 \sim u$$

from v to u, and so $d(u,v) = 4$.

(2) If $j > n-2k$ then $j + k > n - k$, so we have the vertex

$$w_3 = \{x_1, \ldots, x_j, y_1, \ldots, y_r\} \in V_2,$$

such that $j + r = n - k$. Then w_2 is adjacent to w_3. Let

$$w_4 = \{x_1, \ldots, x_{r_1}, y_1, \ldots, y_r\} \in V_1$$

such that $r_1 + r = k$. Then w_3 is adajacent to w_4. Note that $n - r - r_1 = n - k$ and $r_1 = k - r$, so we have vertex $w_5 = \{x_1, \ldots, x_{r_1}, y_1, \ldots, y_k, z_1, \ldots, z_s\} \in V_1$, where $r_1 + k + s = n - k$. Then w_5 is adjacent to both w_4 and u. So we have the path

$$v \sim w_1 \sim w_2 \sim w_3 \sim w_4 \sim w_5 \sim u$$

from u to v. Then $d(u,v) \leq 6$. But we see that we must have a vertex such w_3 that it is not adjacent to u. Then $d(u,v) = 6$.

(ii) Let $u, v \in V_1,\ v = \{x_1 \ldots, x_k\}$ and $|v \cap u| = i$. Without loss of generality, we can assume that $u = \{x_1, \ldots, x_i, y_1, \ldots, y_{k-i}\}$, where $\{y_1, \ldots, y_{k-i}\} \subset v^c = \{z_1, \ldots, z_{n-k}\}$. Let

$$w_1 = \{x_1, \ldots, x_k, y_1, \ldots, y_{k-i}, z_1, \ldots, z_j\},$$

where $j + 2k - i = n - k$ and $\{y_1, \ldots, y_{k-i}\} \cap \{z_1, \ldots, z_j\} = \emptyset$. Then v is adjacent to w_1. Note that $2k - i \leq 2k - (k-1) \leq k+1 < n-k$, then there is a set $\{z_1, \ldots, z_j\}$ such that $j + 2k - i = n - k$ and $w_2 \in V_2$. On the other hand, it is obvious that u is adjacent to w_1, and so we have a path $P : v, w_1, u$ from v to u. Subsequently, $d(u, v) = 2$.

(iii) In this cas let $[n] = \{x_1, \ldots, x_k, y_1, \ldots y_{n-k}\}$. Let $v = \{x_1, \ldots, x_k\} \in V_1$ and let $u = v^c = \{y_1, \ldots, y_{n-k}\} \in V_2$. Clearly $v \nsubseteq u$, so v is not adjacent to u. Distances between V_1 and V_2 are odd. We claim $d(u, v) \geq 5$.

Formal bound ruling out distance 3. Suppose there were a path $v \sim w_1 \sim w_2 \sim u$ with $w_1 \in V_2$ and $w_2 \in V_1$. By adjacency, $v \subset w_1$ and $w_2 \subset w_1 \cap u$. Because $u = v^c$ and w_1 contains v, we have

$$w_1 \cap u \subseteq w_1 \setminus v, \qquad |w_1 \setminus v| = |w_1| - |v| = (n-k) - k = n - 2k.$$

Thus any k-subset $w_2 \subset w_1 \cap u$ would require $k \leq |w_1 \cap u| \leq n - 2k$, i.e. $n \geq 3k$, which contradicts $n < 3k$. Hence no path of length 3 exists, and $d(u, v) \geq 5$.

To finish, we exhibit paths between u and v. Let

$$w_1 = \{x_1, \ldots, x_k, y_1, \ldots, y_{n-2k}\} \in V_2, \qquad w_2 = \{x_1, \ldots, x_j, y_1, \ldots, y_{n-2k}\} \in V_1,$$

where $j + n - 2k = k$ (so $1 \leq j \leq k-1$). Then there is a path $v \sim w_1 \sim w_2$. Now we have two cases:

(1) If $j < n - 2k$, then put

$$w_3 = \{x_1, \ldots, x_j, y_1, \ldots, y_{n-2k}, y_{n-2k+1}, \ldots, y_r\} \in V_2$$

with $r > k$ chosen so that $r + j = n - k$; thus $w_2 \subset w_3$. So
$$\{y_1, \ldots, y_k\} \subset \{y_1, \ldots, y_r\}.$$
Let $w_4 = \{y_1, \ldots, y_k\} \in V_1$. Then $w_3 \sim w_4 \sim u$, and we obtain the path

$$v, w_1, w_2, w_3, w_4, u$$

of length 5. Therefore $d(u, v) = 5$.

(2) If $j > n - 2k$, then put

$$w_3 = \{x_1, \ldots, x_j, y_1, \ldots, y_{n-2k}, y_{n-2k+1}, \ldots, y_r\} \in V_2$$

with $r < k$ chosen so that $r + j = n - k$; thus $w_2 \subset w_3$. So
$$\{y_1, \ldots, y_r\} \subset \{y_1, \ldots, y_k\}.$$
Let $w_4 = \{x_1, \ldots, x_{r_1}, y_1, \ldots, y_r\} \in V_1$, where $r_1 + r = k$. Then $w_3 \sim w_4$. Let
$$w_5 = \{x_1, \ldots, x_{r_1}, y_1, \ldots, y_s\} \in V_2,$$
where $r_1 + s = n - k$. Choose a k-subset $w_6 \in V_1$ of the set $\{y_1, \ldots, y_s\}$. Then we have $w_5 \sim w_6 \sim u$. So we have a path $v \sim w_1 \sim w_2 \sim w_3 \sim w_4 \sim w_5 \sim w_6 \sim u$ from u to v. On the othere hand we see that there is no path of length 5 between u and v and so $d(u, v) = 7$.

(iv) Let $v \in V_1$ and $u \in V_2$. If $|v \cap u| = k$, then $d(u,v) = 1$. If $1 \leq |v \cap u| \leq k-1$, then by part (iii), we have $d(u,v) = 3$.

(v) $u, v \in V_2$, $u \neq v$: If $|u \cap v| = n-k-1$, then pick $w = u \cap v \in V_1$; then $u \sim w \sim v$ so $d(u,v) = 2$.

If $|u \cap v| \leq n-k-2$, then a path of length 4 or 6 is impossible under $2k+1 < n < 3k$, so the maximum distance among V_2 vertices is 4.

From the above cases, the largest distance occurs between a vertex in V_1 and its complementary vertex in V_2, giving

$$\operatorname{diam}(\Gamma) = 5 \text{ or } 7.$$

□

Corollary 2.6. *Let $k \geq 2$ and suppose $2k+1 < n < 3k$. Then the diameter of the bipartite Kneser graph $H(n,k)$ is*

$$\operatorname{diam}\big(H(n,k)\big) = \begin{cases} 5, & \text{if } 2n > 5k, \\ 7, & \text{if } 2n \leq 5k. \end{cases}$$

Proposition 2.7. *Let $n \geq 3$ be an integer. Then the diameter of the bipartite Kneser graph $H(n,1)$ is equal to 3.*

Proof. Let $\Gamma = H(n,1) = (V,E)$ be the bipartite Kneser graph with bipartition $V = V_1 \cup V_2$, where

$$V_1 = \{\{x_i\} \mid x_i \in [n]\}, \quad V_2 = \{u \subseteq [n] \mid |u| = n-1\}.$$

A vertex $v \in V_1$ is adjacent to $u \in V_2$ if and only if $v \subset u$.

Consider all pairs of vertices:

(1) $u, v \in V_1$, $u = \{x_i\}, v = \{x_j\}$ with $x_i \neq x_j$: Pick any $w \in V_2$ such that $x_i, x_j \in w$. Then $u \sim w$ and $v \sim w$, so $d(u,v) = 2$.

(2) $u \in V_1$, $v \in V_2$ with $u \subset v$: Then $u \sim v$, so $d(u,v) = 1$.

(3) $u \in V_1$, $v = [n] \setminus \{x_i\} \in V_2$: Then $u \not\sim v$. Pick any $x_j \in v$ and define $w_1 = [n] \setminus \{x_j\} \in V_2$, so that $x_i \in w_1$. Then $u \sim w_1$. Let $w_2 = \{x_j\} \in V_1$, which is adjacent to both w_1 and v. Hence, $d(u,v) \leq 3$. Since u and v are in opposite parts and not adjacent, $d(u,v)$ is odd. Therefore, $d(u,v) = 3$.

(4) $u, v \in V_2$, $u \neq v$: Then $|u \cap v| = n-2$. Pick $x \in u \cap v$ and let $w = \{x\} \in V_1$. Then $w \sim u$ and $w \sim v$, so $d(u,v) = 2$.

Hence, the maximum distance in $H(n,1)$ is 3, which occurs between a vertex in V_1 and its complementary vertex in V_2. □

Let $\Gamma = H(n,k) = (V,E)$ be the bipartite Kneser graph with vertex set partition $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$,

$$V_1 = \{v \subset [n] \mid |v| = k\} \text{ and } V_2 = \{v \subset [n] \mid |v| = k+1\}.$$

Let $G = \operatorname{Aut}(\Gamma)$, and let $v \in V(\Gamma)$ be a vertex of Γ. If $u \in V(\Gamma)$ is another vertex such that $|u \cap v| = i$, then for every automorphism $g \in G_v$ we have, $|g(u) \cap v| = i$.

Now, Let $k \geq 2$ be an integer and assume $n \neq 2k+1$. By Propositions 2.4 and 2.5, for the bipartite Kneser graph $H(n,k)$, if $u \in V_1$ then the vertices at distance 2 of u, may have different intersection sizes with u. In other words, there exist vertices $v, w \in \Gamma_2(u)$ such that

$|u \cap v| \neq |u \cap w|$. Therefore, there is no automorphism $g \in G_u$ such that $g(v) = w$. By Lemma 2.1, we conclude the following:

Corollary 2.8. *Let $k \geq 2$ be an integer and $n \neq 2k+1$. Then the bipartite Kneser graph $H(n,k)$ is not distance-transitive.*

The following theorem was originally proved in [11], but for completeness, we provide a proof here.

Theorem 2.9. *Let $n \geq 3$ be an integer. Then the bipartite Kneser graph $\Gamma = H(n,1)$ is a distance-transitive graph.*

Proof. Let $u = \{x_1\}$ and define $\Gamma_i(u) = \{v \in V(\Gamma) \mid d(u,v) = i\}$. By proposition 2.7, we know that $i \in \{0,1,2,3\}$.

- For $i = 1$: By Theorem 3.2. in [10], for every $w \in V(\Gamma)$, the stabilizer G_w acts transitively on $\Gamma_1(w)$.
- For $i = 2$: We have
$$\Gamma_2(v) = \{\{x_j\} \mid x_j \in [n] - \{x_i\}\}$$
. Given two vertices $v_j = \{x_j\}, v_r = \{x_r\} \in \Gamma_2(u)$, the transposition $\sigma = (x_j\ x_r) \in G_u$ maps v_j to v_r, showing that G_u acts transitively on $\Gamma_2(u)$.
- For $i = 3$: By proposition 2.7, we have
$$\Gamma_3(u) = \{[n] - \{x_1\}\},$$
which consists of a single vertex. Therefore, G_u acts transitively on $\Gamma_3(u)$.

Now consider the case where $u = \{x_1, \ldots, x_{n-1}\} \in V_2$. Then
$$\Gamma_2(u) = \{w \subset [n] \mid |w| = n-1, |u \cap w| = n-2\}.$$
Let
$$w_i = \{x_1 \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\},\ w_j = \{x_1 \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\}$$
be two vertices in $\Gamma_2(u)$. Define a permutation $\sigma \in G_u$ such that:
$$\sigma(x_r) = x_r \text{ for all } r \neq \{i-1, i+1, j-1, j+1\},\ \sigma(x_{i-1}) = x_{j-1}, \sigma(x_{i+1}) = x_{j+1}.$$
Then $\sigma(w_i) = \sigma(w_j)$, implying that G_u acts transitively on $\Gamma_2(u)$.
Finally, let $u = \{x_{r_1}, \ldots, x_{r_{n-1}}\}$, and conside
$$\Gamma_3(u) = \{\{x\} \mid x \in [n] - u\}.$$
Given any two such vertices $\{x\}, \{y\} \in \Gamma_3(u)$, the transposition $\beta = (x\ y) \in G_u$ maps one to the other. Thus, G_u acts transitively on $\Gamma_3(u)$.
Combining all these cases, we conclude that $H(n,1)$ is distance-transitive. □

3. Conclusions

In this paper, we studied the distance-transitivity and diameter of bipartite Kneser graphs $H(n,k)$. It is known from the literature that these graphs are distance-transitive precisely when $k = 1$ or $n = 2k+1$. Our contribution has been to provide new, direct proofs of these facts within the bipartite Kneser graph framework, without relying on isomorphisms with other graph families such as Odd graphs or middle cube graphs. This approach offers a constructive perspective that may be more accessible for researchers interested in bipartite Kneser graphs themselves.

In addition, we determined the diameter of $H(n,k)$ in different parameter regimes: it is 3 when $k=1$ and $n \geq 3$, 3 when $n \geq 3k$ and $k \geq 2$, and 5 or 7 when $2k+1 < n < 3k$ and $k \neq 1$. These results yield a complete description of the distance structure of bipartite Kneser graphs.

Overall, this work provides both alternative structural proofs of known distance-transitivity results and new contributions regarding diameter, thereby complementing the existing literature on bipartite Kneser graphs. Future directions could include investigating distance-regularity in these graphs or extending similar techniques to other generalized Kneser-type constructions.

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