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Research Paper

TWAIN SECURE PERFECT DOMINATING SETS AND TWAIN SECURE PERFECT DOMINATION POLYNOMIALS OF CYCLES

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ABSTRACT

Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is a dominating set of G, if every vertex in $V \backslash S$ is adjacent to at least one vertex in S. A subset S of V is called a twain secure perfect dominating set of G (TSPD-set) if every vertex $v \in V \backslash S$ is adjacent to exactly one vertex $u \in S$ and $(S \backslash \{u\}) \cup \{v\}$ is a dominating set of G. The minimum cardinality of a twain secure perfect dominating set of G is called the twain secure perfect domination number of G and is denoted by $\gamma_{tsp}(G)$. Let $D_{tsp}(C_n, i)$ denote the family of all twain secure perfect dominating sets of C_n with cardinality i, for $\gamma_{tsp}(C_n) \leq i \leq n$. Let $d_{tsp}(C_n, i) = |D_{tsp}(C_n, i)|$. In this article, we derive a recursive formula for $d_{tsp}(C_n, i)$ and construct $D_{tsp}(C_n, i)$. We consider the polynomial $D_{tsp}(C_n, x) = \sum_{i=\gamma_{tsp}(C_n)}^{n} d_{tsp}(C_n, i) x^i$, which we refer to as the twain secure perfect domination polynomial of cycles using this recursive formula. In this research, we use a recursive technique to generate all twain secure perfect dominating sets of cycles and twain secure perfect domination polynomials of cycles.

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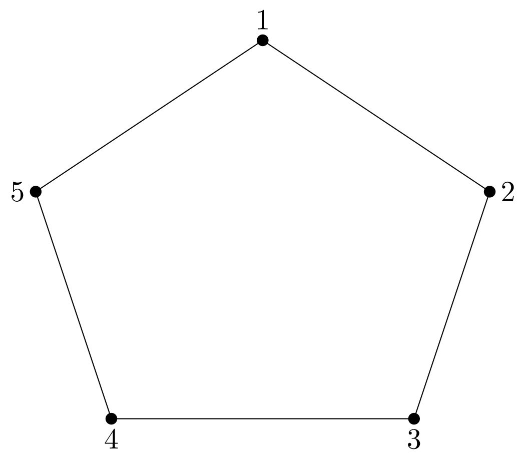
1. Introduction

A finite undirected connected graph without loops or multiple edges is referred to as a graph $G = (V, E)$. G's order and size are shown by the numbers n and m, respectively. For fundamental terms and definitions, see [2]. Let u and v be two vertices. If uv is one of G's edges, then u and v are considered *adjacent*. A vertex v in a graph G has an *open neighborhood* defined as the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, and a *closed neighborhood* defined as $N_G[v] = N_G(v) \cup \{v\}$. A subset $S \subseteq V(G)$ is called a *dominating set* if every vertex $v \in V(G) \setminus S$ is adjacent to at least one vertex $u \in S$. The *domination number*, $\gamma(G)$, of a graph G is the minimum size among all dominating sets of G. A minimum dominating set of a graph G is hence often called as a $\gamma-$ set of G [1]. A dominating set S is called a *secure dominating set* if for each $v \in V(G) \backslash S$ there exists $u \in N(v) \cap S$ such that $(S \backslash \{u\}) \cup \{v\}$ is a dominating set. The *secure domination number*, $\gamma_s(G)$, is the minimum size of a secure dominating set in G. Cockayne et al introduce the concept of secure domination of graphs[3]. A dominating set S is called a *perfect dominating set* if every vertex in $V(G) \backslash S$ is adjacent to exactly one vertex in S. The *perfect domination number* $\gamma_p(G)$ is the minimum cardinality of a perfect dominating set of G. The concept of perfect domination of graphs is introduced by Weichsel [10]. We introduce the concept of twain secure perfect domination of graphs in this work.. A dominating set S is called a *twain secure perfect dominating set* of G (TSPD-set) if for every vertex $v \in V(G) \backslash S$ is adjacent to exactly one vertex $u \in S$ and $(S \backslash \{u\}) \cup \{v\}$ is a dominating set of G. The *twain secure perfect domination number* of G, represented as $\gamma_{tsp}(G)$, is the lowest cardinality of a twain secure perfect dominating set of G. Consider the cycle C_n, which has n vertices. Its $V(C_n) = \{1, 2, ..., n\}$ and $E(C_n) = \{(1, 2), (2, 3), ..., (n - 1, n), (n, 1)\}$. The twain secure perfect domination number of C_n is denoted by $\gamma_{tsp}(C_n)$. The family of all twain secure perfect dominating sets of C_n with cardinality i is denoted as $D_{tsp}(C_n, i)$. Let $d_{tsp}(C_n, i) = |D_{tsp}(C_n, i)|$. Then the twain secure perfect domination polynomial of C_n is defined as $D_{tsp}(C_n, x) = \sum_{i=\gamma_{tsp}(C_n)}^{n} d_{tsp}(C_n, i)x^i$, where $\gamma_{tsp}(C_n)$ is the twain secure perfect domination number of C_n.

Example 1.1. Consider the graph C_5



G
Fig 1.1

In Fig 1.1, $S = \{1, 2, 3\}$ is a twain secure perfect dominating set. For, $V\backslash S = \{4, 5\}$. The vertex $4 \in V\backslash S$ is adjacent to $3 \in S$ and the vertex $5 \in V\backslash S$ is adjacent to $1 \in S$. Also, $(S\backslash\{3\}) \cup \{4\} = \{1, 2, 4\}, (S\backslash\{1\}) \cup \{5\} = \{2, 3, 5\}$ are dominating sets. So, the set $S = \{1, 2, 3\}$ is a twain secure perfect dominating set.

The families of the twain secure perfect dominating sets of cycles are built using a recursive techniques in the following section.

For the smallest integer higher than or equal to x, we use $\lceil x \rceil$ as normal. We refer to the set $\{1, 2, ..., n\}$ in this article as $[n]$.

2. Twain Secure Perfect Dominating Sets of Cycles

The family of twain secure perfect dominating sets of C_n with cardinality i is denoted by $D_{tsp}(C_n, i)$. Also twain secure perfect dominating sets of cycles will be examined. The following lemmas are necessary to support the primary findings of this article.

Lemma 2.1. *For every $n \in \mathbb{N}$,*

(i) If $n \leq 3$, then $\gamma_{tsp}(C_n) = 1$.

(ii) If $n > 3$, $\gamma_{tsp}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{for} \quad n \equiv 0, 1 (mod\ 4) \\ \lceil \frac{n}{2} \rceil + 1 & \text{for} \quad n \equiv 2, 3 (mod\ 4) \end{cases}$.

Proof. (i) When $n \leq 3$, by the definition of twain secure perfect domination, a single vertex dominates the remaining vertices of C_n. Therefore, $\gamma_{tsp}(C_n) = 1$.

(ii) When $n > 3$, let $C_n = \{v_1, v_2, ..., v_{n-1}, v_n, v_{n+1} = v_1\}$. Now we consider four cases.

- $n \equiv 0 (mod\ 4)$. Thus $n = 4k \geq 4$. Then $\{v_1, v_2, v_5, v_6, ..., v_{n-3}, v_{n-2}\}$ is a twain secure perfect dominating set of $2k = \frac{n}{2}$ vertices.
- $n \equiv 1 (mod\ 4)$. Thus $n = 4k + 1 \geq 5$.
  Then $\{v_1, v_2, v_5, v_6, ..., v_{n-8}, v_{n-7}, v_{n-4}, v_{n-3}, v_n\}$ is a twain secure perfect dominating set of $2k + 1 = \lceil \frac{n}{2} \rceil$ vertices.
- $n \equiv 2 (mod\ 4)$. Thus $n = 4k + 2 \geq 6$.
  Then $\{v_1, v_2, v_5, v_6, ..., v_{n-5}, v_{n-4}, v_{n-1}, v_n\}$ is a twain secure perfect dominating set of $2k + 2 = \frac{n}{2} + 1$ vertices.
- $n \equiv 3 (mod\ 4)$. Thus $n = 4k + 3 \geq 7$.
  Then $\{v_1, v_2, v_5, v_6, ..., v_{n-6}, v_{n-5}, v_{n-2}, v_{n-1}, v_n\}$ is a twain secure perfect dominating set of $2k + 3 = \lceil \frac{n}{2} \rceil + 1$ vertices.

Therefore from all cases, $\gamma_{tsp}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{for} \quad n \equiv 0, 1 (mod\ 4) \\ \lceil \frac{n}{2} \rceil + 1 & \text{for} \quad n \equiv 2, 3 (mod\ 4) \end{cases}$.

□

Lemma 2.2. *For every $n \in \mathbb{N}$, $D_{tsp}(C_n, i) = \emptyset$ if and only if $i > n$.*

Proof. Assume that there are n vertices in a cycle C_n, and that every member of $D_{tsp}(C_n, i)$ has at most n vertices. $D_{tsp}(C_n, i) = \emptyset$ for $i > n$, as a result. On the other hand, if $i \geq n+1$, then $D_{tsp}(C_n, i) = \emptyset$ according to the definition of the twain secure perfect dominating set of a cycle C_n. □

Lemma 2.3. *If Y is inside $D_{tsp}(C_{n-1}, i-1)$, then for every $n \geq 3$, there is a $\{x\} \in [n]$ such that $Y \cup \{x\} \in D_{tsp}(C_n, i)$.*

Proof. The twain secure perfect dominating set of G are indicated by Y. Y has at least one vertex labeled $n-1$, $n-2$ or $n-3$, Since $Y \in D_{tsp}(C_{n-1}, i-1)$.

- If $n-1 \in Y$, $Y \cup \{x\} \in X_1$, a twain secure perfect dominating set of C_n. Thus $X_1 \in D_{tsp}(C_n, i)$.
- If $n-2 \in Y$, $Y \cup \{x\} \in X_2$, a twain secure perfect dominating set of C_n. Thus $X_2 \in D_{tsp}(C_n, i)$.
- If $n-3 \in Y$, $Y \cup \{x\} \in X_3$, a twain secure perfect dominating set of C_n. Thus $X_3 \in D_{tsp}(C_n, i)$.

Thus $Y \cup \{x\}$ is a twain secure perfect dominating set of C_n, in each scenario.
Consequently, $Y \cup \{x\} \in D_{tsp}(C_n, i)$. □

Theorem 2.4. *For every $n \geq 5$,*

(*i*) *If $D_{tsp}(C_{n-1}, i-1) \neq \emptyset$ and $D_{tsp}(C_{n-4}, i-2) = \emptyset$, then $D_{tsp}(C_n, i) = \{[n]\}$.*

(*ii*) *If $D_{tsp}(C_{n-1}, i-1) = \emptyset$ and $D_{tsp}(C_{n-4}, i-2) \neq \emptyset$, then $D_{tsp}(C_n, i) = \{\{1, 2, ..., n-3, n-2\}, \{1, 4, ..., n-3, n\}, \{2, 3, ..., n-2, n-1\}, \{3, 4, ..., n-1, n\}\}$.*

(*iii*) *If $D_{tsp}(C_{n-1}, i-1) \neq \emptyset$ and $D_{tsp}(C_{n-4}, i-2) \neq \emptyset$, then $D_{tsp}(C_n, i) = \{\{X \cup \{n\}$ if X ends with $n-1$ or $n-3\} \cup \{X \cup \{n-1\}$ if X ends with $n-2\} \cup \{Y \cup \{n-3, n-2\}$ if Y starts with 1 and $2 \in Y\} \cup \{Y \cup \{n-3, n\}$ if Y starts with 1 and $2 \notin Y\} \cup \{Y \cup \{n-2, n-1\}$ if Y starts with 2 and ends with $n-5\} \cup \{Y \cup \{n-1, n\}$ if Y starts with 3 and ends with $n-4\}\}$, where $X \in D_{tsp}(C_{n-1}, i-1)$ and $Y \in D_{tsp}(C_{n-4}, i-2)$.*

Proof. (*i*) Since $D_{tsp}(C_{n-1}, i-1) \neq \emptyset$ and $D_{tsp}(C_{n-4}, i-2) = \emptyset$, we have $i = n$. Therefore, $D_{tsp}(C_n, i) = \{[n]\}$.

(*ii*) Since $D_{tsp}(C_{n-1}, i-1) = \emptyset$, $D_{tsp}(C_{n-4}, i-2) \neq \emptyset$, we have $n = 4k, i = 2k$, for every $k \in \mathbb{N}$. It is evident that there are $\frac{n}{2}$ elements in the sets $\{1, 2, ..., n-3, n-2\}, \{1, 4, ..., n-3, n\}, \{2, 3, ..., n-2, n-1\}, \{3, 4, ..., n-1, n\}$. By the definition of twain secure perfect domination of C_n, $1, 2, 5, 6, 9, 10$ cover all vertices up to 12 and $1, 4, 5, 8, 9, 12$ cover all vertices up to 12 and $2, 3, 6, 7, 10, 11$ cover all vertices up to 12 and $3, 4, 7, 8, 11, 12$ cover all vertices up to 12 for $n = 12$. Continuing in this manner, we arrive at the conclusion that the twain secure perfect dominant sets are $\{1, 2, ..., n-3, n-2\}, \{1, 4, ..., n-3, n\}, \{2, 3, ..., n-2, n-1\}, \{3, 4, ..., n-1, n\}$.

(*iii*) $D_{tsp}(C_n, i)$ is constructed based on $D_{tsp}(C_{n-1}, i-1)$ and $D_{tsp}(C_{n-4}, i-2)$.
X can be defined as the twain secure perfect dominating sets of C_{n-1} with cardinality $i-1$. The elements of $D_{tsp}(C_{n-1}, i-1)$ ends with $n-1$ or $n-2$ or $n-3$.

- If $n-1 \in X$, then the elements of $D_{tsp}(C_{n-1}, i-1)$ are adjoined to $D_{tsp}(C_n, i)$ through n.
- If $n-2 \in X$, then the elements of $D_{tsp}(C_{n-1}, i-1)$ are adjoined to $D_{tsp}(C_n, i)$ through $n-1$.
- If $n-3 \in X$, then the elements of $D_{tsp}(C_{n-1}, i-1)$ are adjoined to $D_{tsp}(C_n, i)$ through n.

Y can be defined as the twain secure perfect dominating sets of C_{n-4} with cardinality $i-2$. The elements of $D_{tsp}(C_{n-4}, i-2)$ starts with 1 or 2 or 3.

- If Y begins with 1 and $2 \in Y$, then the elements of $D_{tsp}(C_{n-4}, i-2)$ belongs to $D_{tsp}(C_n, i)$ by adjoining $n-3$ and $n-2$.

- If Y begins with 1 and $2 \notin Y$, then the elements of $D_{tsp}(C_{n-4}, i-2)$ belongs to $D_{tsp}(C_n, i)$ by adjoining $n-3$ and n.
- If Y begins with 2 and ends with $n-5$, then the elements of $D_{tsp}(C_{n-4}, i-2)$ belongs to $D_{tsp}(C_n, i)$ by adjoining $n-2$ and $n-1$.
- If Y begins with 3 and ends with $n-4$, then the elements of $D_{tsp}(C_{n-4}, i-2)$ belongs to $D_{tsp}(C_n, i)$ by adjoining $n-1$ and n.

Thus $\{\{X \cup \{n\}$ if X ends with $n-1$ or $n-3\} \cup \{X \cup \{n-1\}$ if X ends with $n-2\} \cup \{Y \cup \{n-3, n-2\}$ if Y starts with 1 and $2 \in Y\} \cup \{Y \cup \{n-3, n\}$ if Y starts with 1 and $2 \notin Y\} \cup \{Y \cup \{n-2, n-1\}$ if Y starts with 2 and ends with $n-5\} \cup \{Y \cup \{n-1, n\}$ if Y starts with 3 and ends with $n-4\}\} \subseteq D_{tsp}(C_n, i)$. (1)

In contrast, let us assume $Z \in D_{tsp}(C_n, i)$. Here all the elements of $D_{tsp}(C_n, i)$ ends with either $n-2$ or $n-1$ or n.

- If $n-2 \in Z$ and $n-1 \notin Z, n \notin Z$, then Z has at least one vertex with the labels $n-6$ or $n-4$. Suppose $n-4 \in Z$, then $Z = Y \cup \{n-3, n-2\}$ for some $Y \in D_{tsp}(C_{n-4}, i-2)$. Suppose $n-6 \in Z, n-4 \notin Z$, then $Z = Y \cup \{n-3, n-2\}$ for some $Y \in D_{tsp}(C_{n-4}, i-2)$.
- If $n-1 \in Z$ and $n \notin Z$, then Z has at least one vertex with the labels $n-2$ or $n-5$. Suppose $n-5 \in Z$, then $Z = Y \cup \{n-2, n-1\}$ for some $Y \in D_{tsp}(C_{n-4}, i-2)$. Suppose $n-2 \in Z$ and $n-5 \notin Z$ then $Z = Y \cup \{n-1\}$ for some $X \in D_{tsp}(C_{n-1}, i-1)$.
- If $n \in Z$, then Z has at least one vertex with the labels $n-1$ or $n-2$ or $n-3$. Suppose $n-1 \in Z$ and $n-2 \notin Z$, then $Z = Y \cup \{n-1, n\}$, for some $Y \in D_{tsp}(C_{n-4}, i-2)$. Suppose $n-1 \in Z$ and $n-2 \in Z$, then $Z = X \cup \{n\}$, for some $X \in D_{tsp}(C_{n-1}, i-1)$. Suppose $n-3 \in Z$, $n-2 \notin Z$ and $n-1 \notin Z$, then $Z = Y \cup \{n-3, n\}$, for some $Y \in D_{tsp}(C_{n-4}, i-2)$.

Thus $D_{tsp}(C_n, i) \subseteq \{\{X \cup \{n\}$ if X ends with $n-1$ or $n-3\} \cup \{X \cup \{n-1\}$ if X ends with $n-2\} \cup \{Y \cup \{n-3, n-2\}$ if Y starts with 1 and $2 \in Y\} \cup \{Y \cup \{n-3, n\}$ if Y starts with 1 and $2 \notin Y\} \cup \{Y \cup \{n-2, n-1\}$ if Y starts with 2 and ends with $n-5\} \cup \{Y \cup \{n-1, n\}$ if Y starts with 3 and ends with $n-4\}\}$ (2)

From (1) and (2), we have, $D_{tsp}(C_n, i) = \{\{X \cup \{n\}$ if X ends with $n-1$ or $n-3\} \cup \{X \cup \{n-1\}$ if X ends with $n-2\} \cup \{Y \cup \{n-3, n-2\}$ if Y starts with 1 and $2 \in Y\} \cup \{Y \cup \{n-3, n\}$ if Y starts with 1 and $2 \notin Y\} \cup \{Y \cup \{n-2, n-1\}$ if Y starts with 2 and ends with $n-5\} \cup \{Y \cup \{n-1, n\}$ if Y starts with 3 and ends with $n-4\}\}$, where $X \in D_{tsp}(C_{n-1}, i-1)$ and $Y \in D_{tsp}(C_{n-4}, i-2)$.

Consequently, the evidence . □

Theorem 2.5. *If $D_{tsp}(C_n, i)$ is a family of twain secure perfect dominating sets with cardinality i, then for every $n \geq 5$, $|D_{tsp}(C_n, i)| = |D_{tsp}(C_{n-1}, i-1)| + |D_{tsp}(C_{n-4}, i-2)|$.*

Proof. Utilizing Theorem 2.4, we examine each of the four scenarios.

Case (i) : If $D_{tsp}(C_{n-1}, i-1) = \emptyset$ and $D_{tsp}(C_{n-4}, i-2) = \emptyset$, then $D_{tsp}(C_n, i) = \emptyset$.

Case (ii) : If $D_{tsp}(C_{n-1}, i-1) \neq \emptyset$ and $D_{tsp}(C_{n-4}, i-2) = \emptyset$, then $D_{tsp}(C_n, i) = \{[n]\}$.

Case (iii) : If $D_{tsp}(C_{n-1}, i-1) = \emptyset$ and $D_{tsp}(C_{n-4}, i-2) \neq \emptyset$, then $D_{tsp}(C_n, i) = \{\{1, 2, ..., n-3, n-2\}, \{1, 4, ..., n-3, n\}, \{2, 3, ..., n-2, n-1\}, \{3, 4, ..., n-1, n\}\}$.

Case (iv) : If $D_{tsp}(C_{n-1}, i-1) \neq \emptyset$ and $D_{tsp}(C_{n-4}, i-2) \neq \emptyset$, then $D_{tsp}(C_n, i) = \{\{X \cup \{n\}$ if X ends with $n-1$ or $n-3\} \cup \{X \cup \{n-1\}$ if X ends with $n-2\} \cup \{Y \cup \{n-3, n-2\}$ if Y

starts with 1 and $2 \in Y\} \cup \{Y \cup \{n-3, n\}$ if Y starts with 1 and $2 \notin Y\} \cup \{Y \cup \{n-2, n-1\}$ if Y starts with 2 and ends with $n-5\} \cup \{Y \cup \{n-1, n\}$ if Y starts with 3 and ends with $n-4\}\}$,where $X \in D_{tsp}(C_{n-1}, i-1)$ and $Y \in D_{tsp}(C_{n-4}, i-2)$.
By constructing $|D_{tsp}(C_n, i)|$ in each case, we get $|D_{tsp}(C_{n-1}, i-1)| + |D_{tsp}(C_{n-4}, i-2)|$. □

3. Twain Secure Perfect Domination Polynomials of Cycles

Twain secure perfect domination polynomials of C_n is denoted by $D_{tsp}(C_n, x)$. In this section we defined twain Secure Perfect Domination Polynomials of C_n and find some results.

Definition 3.1. The family of all twain secure perfect dominating sets of C_n with cardinality i is denoted as $D_{tsp}(C_n, i)$. Let $d_{tsp}(C_n, i)$ should equal$|D_{tsp}(C_n, i)|$. Then the twain secure perfect domination polynomial of C_n is defined as $D_{tsp}(C_n, x) = \sum_{i=\gamma_{tsp}(C_n)}^{n} d_{tsp}(C_n, i)x^i$, where $\gamma_{tsp}(C_n)$ is the twain secure perfect domination number of C_n.

Theorem 3.2. *For every $n \geq 8$, $D_{tsp}(C_n, x) = x[D_{tsp}(C_{n-1}, x) + xD_{tsp}(C_{n-4}, x)]$ with initial values $D_{tsp}(C_4, x) = 4x^2 + x^4, D_{tsp}(C_5, x) = 5x^3 + x^5, D_{tsp}(C_6, x) = 6x^4 + x^6, D_{tsp}(C_7, x) = 7x^5 + x^7$.*

Proof. We have, $|D_{tsp}(C_n, i)| = |D_{tsp}(C_{n-1}, i-1)| + |D_{tsp}(C_{n-4}, i-2)|$.
That is $d_{tsp}(C_n, i) = d_{tsp}(C_{n-1}, i-1) + d_{tsp}(C_{n-4}, i-2)$.
Therefore $d_{tsp}(C_n, i)x^i = d_{tsp}(C_{n-1}, i-1)x^i + d_{tsp}(C_{n-4}, i-2)x^i$.
$\sum d_{tsp}(C_n, i)x^i = \sum d_{tsp}(C_{n-1}, i-1)x^i + \sum d_{tsp}(C_{n-4}, i-2)x^i$.
$\sum d_{tsp}(C_n, i)x^i = x \sum d_{tsp}(C_{n-1}, i-1)x^{i-1} + x \sum d_{tsp}(C_{n-4}, i-2)x^{i-1}$.
$\sum d_{tsp}(C_n, i)x^i = x[\sum d_{tsp}(C_{n-1}, i-1)x^{i-1} + \sum d_{tsp}(C_{n-4}, i-2)x^{i-1}]$.
$\sum d_{tsp}(C_n, i)x^i = x[\sum d_{tsp}(C_{n-1}, i-1)x^{i-1} + x \sum d_{tsp}(C_{n-4}, i-2)x^{i-2}]$.
Therefore, $D_{tsp}(C_n, x) = x[D_{tsp}(C_{n-1}, x) + xD_{tsp}(C_{n-4}, x)]$ with initial values
$D_{tsp}(C_4, x) = 4x^2 + x^4, D_{tsp}(C_5, x) = 5x^3 + x^5, D_{tsp}(C_6, x) = 6x^4 + x^6, D_{tsp}(C_7, x) = 7x^5 + x^7$.
Hence proved. □

Theorem 3.3. *For the coefficients of $D_{tsp}(C_n, x)$, the following characteristics are true.*
(i) $d_{tsp}(C_n, n) = 1$, for every $n \in \mathbb{N}$.
(ii) $d_{tsp}(C_{n+2}, n) = n + 2$, for every $n \geq 2$.
(iii) $d_{tsp}(C_{n+4}, n) = \frac{1}{2}(n^2 + n - 12)$, for every $n \geq 4$.
(iv) $d_{tsp}(C_{n+6}, n) = \frac{1}{6}(n^3 - 3n^2 - 34n + 120)$, for every $n \geq 6$.
(v) $d_{tsp}(C_{4n}, 2n) = 4$, for every $n \in \mathbb{N}$.
(vi) $d_{tsp}(C_{4n+1}, 2n + 1) = 4n + 1$, for every $n \in \mathbb{N}$.
(vii) $d_{tsp}(C_{4n+2}, 2n + 2) = (n + 1)(2n + 1)$, for every $n \in \mathbb{N}$.

Proof. The table of the twain secure perfect dominating sets of cycle C_n indicates this. □

Theorem 3.4. *For every $i \geq 4$ and $n \geq 3$,*
$\sum_{n=i}^{2i} d_{tsp}(C_n, i) = \sum_{n=i-1}^{2i-2} d_{tsp}(C_n, i-1) + \sum_{n=i-2}^{2i-4} d_{tsp}(C_n, i-2)$.

Proof. On i, we use induction. Let's start by assuming that, $i = 4$. Here is what we obtain:
$\sum_{n=4}^{8} d_{tsp}(C_n, 4) = \sum_{n=3}^{6} d_{tsp}(C_n, 3) + \sum_{n=2}^{4} d_{tsp}(C_n, 2)$.
L.H.S: $\sum_{n=4}^{8} d_{tsp}(C_n, 4) = d_{tsp}(C_4, 4) + d_{tsp}(C_5, 4) + d_{tsp}(C_6, 4) + d_{tsp}(C_7, 4) + d_{tsp}(C_8, 4) = 11$.

n\i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1																
2	2	1															
3	3	0	1														
4	0	4	0	1													
5	0	0	5	0	1												
6	0	0	0	6	0	1											
7	0	0	0	0	7	0	1										
8	0	0	0	4	0	8	0	1									
9	0	0	0	0	9	0	9	0	1								
10	0	0	0	0	0	15	0	10	0	1							
11	0	0	0	0	0	0	22	0	11	0	1						
12	0	0	0	0	0	4	0	30	0	12	0	1					
13	0	0	0	0	0	0	13	0	39	0	13	0	1				
14	0	0	0	0	0	0	0	28	0	49	0	14	0	1			
15	0	0	0	0	0	0	0	0	50	0	60	0	15	0	1		
16	0	0	0	0	0	0	0	4	0	80	0	72	0	16	0	1	
17	0	0	0	0	0	0	0	0	17	0	119	0	85	0	17	0	1

TABLE 1. $d_{tsp}(C_n, i)$, the number of twain secure perfect dominating sets of C_n with cardinality i.

R.H.S: $\sum_{n=3}^{6} d_{tsp}(C_n, 3) + \sum_{n=2}^{4} d_{tsp}(C_n, 2) = d_{tsp}(C_3, 3) + d_{tsp}(C_4, 3) + d_{tsp}(C_5, 3) + d_{tsp}(C_6, 3) + d_{tsp}(C_2, 2) + d_{tsp}(C_3, 2) + d_{tsp}(C_4, 2) = 11$. Thus, L.H.S = R.H.S. Consequently, for $i = 4$, the result holds true. We now prove for $i = k$, assuming that the result is valid for every $i < k$. we obtain, $\sum_{n=k}^{2k} d_{tsp}(C_n, k) = \sum_{n=k}^{2k} d_{tsp}(C_{n-1}, k-1) + \sum_{n=k}^{2k} d_{tsp}(C_{n-4}, k-2)$. This is based on Theorem 2.5 and the induction hypothesis. Which implies, $\sum_{n=k}^{2k} d_{tsp}(C_n, k) = \sum_{n=k-1}^{2k-2} d_{tsp}(C_{n-1}, k-2) + \sum_{n=k-2}^{2k-4} d_{tsp}(C_{n-1}, k-3) + \sum_{n=k-1}^{2k-2} d_{tsp}(C_{n-4}, k-3) + \sum_{n=k-2}^{2k-4} d_{tsp}(C_{n-4}, k-4)$. That gives $\sum_{n=k}^{2k} d_{tsp}(C_n, k) = \sum_{n=k-1}^{2k-2} d_{tsp}(C_n, k-1) + \sum_{n=k-2}^{2k-4} d_{tsp}(C_n, k-2)$. Thus, $\sum_{n=i}^{2i} d_{tsp}(C_n, i) = \sum_{n=i-1}^{2i-2} d_{tsp}(C_n, i-1) + \sum_{n=i-2}^{2i-4} d_{tsp}(C_n, i-2)$, for every $i \geq 4$. □

Theorem 3.5. *If $S_n = \sum_{i=\gamma_{tsp}(C_n)}^{n} d_{tsp}(C_n, i)$, then for every $n \geq 8$, $S_n = S_{n-1} + S_{n-4}$ with initial values $S_4 = 5, S_5 = 6, S_6 = 7$ and $S_7 = 8$.*

Proof. Let us take, $S_n = \sum_{i=\gamma_{tsp}(C_n)}^{n} d_{tsp}(C_n, i)$. We obtain $S_n = \sum_{i=\gamma_{tsp}(C_n)}^{n} (d_{tsp}(C_{n-1}, i-1) + d_{tsp}(C_{n-4}, i-2))$ from Theorem 2.5.
$S_n = \sum_{i\gamma_{tsp}(C_n)}^{n} d_{tsp}(C_{n-1}, i-1) + \sum_{i=\gamma_{tsp}(C_n)}^{n} d_{tsp}(C_{n-4}, i-2)$.
$S_n = \sum_{i=\gamma_{tsp}(C_n)-1}^{n-1} d_{tsp}(C_{n-1}, i) + \sum_{i=\gamma_{tsp}(C_n)-2}^{n-4} d_{tsp}(C_{n-4}, i)$.
$S_n = S_{n-1} + S_{n-4}$.
Thus, $S_n = S_{n-1} + S_{n-4}$ starting with $S_4 = 5, S_5 = 6, S_6 = 7$ and $S_7 = 8$. □

4. Conclusions

Twain secure perfect dominating sets and polynomials of cycles are examined and certain properties obtained in this study. We have derived the important relation of $d_{tsp}(C_n, i)$. Using this relation we have to find out the twain secure perfect domination polynomials of cycles. Thus, the study can be applied to any C_n. In the next study pertaining to this idea several graph features will be investigated.

Real world applications:

- **Critical infrastructure:** For nodes where unique control is needed (to avoid confusion or redundancy), but failover security is critical.
- **Cyber-physical systems:** Where each device is managed by a single authority, but there is a secure backup agent in case of failure.

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